The Characteristic Values of Negative Imaginary Systems and Their Riccati-Balanced Representation

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***Abstract*—This brief introduces the definition of the *Characteristic Values* for Negative Imaginary (NI) systems and proposes a *Riccati-based Model Order Reduction* (MOR) procedure specifically tailored for this class of systems. The approach provides a *balanced state-space representation* that preserves the NI property in the reduced models, ensuring *reliability and robustness* even for very high-order systems. A formal connection with the *Positive Real (PR)* and *Bounded Real (BR)* formulations is remarked, revealing structural analogies that enable NI systems to be expressed in a PR-balanced framework. The resulting procedure combines the theoretical rigor of Riccati equations with the well-known robustness of balancing truncation methods, offering a practical and structure-preserving tool for NI model reduction. The effectiveness of the proposed framework is verified through numerical examples, confirming its accuracy and its advantages over conventional optimization-based MOR techniques.**

***Index Terms*—Positive real systems, bounded real systems, negative imaginary systems, characteristic values, model order reduction.**

I. Introduction

A LINEAR Time-Invariant system (LTI) is Negative-Imaginary (NI) if its transfer function matrix $G(s)$ satisfies the following conditions [1], [2]:

(a) All its poles s_p are strictly in the Left Half Plane (LHP), $\Re\{s_p\} < 0$

(b) For all $\omega \geq 0$, the following inequality holds:

$$j\left[G(j\omega) - G^\top(-j\omega)\right] \geq 0 \tag{1}$$

The study of this class of systems has had a great impact over recent decades, in both methodology [3], [4] and applications [5]. A key motivation is their robustness in negative-feedback schemes [6] and in positive feedback [1], which supports damping-enhancement designs for flexible structures with collocated force actuators and position sensors [7]. Applications at both micro and large scales continue to grow [8], [9]. The NI system topic is rich in the current literature. Analytical results also for non-rational Transfer Functions are reported in [10] and [11]. LMI techniques for robust NI control are introduced in [12], while a Riccati-based state feedback design is presented in [13]. Discrete-time NI systems have been studied in [14] and [15]. Fractional-order NI systems have been investigated in [16], and a theory on NI nonlinear systems has been presented in [3]. In this brief, the interest in NI systems is focused on the definition of their *Characteristic Values* and the development of a new state-space balanced representation for them. Although the concept of *Characteristic Values* has been extensively explored in the literature both for Positive Real (PR) systems and for Bounded Real (BR) systems [17], this topic remains unexplored in the context of NI systems. The definition of *Characteristic Values* for NI systems emerges from Riccati equation-based studies, which will be shown to be strictly related to the assessment of Positive Realness and Bounded Realness of auxiliary systems derived from the original NI system [13]. This connection enables a new characterization and leads to the formulation of the Characteristic-Based Realization for NI systems. Such a representation allows the derivation of Reduced-Order Models (ROMs) of the original NI system while ensuring the preservation of the NI property. Although recent works have proposed MOR techniques for NI systems [18], the strategy presented here provides a direct method to obtain reduced-order systems that preserve the NI property in the resulting low-order models, unlike classical open-loop truncation methods [17]. The brief is structured as follows. Section II presents the main mathematical details regding NI systems, as well as PR and BR systems. In Section III, the main results of the brief are proposed. Section IV include numerical examples, which will highlight the effectiveness of the proposed study. The Conclusions summarize the contents of the brief.

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II. Preliminaries

A. Positive Real Lemma

An LTI system in minimal realization form $S(A, B, C, D)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{m\times p}$ with $p = m$, is PR if and only if there exists a positive-definite matrix P such that:

$$PA + A^\top P = -LL^\top;\ PB = C^\top - LW;\ D + D^\top = W^\top W \tag{2}$$

where $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ [17]. The system is assumed to represent a passive impedance/admittance with transfer function given by:

$$G(s) = C(sI - A)^{-1}B + D. \tag{3}$$

Assuming $D + D^\top > 0$, the matrix P is the stabilizing positive-definite solution of the following Riccati equation:

$$PA + A^\top P + (-PB + C^\top)(D + D^\top)^{-1}(-PB + C^\top)^\top = 0 \tag{4}$$

such that the matrix $A_C = A - BK$, with $K = (D + D^\top)^{-1}(B^\top P + C^\top)$, is a Hurwitz Matrix.

B. Bounded Real Lemma

An LTI system in a minimal realization form $S(A, B, C, D)$ is BR if its transfer function $G(s) = C(sI - A)^{-1}B + D$ belongs to $\mathcal{H}_\infty$, and there exists a symmetric positive-definite matrix P such that:

$$PA + A^\top P = C^\top C - LL^\top;\ PB = C^\top D - LW;\ I - D^\top D = W^\top W \quad (5)$$

where $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ [17]. Provided that $I - D^\top D > 0$, the set of conditions in (5) can be rearranged into the following Riccati equation:

$$P\left(A + B(I - D^\top D)^{-1}D^\top C\right) + \left(A^\top + C^\top D(I - D^\top D)^{-1}B^\top\right)P + PB(I - D^\top D)^{-1}B^\top P + C^\top D(I - D^\top D)^{-1}D^\top C + C^\top C = 0. \quad (6)$$

The positive-definite solution P is assumed to be stabilizing for the closed-loop matrix $A_0 = A - BK$, with feedback gain $K = (I - D^\top D)^{-1}(-B^\top P - D^\top C)$.

C. Negative Imaginary Lemma

An LTI system $S(A, B, C, D)$ with transfer function $G(s) = C(sI - A)^{-1}B + D$ and $R = CB + B^\top C^\top > 0$ is NI if and only if $D = D^\top$ and there exists a symmetric matrix $P > 0$ such that:

$$PA + A^\top P = -L^\top L;\ PB - A^\top C^\top = -L^\top W;\ CP + B^\top C = W^\top W \quad (7)$$

where $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ [13]. The matrix P is the solution of the following Riccati equation:

$$PA + A^\top P + (A^\top C^\top - PB) \times (CB + B^\top C^\top)^{-1}(CA - B^\top P) = 0 \quad (8)$$

such that the closed-loop matrix $A_C = A - BK$ is stable, with feedback gain $K = R^{-1}(B^\top P + A^\top C^\top)$.

Remark 1: The Riccati equation (8) can be equivalently rewritten as [13]:

$$PA^0 + A^{0\top}P + PB^0B^{0\top}P + C^{0\top}C^0 = 0 \quad (9)$$

where $A^0 = A - BR^{-1}CA, B^0 = BR^{-1/2}, C^0 = R^{-1/2}CA$.

III. Riccati Balanced Representation of NI Systems

A. Premises

In general, if a system S is represented in state-space form as $S(A, B, C, D)$, its dual system is given by $S_D(A^\top, C^\top, B^\top, D^\top)$. The transfer function matrices of S and S_D are transposes of each other. The duality principle enables the formulation, for the system S_D, of the Riccati equation that is dual to that of the original system S. This holds for both the continuous-time algebraic Riccati equation (CARE) and the filtering algebraic Riccati equation (FARE), including equations (4)–(6). Even though the primal and dual Riccati equations admit different solutions, denoted by P and Π respectively, their product $P\Pi$ has the important property that the square roots of its eigenvalues remain invariant. In this brief, these quantities, defined with respect to the considered class of systems, are referred to as the *Characteristic Values* of the NI system.

B. Details

Remark 2: According to the NI Lemma, $G(s)$ is NI if and only if the system with realization $S_P(A, B, CA, CB)$ is PR. This equivalence follows from comparing equations (4)–(8).

Remark 3: A necessary and sufficient condition for $G(s)$ to be NI is that the system $S_B(A^0, B^0, C^0)$ is BR.

Remark 4: If a system is NI, then its dual is also NI. In fact, the defining condition (1) is preserved under transposition: since the dual system has transfer function $G^\top(s)$, it satisfies

$$j\left[G^\top(j\omega) - G(-j\omega)\right] \geq 0. \quad (10)$$

C. Characteristic Values of NI Systems

Definition 1: Consider the Riccati equations associated with the PR system described in Remark 2 and its dual, under the assumption that the matrix $CB + B^\top C^\top$ is invertible:

$$PA + A^\top P + \left(A^\top C^\top - PB\right)\left(CB + B^\top C^\top\right)^{-1} \times \left(CA - B^\top P\right)^\top = 0 \quad (11)$$

$$\Pi A^\top + A\Pi + \left(AB - \Pi C^\top\right)\left(CB + B^\top C^\top\right)^{-1} \left(B^\top A^\top - C\Pi\right) = 0 \quad (12)$$

where P and Π are the positive definite solutions of the primal and dual Riccati equations, respectively. The *Characteristic Values* of the NI system $S(A, B, C, D)$ are defined as the square roots of the eigenvalues of the matrix product $P\Pi$. These values are derived from the *Characteristic Values* of the associated PR system $S_P(A, B, C\ A, C\ B)$.

Definition 2: Now consider the Riccati equations associated with the BR system described in Remark 3 and its dual:

$$PA^0 + A^{0\top}P + PB^0B^{0\top}P + C^{0\top}C^0 = 0 \quad (13)$$

$$\Pi A^{0\top} + A^0\Pi + \Pi C^{0\top}C^0\Pi + B^0B^{0\top} = 0 \quad (14)$$

where P and Π again denote the solutions of the primal and dual Riccati equations. Since this pair of Riccati equations is an equivalent reformulation of those in (11)–(12), the same matrix product $P\Pi$ is considered. In this case, the *Characteristic Values* of the NI system are obtained from the BR system $S_B(A^0, B^0, C^0)$.

D. NI Balancing Representation

A balanced realization based on the NI *Characteristic Values* can be obtained by solving the Riccati equations (11) and (12), or alternatively the equations (13) and (14). The balancing procedure consists of finding a state transformation matrix T such that, in the transformed coordinates $(\bar{A}, \bar{B}, \bar{C}, D)$, both P and Π are simultaneously diagonalized as:

$$P_d = \Pi_d = \mathrm{diag}(p_1, p_2, \ldots, p_n) \quad (15)$$

where $p_1, p_2, \ldots, p_n$ are the *Characteristic Values* of the NI system, ordered as:

$$p_1 \geq p_2 \geq \cdots \geq p_n > 0. \quad (16)$$

The two couples of Riccati equations (11)–(12) and (13)–(14) are respectively related to specific variational control problems for PR and BR systems. The positive definite solutions of (11)–(12) are associated with the energy stored in the network and its dual. Conversely, the Lyapunov equations in the Open

Loop Balanced representation express only structural properties related to the system controllability and observability.

Proposition 1: Let S^P_{bil} denote the balanced realization of the PR system defined by equations (11)–(12):

$$S^P_{\text{bil}} = \begin{bmatrix} A^P_{\text{bil}} & B^P_{\text{bil}} \\ C^P_{\text{bil}} & CB \end{bmatrix} \tag{17}$$

The NI balanced realization of the original system $S(A, B, C, D)$ is defined as

$$S^{NI}_{\text{bil}} = \begin{bmatrix} A^{NI}_{\text{bil}} & B^{NI}_{\text{bil}} \\ \bar{C} & D \end{bmatrix} \tag{18}$$

where

$$A^{NI}_{\text{bil}} = A^P_{\text{bil}}, B^{NI}_{\text{bil}} = B^P_{\text{bil}}, \bar{C} = C^P_{\text{bil}} A^{P\ -1}_{\text{bil}} = CT$$

In fact, if T is the transformation matrix that yields the balanced representation (17), and since $C^P_{\text{bil}} = CAT$, it follows that $\bar{C} = CT$. Therefore, the equivalence between the representation (18) and that of $S(A, B, C, D)$ is proved. It is assumed

$$S(A^P_{\text{bil}}, B^P_{\text{bil}}, CT, D)$$

to be the Riccati balanced realization of $S(A, B, C, D)$.

For completeness, note that the Riccati equations (13) and (14) are also balanced under the same transformation, where:

$$A^0_{\text{bil}} = A^P_{\text{bil}} - A^P_{\text{bil}} C^P_{\text{bil}};\ B^0_{\text{bil}} = B^P_{\text{bil}} R^{-1/2};\ C^0_{\text{bil}} = R^{-1/2} C^P_{\text{bil}} A_{\text{bil}} \tag{19}$$

Therefore, the PR equations (11)–(12) may be preferred, as they allow a direct expression of A_{bil} and B_{bil}.

Proposition 2: The NI system S^{NI}_{bil} is NI if and only if the system S^P_{bil} is PR, or equivalently, if the system S^B_{bil} is BR.

Remark 5: The transformation that yields (17) and (18) is generated by the same T that balance equations (13) and (14).

E. Towards NI Model Order Reduction

Proposition 3: Let us consider the NI Lemma. If the system (A, B, CA, CB) is PR, then the system (A, B, C, D) is NI.

Consider the system represented in the form (17) and assume that the solution P of the Riccati equation can be written as:

$$P = \text{diag}(P_1, P_2) \tag{20}$$

where

$$P_1 = \Pi_1 = \text{diag}(p_1, .., p_r) \quad P_2 = \Pi_2 = \text{diag}(p_{r+1}, .., p_n). \tag{21}$$

with the *Characteristic Values* ordered as:

$$p_1 \geq p_2 \geq \cdots \geq p_r \geq p_{r+1} \geq \cdots \geq p_n > 0 \tag{22}$$

Let be the following partitioning of the balanced system matrices A^P_{bil}, B^P_{bil}, and C^P_{bil}:

$$A^P_{\text{bil}} = \begin{bmatrix} A^P_{11\,\text{bil}} & A^P_{12\,\text{bil}} \\ A^P_{21\,\text{bil}} & A^P_{22\,\text{bil}} \end{bmatrix}, \quad B^P_{\text{bil}} = \begin{bmatrix} B^P_{1\,\text{bil}} B^P_{2\,\text{bil}} \end{bmatrix}^\top, \quad C^P_{\text{bil}} = \begin{bmatrix} C^P_{1\,\text{bil}} & C^P_{2\,\text{bil}} \end{bmatrix} \tag{23}$$

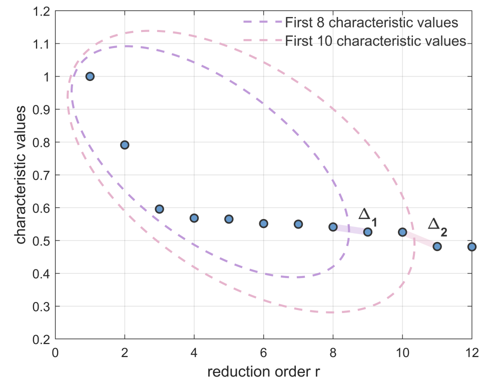


Fig. 1. *Characteristic Values p_i* obtained from the balanced realization in *Example 1*.

where $A^P_{11\,\text{bil}} \in \mathbb{R}^{r \times r}$. The reduced-order system

$$S^r_{\text{bil}}(A^P_{11\,\text{bil}}, B^P_{1\,\text{bil}}, C^P_{1\,\text{bil}}, C^P_{1\,\text{bil}} B^P_{1\,\text{bil}}) \tag{24}$$

is PR, as it is obtained from a balanced realization of a PR system [17]. In accordance with Remark 2, the reduced-order model $S^r_{\text{NI}}(A^P_{11\,\text{bil}}, B^P_{1\,\text{bil}}, C_{1\,\text{bil}} A^{-1}_{11\,\text{bil}}, D)$ is NI. The flow procedure starts from the PR balancing by using Eqs. 11-12, proceeds through Eq. 18, and finally leads to Eqs. 23-24.

Remark 6: Both definitions of *Characteristic Values* for NI systems enable the derivation of a reduced-order model that is guaranteed to be NI. Each value p_i reflects the contribution of the corresponding state to the overall NI property.

Remark 7: The brief's innovations are: (a) the introduction, for NI systems, of a new invariant set of quantities called NI characteristic values; (b) the definition of a new NI balanced representation; (c) the use of points (a) and (b) to develop a novel MOR technique that preserves NI properties.

IV. Numerical Examples

Example 1. Consider a NI system like that in [1], representing a flexible structure with colocated force actuation and position measurements. The system matrices are:

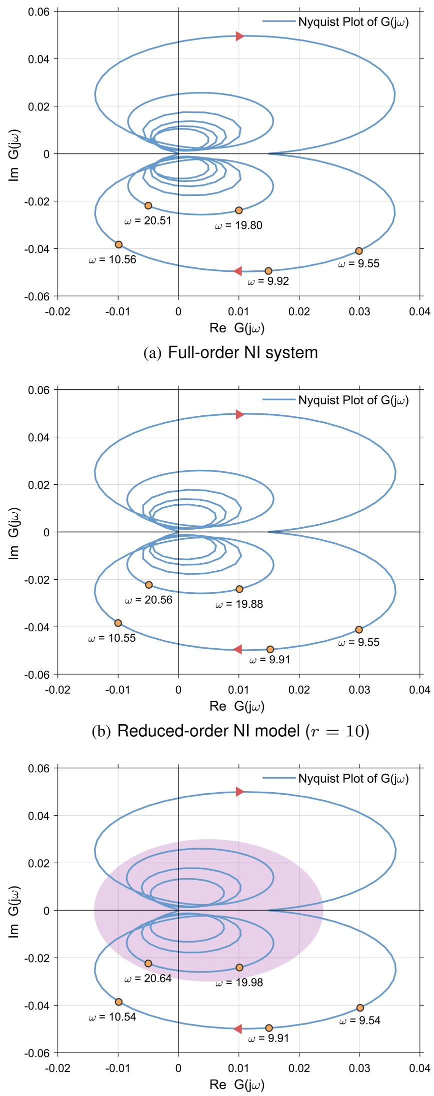
$$A_i = \begin{bmatrix} 0 & 1 \\ -\alpha_i & -2 \end{bmatrix}, \quad \text{with } \alpha_i = i^2 100$$

$$A = \text{diag}(A_1, A_2, \ldots, A_6)$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^\top$$

$$C = \begin{bmatrix} (1\ \varepsilon) & (1\ \varepsilon) & (1\ \varepsilon) & (1\ \varepsilon) & (1\ \varepsilon) & (1\ \varepsilon) \end{bmatrix}$$

The parameter ε is set to approximately 10^{-2} to ensure that $CB^\top \neq 0$. The *Characteristic Values p_i* obtained from the balanced realization are shown in Figure 1. The gap Δ_2 in the same figure justifies selecting a reduced-order model with $r = 10$, discarding the *Characteristic Values* for $r = 11$ and $r = 12$. Figures 2a and 2b display the Nyquist plots of the full-order and ROMs, respectively. The reduced model, obtained using $r = 10$ *Characteristic Values*, preserves the NI property while providing an excellent match in the frequency domain. A satisfactory agreement is also observed for the reduced-order model with $r = 8$, as illustrated in Figure 2c. Although the NI property is still preserved, the approximation degrades slightly especially at high-frequency, as highlighted in the plot.



(a) Full-order NI system

(b) Reduced-order NI model ($r = 10$)

(c) Reduced-order NI model ($r = 8$)

Fig. 2. Nyquist plots of NI systems in *Example 1*.

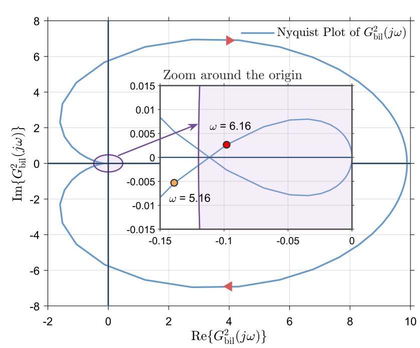


Fig. 3. Red marked NI behavior violation in MOR obtained by classical open-loop balanced truncation in *Example 2*.

Example 2. Consider the system, clearly NI, defined by the transfer function

$$G_{NI}(s) = \frac{N(s)}{D(s)} = \frac{0.002963s^4 + 0.3096s^3 + 1.353s^2 + 2.01s + 1}{s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1}.$$

TABLE I

PHASE MARGIN OF REDUCED-ORDER MODELS

Order r	Transfer Function	Phase Margin [deg]
5 (full-order)	$G^5_{NI}(j\omega)$	27.5894
4	$G^4_{NI}(j\omega)$	26.37
3	$G^3_{NI}(j\omega)$	26.35
2	$G^2_{NI}(j\omega)$	22.42
1	$G^1_{NI}(j\omega)$	n.a.

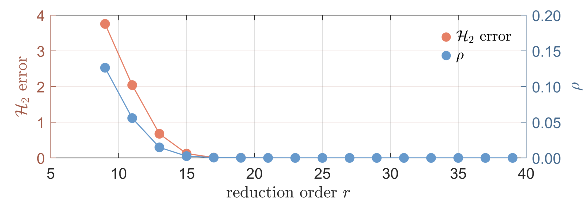


Fig. 4. Discharged *Characteristic Values* sum ρ and $\mathcal{H}_2$ error norm vs. r.

TABLE II

SETS S_α OF RANDOMLY GENERATED PARAMETERS ω_i AND ζ_i

	Set S_1		Set S_2		Set S_3		Set S_4		Set S_5	
i	ω_i	ζ_i	ω_i	ζ_i	ω_i	ζ_i	ω_i	ζ_i	ω_i	ζ_i
1	0.268	0.925	0.695	0.246	0.051	0.642	0.334	0.557	0.788	0.415
2	0.589	0.539	0.634	0.960	0.501	0.530	0.846	0.627	0.084	0.587
3	0.937	0.256	0.726	0.970	0.738	0.275	0.600	0.211	0.639	0.125
4	0.912	0.557	0.762	0.525	0.238	0.760	0.254	0.652	0.865	0.488
5	0.739	0.684	0.447	0.638	0.960	0.166	0.329	0.523	0.276	0.312
6	0.140	0.974	0.307	0.443	0.379	0.097	0.125	0.453	0.667	0.500
7	0.425	0.076	0.665	0.822	0.935	0.803	0.993	0.399	0.444	0.326
8	0.108	0.972	0.319	0.657	0.730	0.795	0.062	0.293	0.566	0.015
9	0.570	0.484	0.505	0.157	0.574	0.362	0.552	0.740	0.553	0.936
10	0.645	0.218	0.305	0.105	0.161	0.886	0.318	0.494	0.482	0.329

The *Characteristic Values* of the system are computed as $p_1 = 1.0$, $p_2 = 0.8546$, $p_3 = 9.18 \times 10^{-2}$, $p_4 = 5.89 \times 10^{-4}$, and $p_5 = 9.55 \times 10^{-7}$. For each reduced-order transfer function G^r_{NI} with $r = 4, r = 3, r = 2, r = 1$, we evaluated the phase margin. The results are reported in Table I. For $r = 1$, the computed phase margin would be positive, however, the result is not reliable because of the severe degradation of the Nyquist plot, hence it is denoted as "n.a." in the table. The trend shown in Table I confirms that, although the ROMs preserve the NI property, the decrease of the phase margin is linked to the elimination of state variables weighted by the *Characteristic Values*. It is worth noting that $p_2 \gg p_3$, which indicates that a reduction to order $r = 2$ can be considered according to the proposed method. The resulting transfer function is

$$G^2_{\rm NI}(s) = \frac{N^2_{\rm NI}(s)}{D^2_{\rm NI}(s)} = \frac{0.0296s + 2.9816}{s^2 + 0.6214s + 0.2982},$$

which preserves the NI property. For comparison, a second-order reduced model was obtained via the classical Lyapunov-based open-loop balanced truncation [17]. It results:

$$G^2_{\rm bil}(s) = \frac{N^2_{\rm bil}(s)}{D^2_{\rm bil}(s)} = \frac{-0.1198\ s + 3.6684}{s^2 + 1.0588s + 0.3710}$$

which clearly does not satisfy the NI property. As shown in Fig.3, the reduced model's Nyquist plot approximately matches the original system; however, the zoom near the origin, in purple, highlights a violation of the NI behavior.

Example 3. The study reported in [18] is taken into consideration. In that work, a mechanical system composed of ten oscillators was analyzed, where both the natural frequencies ω

TABLE III
COMPARISON OF $\mathcal{H}_2$ ERROR NORMS: MABROK–NI–MOR [18] VS. PRESENT–NI–MOR, TESTED ON FIVE RANDOM PARAMETER SETS (S_1–S_5) FOR VARIOUS REDUCED ORDERS r

reduced order r	r=6	r=8	r=10	r=12	r=14
$\mathcal{H}_2$ Mabrok[18]	0.2403	0.2003	0.1434	0.0943	0.0143
$\mathcal{H}_2$ Present S_1	0.2805	0.2753	0.0360	0.0029	0.0003
$\mathcal{H}_2$ Present S_2	0.6719	0.5027	0.0187	0.0022	0.0000
$\mathcal{H}_2$ Present S_3	4.9170	1.1012	0.1279	0.0074	0.0029
$\mathcal{H}_2$ Present S_4	5.5075	3.1245	0.0556	0.0122	0.0005
$\mathcal{H}_2$ Present S_5	2.5091	0.9426	0.3012	0.0429	0.0005

and the damping ratios ζ were randomly selected, and the $\mathcal{H}_2$ errors for ROMs of order $r = 6, 8, 10, 12, 14$ were reported. To compare the procedure proposed in the present brief with that in [18], five sets S_α of ten pairs (ω_i, ζ_i) were generated with a uniform distribution in the interval $(0, 1)$, as listed in Table II. For each of these sets, ROMs with $r = 6, 8, 10, 12, 14$ were derived, and the corresponding $\mathcal{H}_2$ errors were computed, as shown in Table III. The comments on the results presented in Table III are: a) The optimal procedure [18] generally performs well when low-order reduced models are considered (typically for orders $6 \sim 8$). b) For higher ROMs, the proposed direct truncation procedure performs better (orders $12 \sim 14$). This occurs because the optimization-based MOR procedure, involving a larger number of optimization variables, may lead to numerically imperfect results. c) The balancing approximation procedure was introduced to overcome the drawbacks of classical optimization-based MOR strategies, which can produce imprecise solutions [19]. d) Employing the original model expressed in balanced form within the optimization procedure could improve the accuracy of the obtained reduced-order model [20]. Moreover, a large number of mechanical oscillators was considered ($N = 20$, order $n = 40$) and several tests were performed with randomly chosen parameters ω_i and ζ_i, deriving ROMs of different orders. Both the numerical efficiency of the proposed procedure and a simulated empirical relationship between the $\mathcal{H}_2$ error norm and the quantity $\rho = \sum_{i=r}^{n} p_i$, quantifying the discharged *Characteristic Values*, were verified. This relationship, shown in Fig. 4, exhibits the same monotonic trend for both $\mathcal{H}_2$ and ρ with respect to reduced order r, thus suggesting ρ as a potential performance index.

Remark 8: Applying the proposed method to the examples in [21] and [22] confirms improved frequency–response accuracy (magnitude and especially phase), based on additional tests on the corresponding models and MOR procedures.

V. CONCLUSION

In this contribution we proposed the definition of *Characteristic Values* for NI systems and a new Riccati-based MOR procedure specifically tailored to them. The proposed NI-MOR procedure, based on the *Characteristic Values*, provides a reliable and robust framework, even for very high-order systems, ensuring the preservation of the NI property in the reduced models. The method establishes a direct connection between the PR and BR formulations, revealing structural analogies that allow NI systems to be represented in a balanced form. Numerical examples have been presented to verify the effectiveness of the proposed procedure, confirming its accuracy and efficiency across different scenarios. The well-known robustness of balancing truncation methods, compared with optimization-based reduction schemes, further motivates the adoption of the proposed approach for future NI-oriented MOR studies and practical implementations.

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