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ORIGINAL RESEARCH

How to Measure Multidimensional Variation?

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Abstract
The coefficient of variation, which measures the dispersion of a distribution around its mean, is not uniquely defined in the multidimensional case, and so is the Gini index, which measures the dispersion of a distribution in terms of the mean differences among its observations. In this paper, we bridge these two notions of dispersion by proposing a multidimensional coefficient of variation which generalizes the Gini index. We show that our proposed multivariate coefficient of variation coincides with Voinov-Nikulin's coefficient of variation up to a multiplicative factor. We then introduce and study a set of four properties that a multivariate coefficient of variation should have and show that our proposal possesses all of them. Our proposal is practically compared with the existing multivariate coefficients of variation on both real and simulated data.

Keywords Coefficient of variation · Gini index · Multivariate coefficients of variation

1 Introduction

1.1 Measuring the Dispersion of a Set of Values

The coefficient of variation is the standard measure to summarize through a scalar value the dispersion of a set of points in a statistical distribution. For one-dimensional distributions, the universally accepted definition of the coefficient of variation (CV) is as follows:

$$\mathrm{CV}(\mu) = \frac{\sigma}{|m|}, \tag{1}$$

where σ and m are the standard deviation and mean, respectively, of a distribution μ. Albeit the coefficient in Eq. 1 has been used for more than a century to handle one-dimensional data, there is still no universally accepted way to measure the dispersion of a multidimensional

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distribution. In fact, different multivariate extensions of the coefficient of variation CV have been presented throughout time. However, only a few of them enjoy the most basic property that the univariate coefficient of variation possesses, i.e., being scale invariant (see Auricchio et al., 2025b).

A similar issue occurs for another measure of dispersion, the Gini index. The (univariate) Gini index is the average difference between the distribution values, normalized by the mean of the distribution. Given a distribution μ supported over $\mathbb{R}$ with mean $m \neq 0$, the Gini index (Gini, 1914, 1921; Ceriani and Verme, 2012) is defined as

$$\mathrm{G}_1(\mu) := \frac{1}{2|m|} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y| \mu(dx)\mu(dy). \tag{2}$$

Notice that if μ is supported on $[0, \infty)$, we have $0 \leq \mathrm{G}_1(\mu) \leq 1$, making the index G_1 easier to interpret. When $\mathrm{G}_1(\mu) = 0$, μ is a Dirac delta: in economics applications, this describes a situation of equality, in which all individuals have the same wealth. On the other hand, when $\mathrm{G}_1(\mu)$ approaches 1, μ becomes more dispersed, and inequality arises: wealth is concentrated among a few individuals (see, e.g., Lesca and Perny, 2010).

Despite the differences, the coefficient of variation in Eq. 1 and the Gini index in Eq. 2 have many common traits, as they both aim to summarize the information about the dispersion of a distribution through a scalar value that does not depend on the unit of measurement. These similarities become apparent when we consider a Gaussian distribution.

Let $\mu = \mathcal{N}(m, \sigma)$ be a Gaussian distribution with $m, \sigma > 0$. We then have that

$$\mathrm{G}_1(\mu) = \frac{1}{2|m|} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y| \mu(dx)\mu(dy) = \frac{\sigma}{\sqrt{\pi}|m|}. \tag{3}$$

Thus, for a Gaussian distribution, the Gini index is proportional to the coefficient of variation. Note also that, if we consider an alternative "squared" version of the Gini index:

$$\mathrm{G}_2(\mu) := \Big(\frac{1}{2m^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^2 \mu(dx)\mu(dy) \Big)^{\frac{1}{2}}, \tag{4}$$

based on the L^2 norm rather than the L^1 norm as in Eq. 2, we have that

$$\begin{aligned}
\mathrm{G}_2(\mu) &= \Big(\frac{1}{2m^2} \int_{\mathbb{R}} \int_{\mathbb{R}} (x^2 - 2xy + y^2) \mu(dx)\mu(dy) \Big)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}|m|} \Big(\int_{\mathbb{R}} x^2 \mu(dx) - 2m^2 + \int_{\mathbb{R}} y^2 \mu(dy) \Big)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}|m|} \Big(2\sigma^2 \Big)^{\frac{1}{2}} = \frac{\sigma}{|m|},
\end{aligned}$$

which shows that, for any distribution μ, the "squared" Gini index G_2 coincides exactly with the coefficient of variation.

1.2 Related Work

The univariate coefficient of variation is a well-established measure of dispersion that has been widely adopted by the scientific community. Despite its widespread use, a universally accepted method to extend this coefficient to distributions supported over high-dimensional spaces has yet to be developed. This lack of clarity on how a multivariate coefficient of variation (MCV) should be defined has led to multiple definitions: (i) Reyment's coefficient of variation (Reyment, 1960), (ii) Van Valen's coefficient of variation (Van Valen, 1974), (iii) Albert and Zhang's coefficient of variation (Albert and Zhang, 2010), and (iv) Voinov-Nikulin's coefficient of variation (Voinov and Nikulin, 2012). Although these coefficients have been recently unified under a general definition in Colin and Ossikovski (2024), they all exhibit different properties. For example, Voinov-Nikulin's coefficient of variation is the only scale-invariant MCV, i.e., it does not change if we rescale one or more of the components of the random vector under study by changing their respective units of measure (Aerts et al., 2015).

The univariate Gini index was firstly introduced by the Italian statistician Corrado Gini (1914, 1921) along with the less famous index proposed by Pietra (1915). The hunger for measures of dispersion, especially in economics applications, is still alive, as witnessed by the ever-growing amount of works in this direction (Giudici et al., 2025; Betti and Lemmi, 2008; Coulter, 2019; Babaei et al., 2025; Hao and Naiman, 2010). We refer the reader to Tsui (1995); Maasoumi (1986); Banerjee et al. (2020); Eliazar (2018); Eliazar and Giorgi (2020) for an exhaustive review of this topic. Similarly to the coefficient of variation, there is no universally accepted extension of the Gini index to the multivariate case, and for this reason, different higher-dimensional extensions have been proposed over time. The earliest approach relies on differential geometry methods and was developed by Taguchi (1972a, 1972b, 1988) and Lunetta (1972). While theoretically sound, these first efforts had the disadvantage of treating the coordinate variables in a symmetric manner. Subsequent contributions were made by Arnold (2008b, 2008a), Arnold and Sarabia (2018), Gajdos and Weymark (2005), Koshevoy and Mosler (1996; 1997), Decanq (2012), and Mosler (2002).

Unfortunately, as discussed in Arnold and Sarabia (2018), these multivariate extensions are largely guided by elegant mathematical frameworks, but often lack practical applicability and interpretability. This issue has been partially addressed by Sarabia and Jordá (2095), who employed Arnold's definition of the multivariate Lorenz curve to derive closed-form expressions for the bivariate Lorenz surface based on a class of bivariate distributions with fixed marginals (Sarmanov, 1966; Ting Lee, 1996). A different and more practical approach is taken by a recent line of research, which defines a high-dimensional Gini index based on the principal components of the data, rather than on the data itself, as shown in Toscani (2022). This approach has led to the definition of new scale-invariant dispersion measures (Giudici et al., 2025), as well as to a natural way to express the multivariate Gini index as a suitable convex combination of the Gini indices of its principal components (Auricchio et al., 2025b). This has then been fruitfully used in Auricchio et al. (2025a) to define discrepancies that are stable under change of scale. Lastly, a novel approach has been recently presented in Capaldo and Navarro (2025), where the authors extend the classical univariate measure of dispersion to the multivariate setting by defining "dependence–dispersion" indices with covariance representations which can be interpreted in terms of conditional distributions.

1.3 Our Contribution and Structure of the Paper

Building on the previously described line of research and, in particular, on Auricchio et al. (2025b); Giudici et al. (2025), we propose a novel multivariate coefficient of variation (MCV) which leverages the connection between the Gini index and the coefficient of variation in the univariate framework, established in Section 1. Following the approach outlined in Hurley and Rickard (2009), we identify a set of essential properties that a multivariate coefficient of variation should satisfy. These properties serve as criteria for comparing our proposed

MCV with existing multivariate coefficients of variation, highlighting the advantages and limitations of each approach.

The paper is organized as follows. After some preliminary background in Section 2, in Section 3, we define the properties that a multivariate coefficient of variation should have, in analogy with those of the univariate coefficient Eq. 1. In Section 4, we present a Gini-based multivariate coefficient of variation and demonstrate its properties, along with those of Voinov-Nikulin's MCV, to which it is strictly related. In Section 5, we consider the other existing MCVs in terms of their properties. Section 6 presents some examples and simulation studies which help clarify the importance of our proposal. Finally, Section 7 concludes.

2 Background

2.1 Basic Notions

In what follows, we denote with $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. Let $q \geq 1$, and let $\mathcal{P}_q(\mathbb{R}^n)$ denote the class of all probability measures μ on $\mathbb{R}^n$ with finite moment of order q, that is

$$\int_{\mathbb{R}^n} |\mathbf{x}|^q \mu(d\mathbf{x}) < +\infty. \tag{5}$$

Henceforth, we write $\mathbf{X} \sim \mu$ to indicate a random vector $\mathbf{X}$ and its associated probability measure $\mu \in \mathcal{P}_q(\mathbb{R}^n)$ and use $\mathbf{X}$ and μ interchangeably. In particular, given a random vector $\mathbf{X}$ and an MCV γ, we denote the dispersion of $\mathbf{X}$ according to γ with $\gamma(\mathbf{X})$ or $\gamma(\mu)$, depending on which expression is more convenient in the context. Notice that, for every $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, the mean value vector $\mathbf{m} := \mathbb{E}(\mathbf{X}) = \int_{\mathbb{R}^n} \mathbf{x}\, \mu(d\mathbf{x})$ and the covariance matrix $\mathbf{\Sigma}$, defined as

$$(\mathbf{\Sigma})_{i,j} = cov(X_i, X_j) := \int_{\mathbb{R}^n} x_i\, x_j\, \mu(d\mathbf{x}) - \left(\int_{\mathbb{R}^n} x_i \mu(d\mathbf{x})\right) \left(\int_{\mathbb{R}^n} x_j \mu(d\mathbf{x})\right),$$

are well-defined. Moreover, whenever the variance of each X_i is not null, the correlation matrix of μ, namely $\mathbf{P}$, is well-defined and defined as

$$\mathbf{P}_{i,j} = \frac{(\mathbf{\Sigma})_{i,j}}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}}.$$

We remark that $\mathbf{\Sigma}$ and $\mathbf{P}$ are symmetric and positive semidefinite. In what follows, we assume that $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, that $\mathbf{\Sigma}$ is invertible, and that $\mathbf{m} \neq 0$, unless we specify otherwise.

2.2 Whitening

Given a n-dimensional random vector $\mathbf{X}$ whose mean is $\mathbf{m}$ and invertible covariance matrix is $\mathbf{\Sigma}$, we have that the n-dimensional random vector $\mathbf{Y} = \mathbf{W}\mathbf{X}$ has mean $\mathbf{W}\mathbf{m}$ and covariance matrix $\mathbf{W}^\top \mathbf{\Sigma} \mathbf{W}$ for every $n \times n$ matrix $\mathbf{W}$ (see, e.g., Mahalanobis, 2018; Li and Zhang, 1998). We say that $\mathbf{W}$ is a whitening matrix for a random vector $\mathbf{X}$ if the covariance matrix of $\mathbf{W}\mathbf{X}$ is the $n \times n$ identity matrix (namely $\mathbf{Id}_n$), that is $\mathbf{W}^\top \mathbf{\Sigma} \mathbf{W} = \mathbf{Id}_n$ or, equivalently, if

$$\mathbf{W}^\top \mathbf{W} = \mathbf{\Sigma}^{-1} \tag{6}$$

as shown in Li and Zhang (1998).

Owing to the fact that the whitening matrix is not unique, there are a variety of whitening processes that are commonly used, as argued in Kessy et al. (2018). Indeed, it is easy to see that, given a whitening matrix $\mathbf{W}$ for $\mathbf{X}$ and an orthogonal matrix $\mathbf{Z}$, the matrix $\mathbf{W}' = \mathbf{ZW}$ satisfies Eq. 6. However, as shown in Auricchio et al. (2025b), only a few of these processes possess the *scale stability* property, which ensures that the random vector obtained by whitening $\mathbf{X} := (X_1, X_2, \ldots, X_n)$ or $\mathbf{X}' = (X_1, X_2, \ldots, aX_i, \ldots, X_n)$ is the same for every $a > 0$ and $i \in [n] := \{1, 2, \ldots, n\}$.

Remark 1 Note that, if the covariance matrix $\mathbf{\Sigma}$ is not invertible, the analysis can be carried out by restricting attention to the linear subspace spanned by the principal components associated with non-zero eigenvalues, as in Navarro (2016). Equivalently, one may reduce the dimension of the random vector by projecting it onto its effective support, thereby eliminating directions with zero variance (Rao, 1973).

In Auricchio et al. (2025b), it was shown that the zero-phase components analysis whitening transformation applied to the correlation matrix (ZCA-cor whitening) is scale stable. Given any $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, the ZCA-cor whitening process is defined as

$$\mathbf{W}_{\mu}^{ZCA} = \mathbf{P}^{-1/2}\mathbf{V}^{-1/2} = \mathbf{G}^{\top}\Theta^{-1/2}\mathbf{G}\mathbf{V}^{-1/2}, \tag{7}$$

where (i) $\mathbf{V}$ is a diagonal matrix containing the variances of the components of $\mathbf{X}$, (ii) $\mathbf{G}$ is the orthonormal matrix induced by the eigenvectors of $\mathbf{P}$, the correlation matrix of $\mathbf{X}$, and (iii) Θ is the diagonal matrix containing the eigenvalues of $\mathbf{P}$. From now on, any whitening matrix is tacitly assumed to be the ZCA-cor whitening matrix, unless we specify otherwise. In particular, given a random vector $\mathbf{X} \sim \mu$, we denote with $\mathbf{W}_{\mu}^{ZCA}\mathbf{X} =: \mathbf{X}^* \sim \mu^*$ the vector obtained via the ZCA-cor whitening process and refer to the entries of $\mathbf{X}^*$ as the principal components of $\mathbf{X}$.

2.3 Coefficient of Variation and Its High-Dimensional Generalizations

Given a one-dimensional random variable with non-null mean, the standard way to measure its dispersion is through the coefficient of variation, defined as $\mathrm{CV}(\mu) := \sigma/|m|$, where σ is the standard deviation of the random variable and $m \neq 0$ its mean.

To the best of our knowledge, there are four extensions of the coefficient of variation to the multivariate setting. Given a probability measure $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, whose mean is $\mathbf{m}$ and covariance matrix is $\mathbf{\Sigma}$, the known multivariate coefficients of variation are defined as follows:

1. Voinov-Nikulin's coefficient of variation (Voinov and Nikulin, 2012), which is defined as
$$\gamma_{\mathrm{VN}}(\mu) := \sqrt{\frac{1}{\mathbf{m}^{\top}\mathbf{\Sigma}^{-1}\mathbf{m}}}; \tag{8}$$
2. Reyment's coefficient of variation (Reyment, 1960), which is defined as
$$\gamma_{\mathrm{R}}(\mu) := \sqrt{\frac{det(\mathbf{\Sigma})^{\frac{1}{n}}}{\mathbf{m}^{\top}\mathbf{m}}}; \tag{9}$$
3. Van Valen's coefficient of variation (Van Valen, 1974), which is defined as
$$\gamma_{\mathrm{VV}}(\mu) := \sqrt{\frac{\mathrm{tr}(\mathbf{\Sigma})}{\mathbf{m}^{\top}\mathbf{m}}}; \tag{10}$$

4. Albert and Zhang's coefficient of variation (Albert and Zhang, 2010), which is defined as

$$\gamma_{\mathrm{AZ}}(\mu) := \sqrt{\frac{\mathbf{m}^\top \boldsymbol{\Sigma} \mathbf{m}}{(\mathbf{m}^\top \mathbf{m})^2}}. \tag{11}$$

3 Properties of Multivariate Coefficients of Variation

3.1 Multivariate Coefficients of Variation

The aim of a multivariate coefficient of variation is to measure the dispersion of a multivariate statistical distribution. When the underlying random vector represents economic quantities, low dispersion indicates high inequality, and high dispersion indicates low inequality. Because of this connection, the properties that a multivariate coefficient of variation should possess have been studied in the economic literature (see, e.g., Atkinson, 1970) and have been recently summarized in Hurley and Rickard (2009), a reference paper in the information science community.

In this section, we recall such properties. Two of them, i.e., scale stability and the rising tide, are natural extensions of properties that the coefficient of variation

$$\mathrm{CV}(\mu) = \frac{\sigma}{|m|} \tag{12}$$

possesses.

The other two, i.e., coherence and cloning, are properties that naturally arise when considering the dispersion of a multivariate random vector. Coherence requires the MCV to be equal to CV on any one-dimensional distributions with a non-null mean. Cloning requires that the MCV does not change when applied to multiple i.i.d. copies of the same random vector.

The properties defined in this section will be used as guidelines to analyze our proposed multivariate coefficient of variation, in Section 4, and to compare it with the available alternatives, in Section 5.

3.2 Coherence

The first property that a multivariate coefficient of variation should possess is that, when applied to a one-dimensional measure, it coincides with the univariate coefficient of variation in Eq. 12. We name this property coherence.

Definition 3.1 A multivariate coefficient of variation γ is coherent if $\gamma(\mu) = \frac{\sigma}{|m|}$ for every $\mu \in \mathcal{P}_2(\mathbb{R})$ with mean $m \neq 0$ and variance σ^2.

3.3 Scale Invariance

Another important property that an MCV should possess is scale invariance. This property ensures that the dispersion computed by a MCV does not depend on the specific unit of measurement used to describe the data. This property is especially important in economics, where random variables may be expressed in different monetary units.

Definition 3.2 (Scale invariance) Given a random n-dimensional vector $\mathbf{X} \sim \mu$, with $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, and a $n \times n$ invertible matrix $\mathbf{A}$, we denote with $\mu_{\mathbf{A}}$ the probability measure associated with the random vector $\mathbf{Y} = \mathbf{AX}$.

A coefficient of variation γ is scale invariant if

$$\gamma(\mu_{\mathbf{A}}) = \gamma(\mu)$$

for any invertible $n \times n$ matrix $\mathbf{A}$.

Scale invariance ensures that the dispersion of a measure depends only on the probability measure itself, and neither on the unit of measure nor on the coordinates we use to describe the space. Notice that, if γ is scale invariant, the dispersion of any random vector $\mathbf{X}$ is equal to the dispersion of its principal components $\mathbf{X}^*$, that is $\gamma(\mathbf{X}) = \gamma(\mathbf{X}^*)$.

3.4 Rising Tide

The rising tide property, introduced in Hurley and Rickard (2009), states that the dispersion of a positive random variable diminishes when we add to it a positive constant term. It is important to note that this property was introduced for probability distributions supported over $[0, \infty)$. Indeed, in the general case, this property is not well-defined since, given a random variable X with mean $m < 0$, the variable $X + |m|$ has null mean. The same problem occurs when we consider random vectors. Owing to these considerations, we propose the following extension of the rising tide property to the multivariate case.

Definition 3.3 Let γ be a multivariate coefficient of variation and let $\mathbf{X} \sim \mu$ be an n-dimensional random vector.

Denoted with $\mathbf{m}$ and $\mathbf{\Sigma}$ the mean and covariance matrix associated with μ, we say that γ satisfies the rising tide property if

$$\gamma(\mathbf{X} + \mathbf{c}) \leq \gamma(\mathbf{X})$$

for every $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\top \mathbf{\Sigma}^{-1} \mathbf{m} \geq 0$.

Note that the condition $\mathbf{c}^\top \mathbf{\Sigma}^{-1} \mathbf{m} \geq 0$ is necessary to ensure that the mean of the principal components of a random vector $\mathbf{X}$ is not zero. Indeed, given a whitening matrix $\mathbf{W}$ associated to $\mathbf{X}$, we have that the mean of $\mathbf{X}^* = \mathbf{WX}$ is $\mathbf{m}^* = \mathbf{Wm}$. Therefore, when we add a constant value $\mathbf{c}$ to $\mathbf{X}$, we need to ensure that $\mathbf{m}^*$ and the whitened vector $\mathbf{Wc}$ point in the same direction. This happens if and only if $(\mathbf{Wc})^\top \mathbf{Wm} \geq 0$ or equivalently, if

$$\mathbf{c}^\top \mathbf{W}^\top \mathbf{Wm} = \mathbf{c}^\top \mathbf{\Sigma}^{-1} \mathbf{m} \geq 0.$$

Remark 2 When X is a random variable, the condition $\mathbf{c}^\top \mathbf{\Sigma}^{-1} \mathbf{m} \geq 0$ simplifies to $cm > 0$; hence, $c \neq 0$ and m must have the same sign.

Notice that the coefficient of variation CV possesses this property.

Indeed, the random vectors X and $X + c$ have the same standard deviation.

Moreover, if $mc > 0$, we have that

$$\mathrm{CV}(X + c) = \frac{\sigma}{|m + c|} < \frac{\sigma}{|m|} = \mathrm{CV}(\mu).$$

This property does not hold if $mc < 0$.

3.5 Cloning

The *cloning property* states that the value of a multivariate coefficient of variation does not change when the dimension of the system is doubled by adding an *independent copy* of the original random vector. To avoid ambiguity, the duplication is formalized through independent random vectors rather than by repeating the same realization. More formally, let $\mathbf{X} \sim \mu$ be an n-dimensional random vector, and let $\mathbf{Y} \sim \mu$ be an independent random vector with the same distribution as $\mathbf{X}$. The independent coupling $(\mathbf{X}, \mathbf{Y})$ is then the $2n$-dimensional random vector obtained by juxtaposing $\mathbf{X}$ and $\mathbf{Y}$ (see Hurley and Rickard (2009)). The cloning property thus reads as follows.

Definition 3.4 A coefficient of variation γ satisfies the *cloning property* if, for any n-dimensional random vector $\mathbf{X}$ and any independent random vector $\mathbf{Y}$ identically distributed as $\mathbf{X}$, it holds that

$$\gamma(\mathbf{X}) = \gamma\big((\mathbf{X}, \mathbf{Y})\big).$$

4 Gini Index as a Coefficient of Variation

4.1 Bridging the Gini Index with the Coefficient of Variation

In this section, we introduce a new multivariate coefficient of variation, drawing on the connection between the one-dimensional squared Gini index and the coefficient of variation CV shown in Section 1. We will show that the new MCV is proportional to Voinov- Nikulin's coefficient of variation.

We will then exemplify the calculation of the proposed index for two known distributions: the multivariate Gaussian distribution and the multinomial distribution. We will finally show that our proposal yields a multivariate coefficient of variation which satisfies all the properties discussed in the previous section.

Definition 4.1 Let $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ be a probability measure, and let $\mathbf{m}$ be its mean value and $\boldsymbol{\Sigma}$ be its covariance matrix. The squared Gini index of μ is defined as follows:

$$\mathrm{G}_2(\mu) := \left(\frac{1}{2\mathbf{m}^\top \boldsymbol{\Sigma}^{-1} \mathbf{m}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathbf{x}-\mathbf{y})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\mathbf{y}) \mu(d\mathbf{x}) \mu(d\mathbf{y}) \right)^{\frac{1}{2}}. \tag{13}$$

We now demonstrate that G_2 is equal to Voinov-Nikulin's MCV up to a constant multiplicative factor.

Theorem 4.2 *Given any probability measure $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ whose mean is non-null $\mathbf{m} \neq 0$ and invertible covariance matrix $\boldsymbol{\Sigma}$, we have that*

$$G_2(\mu) = \sqrt{n}\, \gamma_{VN}(\mu) = \sqrt{\frac{n}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1} \mathbf{m}}}. \tag{14}$$

Proof By definition, we have that

$$
\begin{aligned}
G_2^2(\mu) &= \frac{1}{2\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{x}-\mathbf{y})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y}) \\
&= \frac{1}{2\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{x} + \mathbf{y}^\top \boldsymbol{\Sigma}^{-1}\mathbf{y} - 2\mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y})\Big) \\
&= \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{x} - \mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y})\Big) \\
&= \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \Big(\int_{\mathbb{R}^n} (\mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{x})\mu(d\mathbf{x}) - \mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}\Big) \\
&= \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \Big(\int_{\mathbb{R}^n} (\mathbf{x}-\mathbf{m})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{m})\mu(d\mathbf{x})\Big)
\end{aligned}
$$

since $\boldsymbol{\Sigma}^{-1}$ is symmetric.

We then rewrite $G_2^2(\mu)$ as follows:

$$
G_2^2(\mu) = \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \int_{\mathbb{R}^n} (\mathbf{W}(\mathbf{x}-\mathbf{m}))^\top (\mathbf{W}(\mathbf{x}-\mathbf{m}))\mu(d\mathbf{x}); \tag{15}
$$

where $\mathbf{W}$ is a whitening matrix.

If we change variable and set $\mathbf{y}^* = \mathbf{W}(\mathbf{x}-\mathbf{m})$, we have that

$$
G_2^2(\mu) = \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \int_{\mathbb{R}^n} ||\mathbf{y}^*||_2^2 \mu^*(d\mathbf{y}^*), \tag{16}
$$

where μ^* is a probability measure whose marginals have null mean and unitary variance.

To conclude, we notice that

$$
\begin{aligned}
G_2^2(\mu) &= \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \int_{\mathbb{R}^n} ||\mathbf{y}^*||_2^2 \mu^*(d\mathbf{y}^*) = \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \int_{\mathbb{R}^n} \sum_{i=1}^n |y_i^*|^2 \mu^*(d\mathbf{y}^*) \\
&= \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \sum_{i=1}^n \int_{\mathbb{R}^n} |y_i^*|^2 \mu^*(d\mathbf{y}^*) = \frac{1}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}} \sum_{i=1}^n 1 = \frac{n}{\mathbf{m}^\top \boldsymbol{\Sigma}^{-1}\mathbf{m}},
\end{aligned}
$$

since each marginal of μ^* has unitary variance and null mean. □

Theorem 4.2 has two important implications.

First, it builds on the identity presented in Section 1 between the squared Gini index and the coefficient of variation, extending it to the multivariate case. Notice that this connection is not trivial: in Eq. 4, the norm used to compute the squared Gini index is the square of the standard Euclidean distance while in Eq. 13, we use the Mahalanobis metric (Mahalanobis, 2018), i.e., $||\mathbf{x}||_M^2 := \mathbf{x}^\top \boldsymbol{\Sigma}^{-1}\mathbf{x}$. However, in one dimension, the Mahalanobis metric boils down to $\frac{x^2}{\sigma^2}$, hence

$$
G_2^2(\mu) = \frac{1}{2m^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^2 \mu(dx)\mu(dy) = \frac{1}{2\frac{m^2}{\sigma^2}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|x-y|^2}{\sigma^2}\mu(dx)\mu(dy)
$$

for any one-dimensional probability distribution with non-null mean m and non-null variance σ^2. Thus, the one-dimensional version of our coefficient of variation proposed in Eq. 13 is consistent with that used in the introduction. Moreover, we notice that the usage of the Mahalanobis distance to quantify the dispersion is perfectly in line with previous works such as Navarro (2016); Chen (2007), where this metric is used to extend the Chebychev inequality to multivariate quantities.

Second, it identifies the squared Gini index as proportional to Voinov-Nikulin's coefficient of variation, with a correction term equal to $\sqrt{n}$. This simple observation points out that, among the several options, Voinov-Nikulin's coefficient of variation is the rightful multidimensional extension of the classic coefficient of variation, albeit corrected by a multiplicative constant.

4.2 Examples

In this subsection, we exemplify the defined multivariate coefficient of variation for two well-established probability distributions: the multivariate Gaussian and the multinomial. For the sake of simplicity, we restrict our study to the case in which all the distributions are supported over $\mathbb{R}^2$ and their covariance matrix is invertible.

4.2.1 First Case in Study: The Gaussian Distribution

Let us first consider the multivariate Gaussian distribution $\mu \sim \mathcal{N}_k(\mathbf{m}, \mathbf{\Sigma})$, for a k-dimensional random vector. For the sake of simplicity, we assume that $k = 2$. Assuming $\mathbf{m} = (m_1, m_2) \neq 0$ and denoting the elements of the covariance matrix as

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & c \\ c & \sigma_2^2 \end{pmatrix}, \tag{17}$$

we have that the determinant of $\mathbf{\Sigma}$ is equal to $\Delta = \sigma_1^2\sigma_2^2 - c^2 \neq 0$. In particular, we have that

$$\mathbf{\Sigma}^{-1} = \frac{1}{\Delta}\begin{pmatrix} \sigma_2^2 & -c \\ -c & \sigma_1^2 \end{pmatrix}. \tag{18}$$

By Eq. 13, we then have that

$$\mathrm{G}_2(\mu) = \sqrt{\frac{2\Delta}{m_1^2\sigma_2^2 + m_2^2\sigma_1^2 - 2cm_1m_2}}. \tag{19}$$

When considering higher-dimensional normal distributions, i.e., $k \geq 3$, one can perform the same computation and observe that the coefficient of variation G_2 remains bounded and non-zero, provided the distribution is non-degenerate. In contrast, the Voinov-Nikulin MCV tends to zero. This behavior is particularly evident when the multivariate random vector has independent components. Indeed, if $\mathbf{X} = (X_1, \ldots, X_k)$ with i.i.d. components X_i, then, owing to the cloning property, G_2 remains equal to $\mathrm{CV}(X_1)$ regardless of k, while γ_{VN} tends to zero as k increases.

4.2.2 Second Case in Study: The Multinomial Distribution

Consider now a multinomial distribution obtained from n trials of an experiment with k possible outcomes. Without loss of generality, consider $k = 3$. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the random vector representing the counts of outcomes in n independent trials, where each trial results in exactly one of three possible outcomes. Let the probability of outcome $i \in \{1, 2, 3\}$ on a single trial be p_i, with

$$p_1 + p_2 + p_3 = 1, \quad \text{and} \quad p_i \geq 0 \text{ for all } i.$$

Then, $\mathbf{X}$ follows a multinomial distribution with parameters n and (p_1, p_2, p_3), denoted

$$\mathbf{X} \sim \text{Multinomial}(n; p_1, p_2, p_3),$$

whose distribution is given by

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \begin{cases} \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}, & \text{if } x_1 + x_2 + x_3 = n, \\ 0, & \text{otherwise.} \end{cases}$$

The expected value and the variance of each component are

$$\mathbb{E}[X_i] = np_i \qquad \text{and} \qquad \text{Var}(X_i) = np_i(1 - p_i),$$

for $i = 1, 2, 3$.

The covariance between different components is given by

$$\text{Cov}(X_i, X_j) = -np_i p_j$$

for every $i \neq j$. Since the covariance matrix of the multinomial distribution is singular, we consider the distribution of the first two components of the random vector, that is (X_1, X_2). In this case, the mean vector is $n(p_1, p_2)$, whereas the covariance matrix has the form

$$\mathbf{\Sigma} = n \begin{pmatrix} p_1(1 - p_1) & -p_1 p_2 \\ -p_1 p_2 & p_2(1 - p_2) \end{pmatrix}, \tag{20}$$

whose determinant is $p_1 p_2(1 - p_1 - p_2)$. A simple computation gives

$$\mathbf{\Sigma}^{-1} = \frac{1}{n(1 - p_1 - p_2)} \begin{pmatrix} \frac{(1-p_2)}{p_1} & 1 \\ 1 & \frac{(1-p_1)}{p_2} \end{pmatrix}, \tag{21}$$

hence

$$m^\top \mathbf{\Sigma}^{-1} m = \frac{n}{(1 - p_1 - p_2)}((1 - p_2)p_1 + 2p_1 p_2 + (1 - p_1)p_2) = \frac{n(p_1 + p_2)}{(1 - p_1 - p_2)}, \tag{22}$$

leading to

$$\text{G}_2(\mu) = \sqrt{\frac{2(1 - p_1 - p_2)}{n(p_1 + p_2)}}. \tag{23}$$

By the same argument used for the Gaussian distribution, we observe that for higher-dimensional multinomial distributions, the Voinov-Nikulin MCV tends to zero as k increases, while G_2 remains stable as long as each $p_i > 0$ (i.e., the distribution is non-degenerate).

4.3 Properties of G_2

In what follows, we show that our proposed MCV, G_2, possesses all the properties highlighted in Section 3. We will also show that Voinov-Nikulin's coefficient γ_{VN} possesses three of them. We summarize the findings of this section in Table 1.

4.3.1 Coherence

The coherence of G_2 follows from the fact that, for $n = 1$, we have that $\mathbf{\Sigma}^{-1} = \frac{1}{\sigma^2}$, thus $\text{G}_2(\mu) = \sqrt{\frac{1}{\frac{m^2}{\sigma^2}}} = \frac{\sigma}{|m|}$. Notice that, since for $n = 1$, $\text{G}_2(\mu) = \gamma_{\text{VN}}(\mu)$ for every probability distribution $\mu \in \mathcal{P}_2(\mathbb{R})$, also Voinov-Nikulin's coefficient of variation is coherent.

Table 1 Summary of the different properties possessed by the multivariate coefficients of variation

Property	G_2	VN	R	AZ	VV
Coherence	Yes	Yes	Yes	Yes	Yes
Cloning	Yes	No	No	Yes	Yes
Scale invariancy	Yes	Yes	No	No	No
Rising tide	Yes	Yes	No	No	No

4.3.2 Scale Invariance

Owing to Theorem 4.2 and to the scale invariance of Voinov-Nikulin's coefficient of variation (see Aerts et al., 2015), we have that G_2 is scale invariant as well.

4.3.3 Rising Tide Property

We now demonstrate that G_2 satisfies the rising tide property. Recall that, if $\mathbf{\Sigma}$ is the covariance matrix of a random vector $\mathbf{X}$, the random vector $\mathbf{X}+\mathbf{c}$ has the same covariance matrix $\mathbf{\Sigma}$ for any $\mathbf{c} \in \mathbb{R}^n$.

Theorem 4.3 *The G_2 multivariate coefficient of variation satisfies the rising tide property.*

Proof of Theorem 4.3 Let $\mathbf{X}$ be a random vector with covariance matrix $\mathbf{\Sigma}$ and mean $\mathbf{m}$. Given $\mathbf{c}$ such that $\mathbf{c}^\top \mathbf{\Sigma}^{-1}\mathbf{m} \geq 0$, we have that

$$\mathrm{G}_2(\mathbf{X}+\mathbf{c}) = \sqrt{\frac{n}{(\mathbf{m}+\mathbf{c})^\top \mathbf{\Sigma}^{-1}(\mathbf{m}+\mathbf{c})}} = \sqrt{\frac{n}{\mathbf{m}^\top \mathbf{\Sigma}^{-1}\mathbf{m} + 2\mathbf{c}^\top \mathbf{\Sigma}^{-1}\mathbf{m} + \mathbf{c}^\top \mathbf{\Sigma}^{-1}\mathbf{c}}}$$
$$\leq \sqrt{\frac{n}{\mathbf{m}^\top \mathbf{\Sigma}^{-1}\mathbf{m}}} = \mathrm{G}_2(\mathbf{X}),$$

which concludes the proof. □

The same argument used to prove G_2 possesses the rising tide property can be adapted to show that Voinov and Nikulin's γ_{VN} possesses it as well.

4.3.4 Cloning

The G_2 multivariate coefficient of variation possesses the cloning property. Indeed, given a random vector $\mathbf{X}$ with mean $\mathbf{m}$ and covariance matrix $\mathbf{\Sigma}$, we have that the independent coupling $(\mathbf{X}, \mathbf{X})$ has mean $(\mathbf{m}, \mathbf{m})$ and covariance matrix and its inverse are

$$\mathbf{\Sigma}_{(\mathbf{X},\mathbf{X})} = \begin{bmatrix} \mathbf{\Sigma} & 0 \\ 0 & \mathbf{\Sigma} \end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma}^{-1}_{(\mathbf{X},\mathbf{X})} = \begin{bmatrix} \mathbf{\Sigma}^{-1}, & 0 \\ 0, & \mathbf{\Sigma}^{-1} \end{bmatrix}, \tag{24}$$

where 0 denotes the $n \times n$ null matrix. Thus, $(\mathbf{m}, \mathbf{m})^\top \mathbf{\Sigma}^{-1}_{(\mathbf{X},\mathbf{X})}(\mathbf{m}, \mathbf{m}) = 2\mathbf{m}^\top \mathbf{\Sigma}^{-1}\mathbf{m}$. In particular, we infer that

$$\mathrm{G}_2\big((\mathbf{X}, \mathbf{X})\big) = \sqrt{\frac{2n}{2\mathbf{m}^\top \mathbf{\Sigma}^{-1}\mathbf{m}}} = \sqrt{\frac{n}{\mathbf{m}^\top \mathbf{\Sigma}^{-1}\mathbf{m}}} = \mathrm{G}_2(\mathbf{X}).$$

Remark 3 Notice that, while G_2 possesses the cloning property, γ_{VN} does not, due to the lack of the corrective term $\sqrt{n}$.

Indeed, a direct computation shows that

$$\gamma_{\mathrm{VN}}((\mathbf{X},\mathbf{X})) = \sqrt{\frac{1}{2\mathbf{m}^\top\boldsymbol{\Sigma}^{-1}\mathbf{m}}} = \frac{1}{\sqrt{2}}\gamma_{\mathrm{VN}}(\mathbf{X}). \tag{25}$$

In particular, the cloning property is a property that our proposal G_2 possesses, and that does not come from γ_{VN}.

4.3.5 Properties of G_1

To conclude this section, we compare the coefficient of variation G_2 with its variant introduced in Auricchio et al. (2025b), which we denote by G_{l_1}, as it is the l_1-norm counterpart to our G_2 function.

Given a random vector $\mathbf{X}\sim\mu$, the Gini index G_{l_1} is defined as follows:

$$\begin{aligned}\mathrm{G}_{l_1}(\mu) &= \frac{1}{2\sum_{l=1}^{n}|(\mathbf{W}_\mu\mathbf{m}_\mu)_l|}\int_{\mathbb{R}^n}\sum_{i=1}^{n}\left|\left(\mathbf{W}_\mu(\mathbf{x}-\mathbf{y})\right)_i\right|\mu(d\mathbf{x})\nu(d\mathbf{y})\\ &= \sum_{i=1}^{n}\frac{|(\mathbf{W}_\mu\mathbf{m}_\mu)_i|}{\sum_{l=1}^{n}|(\mathbf{W}_\mu\mathbf{m}_\mu)_l|}\mathrm{G}_1((\mathbf{W}_\mu\mathbf{X})_i),\end{aligned}$$

where $\mathbf{W}_\mu$ is a scale stable whitening process, $\mathbf{m}_\mu$ is the mean vector associated with $\mathbf{X}\sim\mu$, and G_1 is the classic one-dimensional Gini index applied to the i-th component of the random vector obtained by whitening $\mathbf{X}$.

It is easy to see that if $n=1$, then G_{l_1} is equal to the classic Gini index, G_1.

Owing to Eq. 3, we can infer that G_{l_1} is not coherent since $\mathrm{G}_{l_1}(\mu)\neq \mathrm{CV}(\mu)$ on every one-dimensional Gaussian distribution μ.

It can be shown that G_{l_1} is scale invariant and satisfies the cloning property. Moreover, under the right circumstances, it satisfies the rising tide property, as shown in Auricchio et al. (2025b). However, it is not coherent.

5 Other Multivariate Coefficients of Variation

In this section, we study the properties of the other MCVs introduced in Section 2.3. We do not consider Voinov-Nikulin's coefficient of variation γ_{VN}, as it has already been discussed in Section 4 along with G_2. Moreover, as it was shown in Aerts et al. (2015), only γ_{VN} is scale invariant, so we omit a further analysis of this property.

5.1 Reyment's Coefficient of Variation

We start from the Reyment's coefficient of variation, that is

$$\gamma_{\mathrm{R}}(\mu) = \sqrt{\frac{det(\boldsymbol{\Sigma})^{\frac{1}{n}}}{\mathbf{m}^\top\mathbf{m}}}.$$

Note that this coefficient is coherent, since $\det(\boldsymbol{\Sigma})=\sigma^2$ when $n=1$. Despite being coherent, γ_{R} does not possess any of the other properties outlined in Section 3.

We now show that γ_{R} does not possess the cloning property. Let $\mathbf{X}$ be a random vector with mean $\mathbf{m}$ and covariance $\boldsymbol{\Sigma}$; we have that an independent coupling $(\mathbf{X},\mathbf{X})$ has mean

$(\mathbf{m}, \mathbf{m})$ and covariance matrix as in Eq. 24. We then have that $det(\boldsymbol{\Sigma}_{(\mathbf{X},\mathbf{X})}) = det(\boldsymbol{\Sigma})^2$ and $(\mathbf{m}, \mathbf{m})^\top(\mathbf{m}, \mathbf{m}) = 2\mathbf{m}^\top\mathbf{m}$, thus

$$\gamma_{\mathrm{R}}((\mathbf{X}, \mathbf{X})) = \sqrt{\frac{det(\boldsymbol{\Sigma}_{(\mathbf{X},\mathbf{X})})^{\frac{1}{2n}}}{(\mathbf{m}, \mathbf{m})^\top(\mathbf{m}, \mathbf{m})}} = \sqrt{\frac{det(\boldsymbol{\Sigma})^{\frac{1}{n}}}{2\mathbf{m}^\top\mathbf{m}}} < \sqrt{\frac{det(\boldsymbol{\Sigma})^{\frac{1}{n}}}{\mathbf{m}^\top\mathbf{m}}} = \gamma_{\mathrm{R}}(\mathbf{X}).$$

Therefore, we have that γ_{R} does not satisfy the cloning property.

We then show that γ_{R} does not satisfy the rising tide property via a counterexample. Let us consider a random vector $\mathbf{X} = (X_1, X_2)$ whose mean is $\mathbf{m} = (3, 3)$ and covariance matrix and its inverse are

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{26}$$

Let us now consider the vector $\mathbf{c} = (1, -2)$ and $\mathbf{Y} = \mathbf{X} + \mathbf{c}$. A simple computation shows that $\mathbf{c}^\top\boldsymbol{\Sigma}^{-1}\mathbf{m} = 3 > 0$. To conclude, we need to show that $\gamma_{\mathrm{R}}(\mathbf{Y}) > \gamma_{\mathrm{R}}(\mathbf{X})$. The mean of $\mathbf{Y}$ is $(4, 1)$ while the covariance matrix of $\mathbf{Y}$ remains $\boldsymbol{\Sigma}$ defined as in Eq. 26. We have that $\gamma_{\mathrm{R}}(\mathbf{X}) = \sqrt{\frac{1}{18}} < \sqrt{\frac{1}{17}} = \gamma_{\mathrm{R}}(\mathbf{Y})$; thus, γ_{R} does not satisfy the rising tide property.

5.2 Van Valen's Coefficient of Variation

We consider Van Valen's coefficient of variation, that is

$$gamma_{\mathrm{VV}}(\mu) = \sqrt{\frac{\mathrm{tr}(\boldsymbol{\Sigma})}{\mathbf{m}^\top\mathbf{m}}}.$$

This coefficient is coherent since $\mathrm{tr}(\boldsymbol{\Sigma}) = \sigma^2$ when $n = 1$.

Moreover, γ_{VV} possesses the cloning property. Indeed, given a random vector $\mathbf{X}$ with mean $\mathbf{m}$ and covariance matrix $\boldsymbol{\Sigma}$, we have that the coupling $(\mathbf{X}, \mathbf{X})$ has mean $(\mathbf{m}, \mathbf{m})$ and covariance matrix as in Eq. 24. We then have that $(\mathbf{m}, \mathbf{m})^\top(\mathbf{m}, \mathbf{m}) = 2\mathbf{m}^\top\mathbf{m}$ and $\mathrm{tr}(\boldsymbol{\Sigma}_{(\mathbf{X},\mathbf{X})}) = 2\mathrm{tr}(\boldsymbol{\Sigma})$, thus

$$\gamma_{\mathrm{VV}}\big((\mathbf{X}, \mathbf{X})\big) = \sqrt{\frac{\mathrm{tr}(\boldsymbol{\Sigma}_{(\mathbf{X},\mathbf{X})})}{(\mathbf{m}, \mathbf{m})^\top(\mathbf{m}, \mathbf{m})}} = \sqrt{\frac{2\mathrm{tr}(\boldsymbol{\Sigma})}{2\mathbf{m}^\top\mathbf{m}}} = \gamma_{\mathrm{VV}}(\mathbf{X}).$$

The same example used for γ_{R} can be employed to show that γ_{VV} does not possess the rising tide property.

5.3 Albert and Zhang's Coefficient of Variation

First, we notice that Albert and Zhang's coefficient of variation γ_{AZ} is coherent as we have that $\mathbf{m}^\top\boldsymbol{\Sigma}\mathbf{m} = m^2\sigma^2$ when $n = 1$, hence

$$\gamma_{\mathrm{AZ}}(\mu) = \sqrt{\frac{m^2\sigma^2}{m^4}} = \frac{\sigma}{|m|}$$

when μ is the probability distribution supported over $\mathbb{R}$.

However, γ_{AZ} does not possess the rising tide property. Let $\mathbf{X} = (X_1, X_2)$ be a random vector such that $\mathbf{m} = (1, 0.1)$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}. \tag{27}$$

It is then easy to see that $\gamma_{AZ}(\mathbf{X}) = \sqrt{\frac{2}{(1.001)^2}} < \sqrt{2}$. Let us then consider $\mathbf{c} = (0, 0.99)$. Since $\mathbf{\Sigma}$ is a diagonal matrix, we have that $\mathbf{c}^\top \mathbf{\Sigma}^{-1}\mathbf{m} > 0$. Moreover, the mean of $\mathbf{X} + \mathbf{c}$ is $(1, 1)$ and its covariance matrix remains $\mathbf{\Sigma}$ as in Eq. 27. We then have

$$\gamma_{AZ}(\mathbf{X} + \mathbf{c}) = \sqrt{\frac{(1,1)^\top \mathbf{\Sigma}(1,1)}{((1,1)^\top (1,1))^2}} = \sqrt{\frac{101}{4}} > \sqrt{2} > \gamma_{AZ}(\mathbf{X}),$$

which shows that γ_{AZ} does not possess the rising tide property.

Consider now the cloning property. Given $\mathbf{X}$ a random vector with mean $\mathbf{m}$ and covariance $\mathbf{\Sigma}$, then we have that an independent coupling $(\mathbf{X}, \mathbf{X})$ has mean $(\mathbf{m}, \mathbf{m})$ and covariance matrix as in Eq. 24. We then have that $(\mathbf{m}, \mathbf{m})^\top \mathbf{\Sigma}_{(\mathbf{X},\mathbf{X})}(\mathbf{m}, \mathbf{m}) = 2\mathbf{m}^\top \mathbf{\Sigma}\mathbf{m}$, hence

$$\gamma_{AZ}\big((\mathbf{X}, \mathbf{X})\big) = \sqrt{\frac{2\mathbf{m}^\top \mathbf{\Sigma}\mathbf{m}}{(2\mathbf{m}^\top \mathbf{m})^2}} = \frac{1}{\sqrt{2}}\gamma_{AZ}(\mathbf{X}) \neq \gamma_{AZ}(\mathbf{X}).$$

6 Application

To conclude, we ran three batches of experiments to validate our proposal. In the experiments, we compare our coefficient of variation with previous ones to assess the dispersion of both synthetic and real datasets.

6.1 Simulated Data

In this subsection, we compare our proposal with the other multivariate coefficients of variation on two simulated experiments.

In the first experiment, we compute the multivariate coefficients of variation for a sequence of random samples simulated from multivariate Gaussian vectors of increasing dimension. This experiment complements our study by validating and reinforcing our theoretical results.

In the second experiment, we consider a sequence of random vectors that describes the trajectory of 100 points which move according to a Galton-like time series and compute the multivariate coefficients of variation at the endpoints. The goal of this experiment is to empirically assess the behavior of the MCV as the variance of the random vector increases.

6.1.1 Multivariate Gaussian Distribution

In this first experiment, we consider an equally spaced sequence of dimensions $n = 10, 15, \ldots, 50$, and for each of them, we sample 500 points from a n-dimensional Gaussian vector, with covariance matrix $\mathbf{\Sigma} = 2\mathbf{Id}_n$, where $\mathbf{Id}_n$ is the $n \times n$ identity matrix and with mean either: (i) $\mathbf{m} = (2, \ldots, 2) \in \mathbb{R}^n$ or (ii) $\mathbf{m}$ an n-dimensional vector whose entries are sampled uniformly in $[1, 2]$.

From the theoretical results presented in Sections 4 and 5, we expect the five coefficients of variation to converge at different limits. Figure 1 presents the results for the simulated data. Figure 1 shows that, in line with the theoretical results, (i) γ_{VN}, γ_{AZ}, and γ_R converge to zero at a rate equal to $\frac{1}{\sqrt{n}}$, as the dimension n increases; regardless of the mean values, (ii) γ_{VV} and $G_2 = \gamma_{CorrVN}$ converge to $\sqrt{\frac{\sigma^2}{m^2}} = \frac{1}{\sqrt{2}} \sim 0.71$ when $\mathbf{m} = (2, \ldots, 2)$ and to $\frac{\sqrt{2}}{\frac{3}{2}} = \frac{2\sqrt{2}}{3} \sim 0.94$ when the means are sampled from a uniform distribution. (iii) γ_{VN}, γ_{AZ},

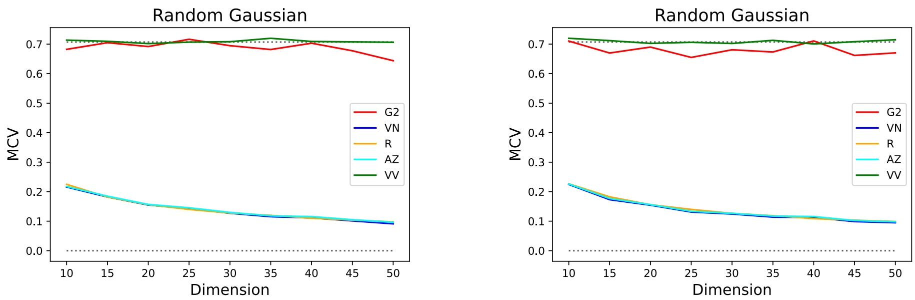

Fig. 1 On the left, the values attained by all the MCVs (where G_2 denotes the 2-Gini, VN is Voinov-Nikulin's MCV, R is Reyment's MCV, AZ is Albert and Zhang's MCV, and VV is Van Valen's MCV) for the n-dimensional independent Gaussian samples with means $\mathbf{m} = (2, \ldots, 2)$. The dotted line indicates the theoretical limits to which the MCV should converge. On the right, the same results for the n-dimensional independent Gaussian samples with means sampled from a uniform distribution

and γ_{R} converge to a value that is independent from the means of the distribution, whereas the limiting value of γ_{VV} and γ_{CorrVN} depends on the means, with a higher value in the case of a variable mean, whose sequence fluctuates more.

6.1.2 Galton-Like Trajectories

In the second experiment, we study the behavior of MCVs when the eigenvalues of the covariance matrix associated with the input random vector gradually increase. In particular, we consider a set of particles whose position changes at every time step, under the effect of a random variable. More precisely, if the position of a particle at time step t is x_t, at the next time step, the particle will move one unit to the right with probability 0.5 or one unit to the left with probability 0.5. Given a time horizon T, each initial position of a particle induces a random trajectory of T points, which we consider as a T-dimensional vector. We denote such trajectories as Galton-like trajectories, as they are inspired by the experiments of Sir Francis Galton. We consider 100 particles, with different initial points and, correspondingly, 100 T-dimensional random trajectories, where T is the time horizon we consider: the number of times we update the position of each particle. In our setting, we consider $\mathrm{T} \leq 90$ and study

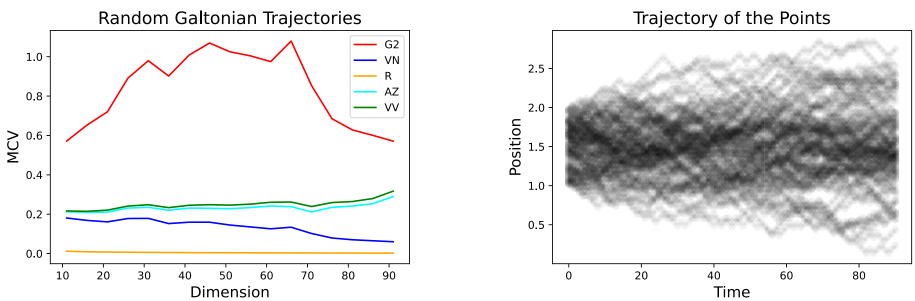

Fig. 2 On the left, the values of the multivariate coefficients of variation (where G_2 denotes the 2-Gini, VN is Voinov-Nikulin's MCV, R is Reyment's MCV, AZ is Albert and Zhang's MCV, and VV is Van Valen's MCV) for the Galtonian trajectories. On the right, a visualization of the Galtonian trajectories

Table 2 Univariate CV for ESG scores

	ESG	E.Sc	S.Sc	G.Sc
Mean	0.65	0.76	0.51	0.62
Sdev	0.11	0.17	0.23	0.15
CV	0.16	0.22	0.45	0.24

how the values of the multivariate coefficients of variation (MCV) change as time increases. The starting positions of each particle are sampled uniformly from $[1, 2]$. In Fig. 2, we report the evolution of the five MCVs of the sampled Galtonian trajectories, for a sequence of equally spaced times $T = 10, 15, \ldots, 85, 90$ along with the simulated trajectories.

Figure 2 shows that γ_{R} and γ_{VN} converge to zero as T increases. Although this is in line with the theoretical results presented in Sections 4 and 5, it does not seem intuitive, as the initial points are different. On the other hand, γ_{VV}, γ_{CorrVN}, and γ_{AZ} converge, although slowly, to a non-zero value. Note that γ_{AZ} does not converge to zero, which is different from the results obtained in the first experiment. This is due to the fact that the entries of the Galton trajectories are not independent.

In this subsection, we consider comparing our proposal to Voinov-Nikulin's index on a set of real data that concerns an important economic problem: the sustainability of companies, as measured by their environmental, social, and governance (ESG) factors.

We consider annual data from a set of small and medium enterprises across various sectors, in the period from 2020 to 2022. The data, provided by modefinance, a fintech rating agency, contains the ESG scores of the considered companies. The data also contain the classification of each company in the economic sectors, following the Global Industry Classification Standard (GICS).

The resulting data consists of a total of $1,062$ observations, corresponding to $1,062$ different small and medium enterprises (SMEs). Table 2 reports a summary analysis of the ESG metrics for the considered companies.

From Table 2, note that the overall ESG scores have the lowest variability (CV = 0.16). Environmental scores (E.Sc) and governance scores (G.Sc) have a slightly higher variability. Finally, social scores (S.Sc) have a much higher variability.

We would like to assess the overall variability of the data and how it changes along the three considered dimensions. The assessment will have two main practical implementations. On one hand, a multidimensional coefficient of variation will inform us on which ESG pillar (E, S, or G), or their combination, is more variable and, thus, more difficult to predict and control. On the other hand, it will tell us for which pillar or combination there is a more unequal implementation across companies.

To this aim, Table 3 compares the Voinov-Nikulin with our proposal to measure the multidimensional coefficient of variation for different combinations of E,S,G, scores.

From Table 3 note that the Voinov-Nikulin coefficient of variation indicates a smaller variation than that measured at the univariate level, and especially when all three dimensions are considered, whereas our proposed Gini CV indicates a variation which is similar to the

Table 3 Multivariate CV for ESG scores

	E,S	E,G	S,G	E,S,G
Voinov-Nikulin CV	0.21	0.17	0.22	0.17
Gini CV	0.29	0.24	0.31	0.29

mean of the univariate measures, regardless of the dimension. This difference depends on the $\sqrt{n}$ correction factor, present in our proposal, but not in Voinov and Nikulin's coefficient.

6.2 Application to the ESG Scores

In this subsection, we validate our methodology by comparing the five multivariate coefficients of variation (MCVs), namely the squared Gini (G_2), Voinov-Nikulin's MCV (VN), Reyment's MCV (R), Albert and Zhang's MCV (AZ), and Van Valen's MCV (VV), computed from ESG scores with those obtained from a set of financial (FIN) indicators. In particular, we examine the concentration of environmental, social, and governance (ESG) indicators across firms within the same national market, and complement this with a parallel analysis of financial metrics—namely total assets, shareholder funding, and total turnover (collectively denoted as "FIN"). This framework enables us to evaluate how the proposed sparsity indices capture heterogeneity in both financial and non-financial performance across different sectors, namely the consumer sector, the financial sector, the health and utilities sector, the manufacturing sector, and the technology and communication sector. The goal is to assess not only the overall variability within each domain, but also how variability differs between sustainability-related and financial metrics across small and medium enterprises (SMEs). We consider annual data from the same set of 1062 SMEs over the period from 2020 to 2022.

In Tables 4 and 5, we report the MCVs for selected combinations of ESG and FIN metrics, respectively. From our results, we are able to infer several patterns. First, financial indicators exhibit substantially higher variability than ESG scores across all sectors, reflecting more heterogeneous performance among firms. Within ESG metrics, the consumer, the manufacturing, and the health and utilities sectors display the highest variation, whereas the financial and the technology and communication sectors are more uniform. Financial metrics show a similar but more pronounced pattern: the manufacturing and the health and utilities have the largest MCVs, indicating uneven financial outcomes, while the financial and the technology sectors are comparatively less sparse. Across both domains, the choice of MCV method influences the magnitude of variability, with G_2 and VV generally producing the highest values.

Overall, our results suggest that ESG performance is more consistent across SMEs than financial performance. To validate this point, we compute the correlation between the values identified by the MCVs and their rankings (i.e., the ordering they induce on the five sectors considered). Table 6 reports the correlation matrices of the values of multivariate coefficients of variation computed over the ESG indicators, both for the raw values (left) and for the rankings (right), while Table 7 reports the same values computed for the FIN indicators. The results show that for this dataset, the different MCV measures are highly positively

Table 4 Multivariate coefficients of variation of ESG scores by market sector

Sector	G_2	VN	R	AZ	VV
Consumer	0.317	0.183	0.179	0.193	0.320
Financials	0.120	0.069	0.091	0.146	0.263
Health utilities	0.252	0.145	0.155	0.158	0.281
Manufacturing	0.274	0.158	0.154	0.166	0.279
Technology and communication	0.161	0.093	0.114	0.114	0.218

Table 5 Multivariate coefficients of variation of FIN scores by market sector

Sector	G_2	VN	R	AZ	VV
Consumer	2.989	1.726	0.707	1.799	1.953
Financials	1.672	0.965	0.259	1.342	1.379
Health utilities	4.395	2.537	0.633	3.065	3.275
Manufacturing	6.129	3.539	1.354	3.641	3.944
Technology and communication	2.086	1.204	0.433	1.722	1.828

Table 6 Correlation matrices of ESG-based multivariate coefficients of variation. On the right, the correlation between the values, and on the left, the correlation between rankings

	G_2	VN	R	AZ	VV
G_2	1.000	1.000	0.900	0.900	0.800
VN	1.000	1.000	0.900	0.900	0.800
R	0.900	0.900	1.000	0.800	0.900
AZ	0.900	0.900	0.800	1.000	0.900
VV	0.800	0.800	0.900	0.900	1.000
G_2	1.000	1.000	0.993	0.810	0.779
VN	1.000	1.000	0.993	0.810	0.779
R	0.993	0.993	1.000	0.779	0.759
AZ	0.810	0.810	0.779	1.000	0.992
VV	0.779	0.779	0.759	0.992	1.000

Table 7 Correlation matrices of FIN-based multivariate coefficients of variation. On the right, the correlation between the values, and on the left, the correlation between rankings

	G_2	VN	R	AZ	VV
G_2	1.000	1.000	0.900	1.000	1.000
VN	1.000	1.000	0.900	1.000	1.000
R	0.900	0.900	1.000	0.900	0.900
AZ	1.000	1.000	0.900	1.000	1.000
VV	1.000	1.000	0.900	1.000	1.000
G_2	1.000	1.000	0.932	0.979	0.983
VN	1.000	1.000	0.932	0.979	0.983
R	0.932	0.932	1.000	0.852	0.864
AZ	0.979	0.979	0.852	1.000	1.000
VV	0.983	0.983	0.864	1.000	1.000

correlated, with correlations among the raw values ranging from 0.8 to 1.0, and correlations among rankings generally above 0.75.

6.3 Market Concentration Analysis

To conclude, we run another study on the behavior of the examined five multivariate coefficients of variation (MCVs) when used to analyze international equity markets, focusing on their levels, dependence structure, and ranking consistency. For each country, we model a company based in the country as a three-dimensional vector. Each entry of the vector contains market capitalization, number of employees, and total revenues. We thus assess the sparsity of the market by computing the five multivariate coefficients of variation for every country. Data is taken from the site companiesmarketcap.com, which contains, at the moment, the 8081 largest companies in the world by capitalisation.

Table 8 Multivariate coefficients of variation (MCVs) by market

Market	G_2	VN	R	AZ	VV
Australia	4.587	2.648	0.159	2.957	3.066
Austria	2.447	1.413	0.167	1.623	1.794
Belgium	4.271	2.466	0.112	2.747	2.806
Brazil	2.233	1.289	0.155	1.880	2.352
Canada	4.112	2.374	0.158	2.517	2.653
China	3.588	2.072	0.183	2.235	2.542
Denmark	3.697	2.135	0.169	2.762	2.869
Finland	3.052	1.762	0.123	1.940	2.046
France	4.024	2.324	0.231	2.949	3.201
Germany	5.150	2.973	0.239	3.059	3.469
India	4.114	2.375	0.203	2.383	2.548
Israel	2.883	1.665	0.117	1.800	1.920
Italy	3.525	2.035	0.139	2.352	2.530
Japan	3.425	1.977	0.162	2.249	2.434
Netherlands	3.015	1.741	0.183	2.010	2.393
Norway	2.751	1.588	0.140	2.947	3.175
Poland	3.104	1.792	0.177	2.438	2.766
Saudi Arabia	5.182	2.992	0.099	9.840	9.845
South Korea	3.188	1.841	0.087	2.248	2.420
Spain	3.336	1.926	0.141	2.225	2.358
Sweden	3.575	2.064	0.180	2.353	2.490
Switzerland	4.242	2.449	0.148	2.672	2.882
United Arab Emirates	4.613	2.663	0.018	3.064	3.072
UK	4.472	2.582	0.228	2.947	3.205
USA	7.681	4.434	0.259	7.238	7.493

6.3.1 Cross-Market Heterogeneity

The results reported in Table 8 reveal substantial heterogeneity in multivariate dispersion across markets. Developed markets such as the USA, Germany, Switzerland, and the UK tend to exhibit higher values of G_2, VN, AZ, and VV, reflecting elevated joint variability across assets. In contrast, smaller or less volatile markets (e.g., Norway, Austria, and Brazil) display systematically lower values across all MCVs. A notable feature is the markedly different scale of R, whose values are an order of magnitude smaller than those of the other indices. This suggests that R captures a fundamentally different aspect of multivariate variability and should not be directly compared in magnitude with G_2, VN, AZ, or VV. This suggestion is also reinforced by the fact that R is the only multivariate coefficient of variation that does not possess any property aside from being consistent. Nevertheless, its cross-market variation remains economically meaningful, as evidenced by its non-negligible association with the other measures.

6.3.2 Linear Dependence Among MCVs

Then consider the correlation between the values attained by the multivariate coefficients of variation, reported in Table 9. The correlation matrix highlights a very strong linear dependence between several of the proposed MCVs. First, in accordance with Theorem 4.2, the correlation between G_2 and VN is equal to one. Second, we notice that AZ and VV exhibit an extremely high correlation, indicating that these two coefficients respond almost identically to changes in multivariate dispersion across markets. Both indices also display strong positive correlations with G_2 (and thus also VN), reinforcing the view that they belong to the same family of dispersion measures driven primarily by scale effects. By contrast, the coefficient R shows only moderate correlations with the remaining MCVs, confirming the suggestion that R captures aspects of multivariate variability that are not fully aligned with scale-based measures.

6.3.3 Correlation of the Rankings Induced by the MCVs

Further insight is provided by the covariance matrix of ranks reported in Table 10. As expected, a perfect agreement in market rankings is observed between G_2 and VN. Consistent with the correlation analysis based on the raw values, the rank covariance between AZ and VV is close to unity, indicating that these two coefficients induce an almost identical ordering of markets. By contrast, the MCV R exhibits the weakest association with the remaining measures, confirming its distinct ranking behavior.

Table 9 Correlation matrix of multivariate coefficients of variation

	G_2	VN	R	AZ	VV
G_2	1.000	1.000	0.383	0.825	0.775
VN	1.000	1.000	0.383	0.825	0.775
R	0.383	0.383	1.000	0.278	0.434
AZ	0.825	0.825	0.278	1.000	0.950
VV	0.775	0.775	0.434	0.950	1.000

Table 10 Covariance matrix of the ranks of multivariate coefficients of variation

	G_2	VN	R	AZ	VV
G_2	1.000	1.000	0.419	0.704	0.719
VN	1.000	1.000	0.419	0.704	0.719
R	0.419	0.419	1.000	0.058	0.101
AZ	0.704	0.704	0.058	1.000	0.999
VV	0.719	0.719	0.101	0.999	1.000

7 Conclusions

We have shown how to leverage the relationship between the Gini index and the coefficient of variation, in the univariate case, to derive a multidimensional coefficient of variation, G_2, which is coherent and scale invariant, and satisfies both the rising tide and the cloning property.

We have demonstrated that G_2 is proportional to Voinov and Nikulin's coefficient of variation, which is also coherent and scale invariant, and satisfies the rising tide property, but not the cloning property.

The implications of our results are two-fold. On one hand, we supply a theoretical justification to prefer G_2 and Voinov and Nikulin's coefficient over the other available MCV. On the other hand, we show that our G_2 improves Voinov and Nikulin's, satisfying the cloning property and, more generally, having a better behavior as the dimension of the random vector analyzed increases.

From an applied viewpoint, our results can be extended to multivariate data analysis problems that involve statistical comparison among different groups, following the recent work of Ditzhaus and Smaga (2025), who has compared the examined coefficient of variation in a one-way and two-way factorial classification designs.

From a computational viewpoint, computing the trace of a matrix is much cheaper than computing its determinant or its inverse matrix, especially as the dimension of the matrix increases. Indeed, in our second simulated experiment, we have considered very large matrices, of dimension up to 90; however, we were able to compute all coefficients of variation in a matter of seconds, using a personal laptop and a standard Python software implementation. This suggests that, from a computational viewpoint, no MCV has a significant computational advantage over the others.

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Data Availability Data will be made available upon request.

Declarations

Ethical Approval The authors declare that the paper is compliant with Ethical Standards. The paper is aligned with ethical conduct standards.

Conflict of Interest The authors declare no competing interests.

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