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INVITED PAPER

Robust inference for step-stress experiments under interval-censoring

Narayanaswamy Balakrishnan[1,2] · María Jaenada[3] · Leandro Pardo[4]

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Abstract
Many moderns products have a long life before failure. Reliability analyses for such highly reliable devices therefore present a practical challenge as obtaining sufficient failure information to adequately assess lifetime behavior will require extended experimental duration. As an alternative, accelerated life testing (ALT) is commonly used to shorten the time to failure of units under test, with the results subsequently extrapolated to normal operating conditions. This paper provides a comprehensive review of robust inferential methods based on density power divergence for analyzing step-stress ALT data. Point estimates and approximate confidence intervals for model parameters, along with robust estimates of some important lifetime characteristics are developed for general lifetime distributions. Subsequently, explicit expressions are derived for four most prominent parametric lifetime distributions: exponential, Weibull, gamma, and lognormal. A semi-parametric approach based on the proportional hazards model and a competing risks scenario are also discussed as extensions of the proposed model. Throughout the manuscript, several open problems are highlighted, along with significant gaps in the literature, to motivate readers and also to promote further research in this important research area. Moreover, to illustrate the importance of step-stress ALTs and the practical utility of robust estimators, we also present some real data sets used in the literature and analyze one of them using robust methods. By analyzing

N. Balakrishnan, M. Jaenada and Leandro Pardo have contributed equally to this work.

✉ Leandro Pardo
lpardo@mat.ucm.es

Narayanaswamy Balakrishnan
bala@mcmaster.ca

María Jaenada
mjaenada@ucm.es

1 Department of Mathematics and Statistics, McMaster University, 1280 Main St W, Hamilton L8S 4K1, ON, Canada

2 Department of Mathematics, Atilim University, Kizilcasar, 1184, Ankara 06830, Turkey

3 Department of Statistics and O.R., National University of Distance Education, Senda del Rey, 11, Madrid 28040, Spain

4 Department of Statistics and O.R., Complutense University of Madrid, Plaza Ciencias, 3, Madrid 28040, Spain

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real data, we demonstrate the stability of the Minimum Density Power Divergence Estimator (MDPDE) for different values of the tuning parameter in the presence of outliers. We also analyze the implications of distributional assumptions on parameter estimation. Confidence intervals, including transformed intervals, are examined, with transformed intervals resulting in confidence levels close to nominal level and also provide better interpretability. Our results highlight the importance of robust estimation techniques in the presence of data contamination and also in a careful selection of parametric models for modeling the lifetime data, as these choices significantly influence predictions of lifetime characteristics under normal operating conditions.

Keywords Accelerated life-tests · Density power divergence · Interval-censoring · Reliability · Robustness

1 Introduction

Most of the products are highly reliable these days with large mean-times-to-failure (MTTF). Conducting reliability experiments for these highly reliable products under normal operating conditions may require a very long experimental time to gather sufficient information for developing accurate inference, which would also result in high experimental costs. In this regard, experimental designs under normal-use conditions require a compromise between accuracy, time and costs; the greater number of failures is required, the more experimental time and consequently higher cost may become necessary. Alternatively, one may identify one or more external stress factors influencing the wear and tear of the product under test. In such cases, accelerated life tests (ALTs) can be applied to induce higher number of failures within a shorter time frame, thereby reducing both experimentation time and cost. A model relating life characteristic to stress then need to be fitted to the accelerated failure times and then extrapolate it to estimate the lifetime characteristics under normal use condition. Different ALT designs can be adopted depending on the objective of the analysis, the nature of the product, and the budget. Time or budget constraints may also need to be considered, but ultimately, the final test design will depend on the product's characteristics and all experimental limitations. ALTs are commonly used in reliability testing of highly reliable devices. Numerous examples of their applications have been discussed in the literature, illustrating their effectiveness in providing adequate data for ensuring product lifetime statistical analyses. Some real-life examples describing step-stress experiments are described later in Sect.3.

On the other hand, censoring occurs frequently in life-testing whenever the experimenter does not observe failure times of all units placed on the life-test. Since failure data is only observed for a bounded period of time, it is common for some products not to have failed by the end of the experiment, resulting in right-censoring. The two most common censoring schemes are Type-I censoring, in which the experiment ends at a fixed time, and Type-II censoring, in which the experiment ends after a fixed number of failures have been observed. A substantial body of research has been devoted to developing statistical methods for reliability analysis under both types of censoring. Censoring can occur either intentionally or unintentionally, and may be caused by time constraints on the test duration, requirements for a minimum number of observed failures, or the structure of a technical system. More generally, for example, in progressive censoring, some units get removed at certain fixed or random times during the experiment. Two important references on progressive censoring and associated inferential results are Balakrishnan and Cramer (2014) and Balakrishnan (2007).

Certain experimental or product constraints can also lead to other types of censoring. For instance, due to measurement availability, costs, and other constraints, experimenters may sometimes be unable to monitor test units continuously, but can do only intermittent checks. This is the case of experiments in which manual checks are required to assess the functional state of a device, or in cases involving non-destructive one-shot devices (see Cheng and Elsayed (2018), Balakrishnan et al. (2023c)). In such situations, only the number of failures occurred before each inspection can be recorded, leading to interval-censored data. Naturally, the probabilistic structure of the resulting incomplete data heavily depends on the censoring mechanism, necessitating the development of suitable inferential procedures. A vast body of literature has addressed both estimation procedures and the design of optimal test plans for ALTs under interval censoring. The optimal allocation of the stress level change for simple step-stress ALTs with exponential lifetimes and CE model has been examined in Bai et al. (1989) and Xiong and Ji (2004), while the optimal length of stress intervals under Type-I progressive censoring was studied in Wu et al. (2008). The work of Yang and Pan (2013) discussed optimal plans based on the proportional hazard model for the failure-time distribution. Moreover, optimal plans for multiple-step stress experiments under progressive Type-I interval censoring were proposed in Zhao and Bordes (2017). Additional contributions include the discussion of optimal ALT planning with random removals in Tse et al. (2008); Ding and Tse (2013), and the determination of optimum inspection times under progressive Type-I interval censoring in Hassan et al. (2014).

Subsequent developments expanded this line of research. Optimal design under step-stress models was analyzed in Bobotas and Kateri (2019), and Bayesian inference for interval-inspected setups was developed in Kohl and Kateri (2019). More recently, the theory of optimal plans for progressively censored step-stress ALTs under interval inspection has been further extended in Han and Bai (2022).

With respect to estimation methods, least-squares estimates based on a generalized linear model (GLM) approach for the same experimental setup were developed in Lee and Pan (2012b). MLEs for step-stress and constant-stress ALTs under progressive Type-I censoring with interval inspection were obtained using the EM algorithm in Han and Bai (2019). Exact inference procedures for progressively Type-I censored step-stress ALTs assuming exponential lifetimes were later developed in Han and Bai (2025). Finally, a comparison between continuous and interval inspections in step-stress ALTs was conducted in Hakamipour (2021), providing valuable insights into the relative efficiency of both inspection schemes. Most of the results mentioned thus far are based on the likelihood-based estimates of the model parameters under the assumed distribution for the product lifetimes. While MLEs are known to be efficient, they are also known to be non-robust.

While ALT with interval-censoring is relatively easy to implement, interval censoring results in a loss of information, impacting test planning, data analysis, and the accuracy of product lifetime predictions. In this context, a single outlying observation, such as an abnormal failure count, can significantly affect the estimation of lifetime parameters and hence have an impact on the conclusions of the analysis. Recent studies have addressed this issue by developing robust inferential techniques for step-stress ALTs under interval-censoring, based on the density power divergence (DPD) method (see Balakrishnan et al. (2022c, 2023a, b, c, 2024a, b)). The present paper offers a comprehensive overview of existing robust inferential techniques based on the DPD for step-stress tests under interval-censoring, focusing on four commonly used parametric families: exponential, Weibull, gamma, and log-normal. An in-depth discussion of the step-stress model and existing life-stress relationships, and the robust DPD framework is provided here, along with numerical analysis and practical real-world dataset analyses. We establish the theoretical properties of the robust methods for general lifetime distributions and conduct a model-misspecification sensitivity analysis to assess the impact of assuming each of four candidate lifetime distributions in real data -even when goodness-of-fit tests fail to reject them. Additionally, several open problems of interest are also highlighted throughout the text to motivate and guide readers in addressing some of the remaining gaps in the literature. Therefore, this paper provides a comprehensive and unified methodology for research on robust methods for step-stress reliability models, as well as clear, consistent tools for reliability practitioners, reflecting the importance of practical utility of this work.

2 Step-stress model

Reliability analyses under normal operating conditions are generally conducted at normal use stress levels, ensuring that the observed failures reflect the true failure behaviour under real-use conditions. However, for highly reliable devices, conducting reliability tests under normal operating conditions may be inefficient as very few failures would be observed. Instead, one can identify various environmental or external stress factors that influence the reliability of such products under test, including factors such as humidity, voltage, temperature, exposure to contaminants, mechanical forces, or vibrations. These environmental stress factors can significantly impact the wear and tear of the product, and the specific choice of the stress factor would depend on the nature of the product. ALTs aim to induce a higher number of observed failures by subjecting the test units to higher-than-normal stress levels. In this process, more failures could likely be obtained within a limited test time. These increased stress conditions simulate accelerated wear and tear on the product and may lead to early failures. The aim of such testing is to quickly obtain data which, properly modelled and analyzed, would yield desired information on product life or performance.

Because the primary goal of the analysis is to estimate the reliability of products under normal operating conditions, results from ALTs would then typically be extrapolated to normal operating conditions. Engineers have long used accelerated testing for diverse products and there exists an extensive literature and methods for statistical analysis of the resulting data. Moreover, ALTs are known to be efficient in capturing valuable lifetime information, especially when there is a need to shorten the life-testing experiment. From Viertl (1988) and Nelson (1990), it can be readily seen that many authors have studied statistical tools for analyzing the data arising from different ALT designs.

There are various ALT designs, such as constant-stress, step-stress, progressive-stress, and cyclic-stress designs. Each stress loading design possesses its own advantages and disadvantages concerning experimental feasibility and statistical analysis accuracy. For instance, Han and Ng (2014) compared constant-stress and step-stress ALTs using three optimality criteria: minimizing the variability of the estimator, maximizing the determinant of the Fisher information matrix, and minimizing the trace of the Fisher information matrix. The study concluded that the step-stress accelerated life test model was more efficient than the constant-stress model when using equal sample sizes. Additionally, the application of increased stress from different designs would impact the device's lifetime differently, necessitating the use of distinct statistical models to accurately fit the observed data.

The step-stress design increases the stress level at which units under test are exposed to at certain pre-specified times, and maintains the stress level constant between two subsequent times of stress change. That is, the stress level is repeatedly increased and held, until the test unit fails or the experiment terminates. A simple step-stress accelerated life testing uses only two stress levels. This simple design has been widely studied as a first approach to the general framework, for example, in Bai et al. (1989); Bai and Chun (1991); Fard and Li (2009); Yuan et al. (2011); Han (2019, 2020). When the number of stress levels considered is greater than two, say k different stress levels, the step-stress design is commonly referred to as a multiple k-step-stress ALT.

Step-stress ALT designs are commonly motivated by industrial life-testing problems and the vast majority of the literature is related to material reliability studies. Indeed, step-stress tests have been widely used in electronics applications to reveal failure modes. Nonetheless, they can be also applied in many other fields, such as in biomedical applications. An introduction to step-stress ALT models can be found in the basic monographs by Bagdonavicius and Nikulin (2001); Meeker and Escobar (2014), and Nelson (2009). For an overview on step-stress testing, we refer to Gouno and Balakrishnan (2001b); Xiong (2003); Gouno (2004). Step-stress models under exponential lifetimes have been discussed in great detail by Balakrishnan (2009).

Accelerated design approaches, including step-stress ALT, offer several key benefits while also presenting certain drawbacks. Specifically, step-stress ALTs are valued for their efficiency in time, often significantly shortening test durations without sacrificing the precision of outcome estimates. This method stands out for its practicality compared to constant-stress ALTs, as it necessitates fewer test units, making it a more feasible option in many practical scenarios. Additionally, the adaptability of step-stress ALTs is noteworthy. They provide a flexible testing framework, particularly suitable for new products where adequate testing stress levels might be unknown. This flexibility allows for the adjustment of stress levels and transition times based on real-time failure data, enhancing the test's responsiveness to unforeseen variables. Moreover, by potentially decreasing the length of the test and initially applying lower stress levels, the overall costs of testing can be reduced. This is despite the higher expenses that might be incurred when testing units under stress levels exceed normal operating conditions. Optimal step-stress accelerated degradation plans, as suggested by certain studies, have shown to require fewer samples and less testing time, thereby lowering

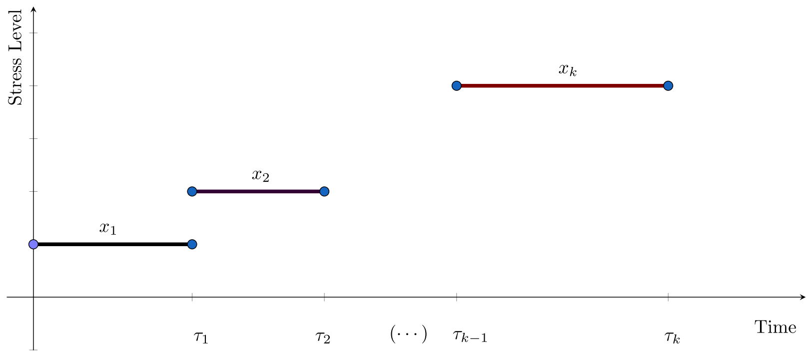


Fig. 1 Scheme of the experimental stress loading design from a multiple step-stress design with k stress levels

the total cost of testing as compared to traditional degradation tests. Another drawback to consider is the need to accurately understand the relationship between stress and the lifetime distribution in order to extrapolate the results to normal operating conditions. Thus, step-stress models must account for the life-stress relationship through the cumulative effect of stress exposure, which makes them generally quite complex.

Let us introduce some key notation to be used here concerning step-stress tests. We consider k stress levels, increased at the pre-fixed times $\tau_1, ..., \tau_{k-1}$. If the experiment design is reduced to simple step-stress, then the number of stress levels is set to $k = 2$. In some step-stress experiments, the first stress level is the stress level under normal operating conditions, generally denoted by x_0. Such experiments are known as step-stress partially accelerated (SSPALT) and we will identify $x_1 = x_0$.

A random sample of N devices are placed under an initial stress level x_1. At time τ_i, the stress is switched from x_i to x_{i+1}, $i = 1, ..., k-1$, and the experiment gets terminated at a fixed time τ_k. Generally, stress levels are considered to be ordered $x_0 < x_1 < \cdots < x_k$ and so the stress levels get increased at the time of stress change. Note that, since the termination time is fixed, the observed data is Type-I censored. If alternatively the termination time was dependent on a fixed number of failures, the data would be Type-II censored. In this work, we will focus on inference under Type-I censoring. Figure 1 shows a scheme of the experimental stress loading design.

Let $T_1, \ldots, T_N$ denote independent and identically distributed variables describing the failure times. The failure times of test items under step-stress experiments are then described by the respective order statistics $T_{1:N} \leq \cdots \leq T_{N:N}$. It is reasonable to assume that the lifetime distribution of the devices is continuous. Then, the observed failure times (which may be censored) are denoted by $(t_{1:N}, \ldots, t_{N:N})$ and are realized under the corresponding stress levels of the experiment. The number of failures recorded under the stress level x_i, $i = 1, ..., k$, is denoted by N_i. Therefore, the total number of failures observed during the experiment is given by $N_1 + \cdots + N_k$, and the remaining devices' failure times are then right-censored.

Figure 2 illustrates a k-stress-step ALT observed data.

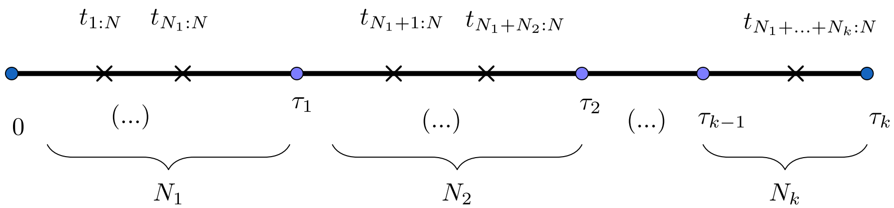


Fig. 2 Observed data from a multiple step-stress design with k stress levels

3 Motivating datasets

To demonstrate the practical applicability of the step-stress model, we now list several real-world datasets that have been analyzed in the literature. It should be mentioned here that some of these datasets were originally collected under continuous monitoring and have been transformed into interval-monitoring for illustrative purposes.

Example 1. Medium Power Silicon Bipolar Transistors Dataset
A temperature multiple step-stress ALT with $n = 31$ medium power silicon bipolar transistors was carried out in Gouno and Balakrishnan (2001b). The Arrhenius model with censored data was considered to describe the temperature stress. The devices were successively exposed to 10 temperature levels of 120, 140, 160, 180, 190, 200, 210, 220, 230, 240 degrees during equal time intervals of length 168 h. At each time of stress change, the number of failures were recorded, yielding the observed vector of numbers of failures to be: 0, 0, 0, 2, 5, 5, 3, 3, 0 and 9.

Example 2. Light Bulb Dataset
An accelerated life testing experiment in the Quality and Reliability Engineering Laboratory of the Industrial and Systems Engineering Department at Rutgers University so as to examine the reliability of light bulbs (see Zhu (2010)). Two sets of 32 miniature light bulbs were placed in a temperature and humidity chamber where humidity was held constant, and the long term failure due to bulb filament fatigue was then studied. When the switch was turned on, full current suddenly flowed to the filament at the speed of light. This sudden massive vibration caused the filament to wildly bounce causing fatigue behaviour of the filament which resulted in breakage of the filament. Long term failure occurred when the filament eventually becomes so fatigued that its electrical resistance increases to the point that current would not flow. Each light bulb was connected with a resistor, across which Voltage was measured to monitor the status of the light bulbs. Normal operating condition of the light bulbs is 2V. To carry out the step-stress ALT, they applied 2.25V for 96hr and then increased the voltage to 2.44V. The step-voltage test got stopped at 140hr. Failure times observed during the experiment are as follows:

12.07, 19.5, 22.1, 23.11, 24, 25.1, 26.9, 36.64, 44.1, 46.3, 54, 58.09, 64.17, 72.25, 86.9, 90.09, 91.22, 102.1, 105.1, 109.2, 114.4, 117.9, 121.9, 122.5, 123.6, 126.5, 130.1, 14 17.95, 24, 26.46, 26.58, 28.06, 34, 36.13, 40.85, 41.11 42.63, 52.51, 62.68, 73.13, 83.63, 91.56, 94.38, 97.71, 101.53, 105.11 112.11, 119.58 ,120.2, 126.95, 129.25, 136.31.

The remaining 11 light bulbs continued to provide light when the experiment got terminated. To illustrate the performance of the step-stress ALT model under interval-censoring, Balakrishnan et al. (2023a) transformed the collected data into grouped data with inspection times $t = 25, 50, 96, 110, 120, 140$.

Example 3. Solar Lighting Devices Dataset

Solar lights are known to be resilient and environmental-friendly by design and so they offer an attractive lighting alternative for outdoor and indoor places. Conversely, there is an unwanted effect of extreme weather on the reliability of such devices.

In particular, the efficiency of solar lights generally decreases as the temperature rises, due to a voltage decrease.

Under normal operating conditions, solar lighting devices are expected to last for hundreds of hours, and therefore an ALT must be carried out to infer the effect of temperature on solar lighting devices.

For this purpose, Han and Kundu (2014) conducted a simple step-stress accelerated life testing experiment. A set of $n = 35$ solar light prototypes were placed on equal environmental conditions, and the temperature was increased at a pre-specified time $\tau_1 = 5$ (in hundred hours) from normal operating temperature $(293K)$ to $353K$. The experiment was terminated at $\tau_2 = 6$ (in hundred hours), when only 4 devices were still functioning. The time of stress-change were chosen in advance by the experimenter. Failure times observed during the experiment are as follows:

0.140, 0.783, 1.324, 1.582, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.396, 4.612, 4.892, 5.002, 5.022, 5.082, 5.112, 5.147, 5.238, 5.244, 5.247, 5.305, 5.337, 5.407, 5.408, 5.445, 5.483, 5.717.

The collected data were transformed into intermittent inspecting devices with inspection times $t = 1.5, 3, 5, 5.2, 5.4, 6$ (in hundred ours) in Balakrishnan et al. (2023a), by counting the number of failures within each inspection interval as if the exact failure times were not available. The resulting observed data are given by the counts of failures: 3, 8, 5, 5, 5, 5 and 4. Furthermore, Han and Bai (2020) determined the optimal step duration under interval inspection for a similar setup, with a sample size of $n = 30$ prototypes and a total test duration of $\tau_k = 20$ (in hundred hours), to assess the reliability characteristics of a solar lighting device in the second phase. The dominant failure mode of the device was controller failure, and the stress factor was temperature, which was increased during the test from $293K$ to $353K$, with the normal operating temperature set at $293K$. The optimal test duration for this experiment under interval-censoring ranged from 995 to 1080 hours.

Example 4. Electronic Components Dataset

The electronic components dataset was studied by Wang and Fei (2003) to get the reliability indices of a kind of electronic components at the normal temperature of $x_0 = 25°C$. $N = 100$ items from a batch of products were randomly selected for a simple SSALT ($k = 2$), with two stress levels $x_1 = 100°C$ and $x_2 = 150°C$. In the original experiment, the stress level rises when 30 products had failed and the test continues until 20 more products had failed, obtaining a total of 50 failures. Their failure times are as follows:

- Failure times at the first stress level x_1: 32, 54, 59, 86, 117, 123, 213, 267, 268, 273, 299, 311, 321, 333, 339, 386, 408, 422, 435, 437, 476, 518, 570, 632, 666, 697, 796, 854, 858, 910.
- Failure times at the second stress level, x_2: 16, 19, 21, 36, 37, 63, 70, 75, 83, 95, 100, 106, 110, 113, 116, 135, 136, 149, 172, 186.

Intermittent-monitoring were analyzed in Balakrishnan et al. (2023a), with monitoring times $t = 270, 430, 600, 910, 975, 1015, 1040, 1096$ and time of stress change, τ_1, pre-fixed at $t = 910$.

We aim to analyze the influence of temperature in some electronic components such as solar lights, medium power silicon bipolar transistors and LED lights. Allowing systems to run for prolonged periods of time in high temperatures can severely decrease the longevity and reliability of some electronic devices. Therefore, in order to get the most out of an electronic component in different environments or otherwise choose the best material for a certain function, it will of be interest to study its reliability based on the temperature.

Recall that the step-stress model enables one to obtain accurate estimates with fewer sample sizes than constant-stress ALTs. Therefore, the model is especially useful for the presented industrial experiments, which traditionally have small sample sizes.

Example 5. Automotive Part Dataset

Failure data set from an automotive part is a dataset of industrial data given in the reference book of ALTA software (ReliaSoft 2014) and first analyzed in Lee and Pan (2012a). The experiment was performed to hasten failures for a particular automotive part under four levels of stress. The combination of applied stresses is quantified in terms of a "percentage stress" as compared with the use condition; that is, if the use condition is defined as $x_0 = 100\%$, other stress levels are quantified as percentages over the use condition, such as 125%, 175%, 200% and 250%. The data consist of 10 failures out of 12 test units for four stress levels, and the numbers of failures at each stress level are (0, 2, 3, 5), respectively. These units are tested at stress level 1 for $200h$, stress level 2 for $100h$, stress level 3 for $50h$ and stress level 4 for $25h$. The test is terminated after obtaining the 10th failure at 375 h. Table 1 presents the data set for the fourth-level SSALT with type II censoring.

Table 1 Failure data set for an automotive part

Number	Failure time (h)	Final stress level	Stress V (%)
1	252	2	175
2	280	2	175
3	320	3	200
4	328+	3	200
5	335	3	200
6	354	4	250
7	361	4	250
8	362	4	250
9	368	4	250
10	375	4	250
11	375+	4	250
12	375+	4	250

The life-stress function is an inverse power function given by $\lambda_i = (a/V_i)^b$, where V_i is the percentage stress.

Example 6. Light Emitting Diodes (LEDs) Dataset

Light-emitting diodes dataset comes from a real life experiment with $k = 4$ stress levels that was conducted in the Quality and Reliability Engineering Laboratory at Rutgers University. The purpose was to estimate the lifetime of light-emitting diodes (LEDs).

Temperature has a tangible effect on the material and output of an LED light. Cold environments favour good functioning of LEDs, and light output diminishes with temperature increment. The effect of temperature in LEDs was examined in Zhao and Elsayed (2005) through a multiple step-stress ALT experiment with four levels. The LEDs are considered to be working properly as long as the change in the light intensity does not exceed a pre-fixed threshold level. Then, they placed 27 LED units in a temperature and humidity chamber and the humidity and current in the circuit were held constant at 70% RH and 28 mA, respectively. Four temperature levels were considered during the test. According to expert engineering judgment, temperatures less than $80°C$ are not appropriate since it is difficult to observe the LED failure in a reasonable test time. Conversely, temperatures above $200°C$ are not recommended either as extrapolation can not be justified. Therefore, four temperature levels ranging from 80 to 200° were considered for the ALT as $T = 363K, 413K, 433K$ and $448K$. The normal operating temperature for LEDs is $50°C$. The times of stress change were fixed at $\tau_1 = 300$, $\tau_2 = 500$ and $\tau_3 = 600$ hours, and the test got terminated at $\tau_4 = 720$ hours. Recorded failure times are provided below in Table 2.

Table 2 The light-emitting diode (LED) data set, based on a step-stress ALT experiment of $k = 4$ stress levels. The data (failure times in hours) are given per stress-level i, along with the stress (temperature in Kelvin, T_i), the transformed x_i of the log-link function, the τ_i, and the number of progressively censored items r_i, $i = 1, \ldots, s$

Stress-Level	T_i (K)	$x_i = 323/T_i$	τ_i	Failure Times (hours)	r_i
1	363	0.8898	300		1
2	413	0.7821	500	347, 397, 432, 491	2
3	433	0.7460	600	512, 567, 574, 588, 597	2
4	448	0.7210	720	603, 605, 615, 633, 634, 637, 644, 653, 675, 684, 699, 706, 718, 720	4

Table 3 Step-stress pattern after step 4

Step	5	6	7	8	9	10	11
Kilovolts	26.0	28.5	31.0	33.4	36.0	38.5	41.0

These continuously monitored data were analyzed by a frequentist approach in Zhao and Elsayed (2005) and in the work of Sha and Pan (2014) by frequentist as well as Bayesian approaches. Later, Kohl and Kateri (2019) adapted the data to interval monitoring setup by using just the number of failures at each stress level instead of the exact failure times, and compared their results with the one obtained with continuous monitoring. More recently, Balakrishnan et al. (2022b) also used this real data to illustrate the use of robust estimators under interval censoring.

They both transformed the data to intermittent tested data in which the inspection times coincide with the times of stress change.

Example 7. Cryogenic Cable Insulation Dataset
The data set was obtained from a step-stress test of cryogenic cable insulation experiment and analyzed for the first time in Nelson (1990) [Table 2.1 of Chapter 10]. Each specimen was first stressed for 10 minutes each at voltages of 5 kV, 10 kV, 15 kV, and 20 kV before it went into step 5. Thereafter, one group of specimens was stressed 15 minutes at each step as given in Table 3.

Three other groups were held for 60, 240, and 960 minutes at each step. Thus, there were four step-stress patterns. The stress on a specimen is the natural logarithm of the ratio between the voltage and the insulation thickness. The original data were observed as exact failure times. In order to demonstrate the estimation process under interval-censoring, Xiong and Ji (2004) and Wu et al. (2006) grouped the failure time data according to the intervals formed by consecutive stress-change times. Each interval has the length of 960 minutes. There are $n = 9$ specimens simultaneously placed on a five-stage step-stress ALT.

There were five censored failure times in the data set. The grouped and censored data are summarized bellow in Table 4.

Table 4 Failure counts for cryogenic cable insulation dataset under interval-censoring

Holding time (min)	Final step	Count (uncensored)	Thickness (mil)
15	9	3	27
60	10	1	29.5
60	10	0	28
240	9	2	29
240	10	2	29
240	10	1	30
960	5	1	30
960	5	0	30
960	6	1	30
960	7	3	30
960	8	1	30
960	9	1	30

Example 8. Kinetics dataset

Some materials suffer kinetic reactions resulting in failures when a sufficient amount has accumulated. Further, the source of potential failure may depend on some stress at which materials are exposed to, such as temperature, humidity or voltage. The kinetics governing a failure mode in epoxy-glass printed wiring boards can be analyzed thorough ALT by subjecting the epoxy-gass wiring board to high humidity levels.

The kinetics dataset was presented in Luvalle and Hines (1992), who carried out a 2-step stress experiment with $N = 144$ test coupons tested under two humidity levels, $x_1 = 65\%$ of relative humidity until $\tau_1 = 2023$ hours when the relative humidity was increased to $x_2 = 85\%$. The only information available was recorded at certain monitoring times so that the experiment conforms exactly to the interval-censoring set-up.

We analyze this kinetic data for studying failures in printed wiring boards accelerated by the effect of relative humidity at fixed high voltage and temperature.

Each experimental test coupon was mounted in hermetically sealed jars in which a saturated salt solution was used to maintain the relative humidity at a constant level. Each test coupon had three sense points on it at which failures could be measured.

An automatic data acquisition system monitored the resistance between isolated sets of copper plates through holes periodically. At each inspection time, the number of devices failing within the preceding time interval was recorded and consequently the failure times are interval-censored. Some previous studies chose the left end of the interval as the failure time to simplify their analyses and then applied step-stress ALT models to fit those data. In contrast, interval-monitoring set-up would allow us to directly estimate based on the naturally observed interval-censored data.

Table 5 Simulated data set from a simple step-stress model with $n = 20$, $\theta_1 = e^{2.5} \approx 12.1825$, and $\theta_2 = e^{1.5} \approx 4.4817$ for $\tau_1 = 5$ (see Xiong (1998))

Simulated failure times (hours)
2.01, 3.6, 4.12, 4.34, 5.04, 5.94, 6.68, 7.09, 7.17, 7.49, 7.6, 8.23, 8.24, 8.25, 8.69, 12.05

Further, the inspection times can be reduced from 77 to 58 by eliminating some inspection times that were less than 2 hours apart, to avoid a large number of intervals with very few failures, which could pose difficulties while developing inference. The standard accelerated failure time regression models based on parametric distributions such as exponential, Weibull or gamma will be adequate for modelling some simple chemical processes that lead to failure (LuValle et al. 1988), but would be inadequate for more complex kinetics processes.

Example 9. Simulated dataset from step-stress model
This is a classical data set of Xiong (1998) analyzed in step-stress models. The complete data are provided in Table 5. It is a data set of $n = 20$, simulated from an originally type-II censored simple step-stress model assuming exponential lifetimes for the testing items at each stress level with exponential parameters $\theta_1 = e^{2.5}$ and $\theta_2 = e^{1.5}$, $\tau_1 = 5$ and continuous monitoring.

This dataset has been used in step-stress literature to illustrate both continuous and interval monitoring schemes.

4 Impact of stress level changes

Step-stress ALTs require a model relating the lifetime distribution at a current stress level to the proceeding stress levels. There are three fundamental models for examining the effect of increased stress levels on the lifetime distribution of a device: The tampered random variable model proposed by Goel (1971) and DeGroot and Goel (1979), the cumulative exposure model of [92] and Nelson (1980), and the tampered failure rate model proposed by Bhattacharyya and Soejoeti (1989). All these models of step-stress designs have been discussed extensively in the literature. Here, we briefly review the main principles underlying the three models. It is important to note that we assume testing is conducted in an ordered stress sequence, meaning that the stress levels are progressively increased. Furthermore, the higher the stress level, the faster the deterioration on reliability of the devices.

4.1 Cumulative exposure model

Cumulative exposure (CE) model, first proposed by Sedyakin (1966) for the simple step-stress model and later generalized by Bagdonavicius and Nikulin (2001) and Nelson (1990) to multiple stresses, assumes that the residual lives of the experimental units depend only on the cumulative exposure the units have experienced thus far, with no memory of how this exposure got accumulated. The CE model is the most commonly used model in engineering for relating the lifetime distribution at subsequent stress levels. A synthesis of CE model for a general scale family can be found in Balakrishnan (2009).

To construct a model for the cumulative distribution function of the lifetime of experimental units under this setup, let us first describe the CE model for a simple step-stress testing experiment in which the experimental units are tested at an initial stress level of x_1 until a prefixed time τ_1, at which point the stress level is changed to x_2 and the life-test continues on here after. Let G_1 and G_2 denote the lifetime cumulative distribution function (c.d.f.) at constant stress levels x_1 and x_2, respectively, both having an assumed parametric or semi-parametric form, and let G_T denote the lifetime c.d.f. of a device during the whole step-stress experiment. We assume that both c.d.f.'s G_1 and G_2 belong to the same family of distributions. Then, the effect of change of stress from x_1 to x_2 is to modify the lifetime distribution at stress level x_2 from $G_2(t)$, the c.d.f. under the current stress at time t if the device were maintained at that stress level from the beginning, to $G_2(t+s)$, where the 'shifting time' $s<0$ is such that the c.d.f. is continuous throughout its domain; that is,

$$G_1(\tau_1) = G_2(\tau_1 + s).$$

Therefore, the c.d.f. of the devices' lifetime over the experiment can be expressed as a piecewise function as follows:

$$G_T(t) = \begin{cases} G_1(t), & 0 < t < \tau_1 \\ G_2(t+s), & \tau_1 \le t < \infty \end{cases}.$$

If the assumed distributions $G_1(t)$ and $G_2(t)$ belong to a scale family, with scale parameters λ_1 and λ_2, respectively, then from the assumption of absolute continuity of the c.d.f. of the lifetime, we can readily observe that the shifting time s is such that

$$s = \frac{\lambda_2}{\lambda_1}\tau_1 - \tau_1,$$

and that

$$G_T(t) = \begin{cases} G_1(t), & 0 < t < \tau_1 \\ G_2(t + \frac{\lambda_2}{\lambda_1}\tau_1 - \tau_1), & \tau_1 \le t < \infty \end{cases}.$$

Now, consider a multiple step-stress ALT with k ordered stress levels $x_1 < x_2 < \cdots < x_k$ changed at prefixed times

$$\tau_1 < \tau_2 < \cdots < \tau_{k-1},$$

respectively, as presented in Sect.2. Generalizing the above CE model to multiple stresses, the c.d.f. of the devices' lifetime is given as a piecewise function as

$$G_T(t) = G_i(t+s_i), \qquad \tau_{i-1} \le t < \tau_i, \quad i = 1, ..., k, \tag{1}$$

where $s_i < 0$ is a “shifting time” representing the wear of the product until the time of stress change, with $s_1 = 0$, $\tau_0 = 0$ for mathematical convenience. The lifetime can be extended by setting $\tau_k = \infty$ in Equation (1). The shifting time should be chosen so that the lifetime distribution G_T is continuous at the times of stress change, that is, for each time of change τ_i, $i = 1, ..., k$, we have

$$G_{i+1}(\tau_i + s_{i+1}) = G_i(\tau_i + s_i), \qquad i = 1, ..., k-1. \tag{2}$$

Similarly, if we assume that the parametric distribution G_i under constant stress x_i depends on a scale parameter, say λ_i, that varies with the stress level, as is the case for exponential, Weibull, or gamma distributions, then it can be shown that the shifting time has a closed form given by

$$s_i = \lambda_i \sum_{l=1}^{i-1} \left(-\frac{1}{\lambda_{l+1}} + \frac{1}{\lambda_l} \right) \tau_l, \; i = 2, ..., k. \tag{3}$$

Note that we could alternatively define the shifting time as

$$s_i = \lambda_i \sum_{l=1}^{i-1} \left(\frac{\tau_l - \tau_{l-1}}{\lambda_l} \right) - \tau_{i-1}, \; i = 2, ..., k, \tag{4}$$

and the above two expressions are equivalent.

Of course, the CE model could be generalized to a more general scenario wherein other distribution parameters, different from the scale parameter, also depend on the stress level. But, in this case, the shifting time may not have an explicit expression, and so numerical methods may be needed for solving (2). Typically, statistical models under a step-stress setup assume that the remaining lifetime distribution parameters remain constant throughout the different stress levels. This assumption is known as the *homogeneity condition* and is widely adopted in practice.

Thus, the CE model forms a composite c.d.f. of the form

$$G_T(t) = G_i \left(t + \lambda_i \sum_{l=1}^{i-1} \left(-\frac{1}{\lambda_{l+1}} + \frac{1}{\lambda_l} \right) \tau_l \right) \text{ if } \tau_{i-1} \leq t < \tau_i, \quad i = 1, ..., k, \tag{5}$$

where the last part of the function can be extended up to ∞. The reliability, density and hazard functions can all be derived from Equation (5) as follows. The reliability of the product at time t is

$$R_T(t) = 1 - G_T(t) = 1 - G_i \left(t + \lambda_i \sum_{l=1}^{i-1} \left(-\frac{1}{\lambda_{l+1}} + \frac{1}{\lambda_l} \right) \tau_l \right) \text{ if } \tau_{i-1} \leq t < \tau_i.$$

Taking derivatives on the above expressions, the probability density function (p.d.f.) is given by

$$g_T(t) = \frac{\partial G_T(t)}{\partial t} = g_i\left(t + \lambda_i \sum_{l=1}^{i-1}\left(-\frac{1}{\lambda_{l+1}} + \frac{1}{\lambda_l}\right)\tau_l\right) \text{ if } \tau_{i-1} \le t < \tau_i,$$

where g_i denotes the p.d.f. under constant stress x_i.

4.2 Tampered random variable model

The tampered random variable (TRV) model was introduced in Goel (1971) and DeGroot and Goel (1979), and optimal tests under a Bayesian framework were therein discussed. Later, Miller and Nelson (1983) obtained the optimal time for changing the stress level from x_1 to x_2, assuming exponentially distributed life times and Bai et al. (1989) extended the results of Miller and Nelson to the case of interval-censoring. Estimation methods for simple step-stress models under TRV were discussed by Goel (1971) and Kundu and Ganguly (2017). The theory was recently extended to multiple stress levels by Sultana and Dewanji (2023) and Koley et al. (2023). Optimal design for a simple step-stress ALT under TRV modeling for exponential and lognormal distributions under type-I censoring were discussed in Bai and Chung (1992) and Bai and Kim (1993).

The TRV models scale down the remaining lifetime at the successive stress levels. For a simple step-stress ALT, it is assumed that the effect of change of the stress level from x_1 to x_2 at time τ_1 is equivalent to multiplying the remaining life of the experimental unit at time τ_1 by an unknown positive constant, β_1. That is, if we denote $T_{TRV}^{(1)}$ for the random variable representing the life time under stress x_1, then the overall lifetime, denoted by the "tampered" random variable T_{TRV}, is defined as

$$T_{TRV} = \begin{cases} T_{TRV}^{(1)} & \text{if } T_{TRV} \le \tau_1 \\ T_{TRV}^{(2)} = \tau_1 + \beta_1(T_{TRV}^{(1)} - \tau_1) & \text{if } \tau_1 < T_{TRV} \end{cases}, \tag{6}$$

where the scale factor β_1, called the tampering coefficient, depends on both levels of stress x_1 and x_2 and possibly on the time of change τ_1 as well.

4.3 Tampered failure rate model

The tampered failure rate (TFR) model was proposed in Bhattacharyya and Soejoeti (1989) as a hazard rate based approach which assumes that the effect of changing stress level is to multiply the initial failure rate function by a positive factor. That is, if we denote by $h_i(t)$ the hazard rate function of a product lifetime under a constant stress x_i, the acceleration of failure when the stress is raised from a lower level to a higher level is reflected in the hazard rate function as

$$h_{i+1}(t) = \alpha_{i+1} h_i(t), \quad \tau_i \le t < \tau_{i+1}. \tag{7}$$

Note that the tampered factor $\alpha_{i+1} = \alpha(x_{i+1})$ depends on the current and new stress levels, but not on the time of change. Assumption (7) can be easily transformed into an accumulated tampered factor depending on the new and initial stress levels, but regardless of the in-between stresses.

Under the TFR model, the c.d.f. of the lifetime G_T can be represented as

$$G_T(t) = 1 - \exp\left\{-\int_0^t \sum_{i=1}^{k} h_i(w)\,\mathrm{I}_{[\tau_{i-1},\tau_i)}(w)dw\right\}, \tag{8}$$

where $\mathrm{I}(\cdot)$ denotes the indicator function.

Later, Madi (1993) adopted the TFR model for a multiple k-step-stress ALT and analyzed complete lifetime data under Weibull baseline distribution with common shape parameter. Under this framework, Khamis and Higgins (1998) derived the MLEs for the scale and common shape parameters using a log-link model, thereby introducing the Khamis-Higgins (KH) model. Hypothesis testing for the log-link parameters were derived in Khamis and Higgins (1999). The relationship between the TFR and KH models was discussed in Xu and Tang (2003), noting that the KH model is a special case of TFR model under Weibull distributions. Wang and Fei (2003) proved the uniqueness of the MLE under TFR for the Weibull distribution, and Kateri and Kamps (2017) studied MLEs and related issues using the TFR modeling approach under a general scale family for Type-I and Type-II censored data from k-step-stress ALT experiment.

The TFR and CE models coincide under exponential lifetimes (see Wang and Fei (2004)). For non-exponential distributions, Kateri and Kamps (2015) empirically compared the reliability estimates under the CE and TFR models and demonstrated that both models produce quite similar estimates of the reliability of devices. Besides, each approach for relating the lifetime distribution at different stress levels has its own advantages and disadvantages in describing the true distribution underlying the data and may result in different mathematical models. The selection of an appropriate model to represent the impact of stress changes depends on the specific product that is under test. Particularly, the CE model is the most commonly used model in engineering for relating the lifetime distribution at subsequent stress levels due to its natural motivation, and we will therefore adopt it here in the subsequent sections. However, developing robust estimators for the Tampered Failure Rate and Tampered Random Variable models would also be of great interest, and so we list this as our first open problem.

Open Problem 1 (Robust inference under Tampered Failure Rate and Tampered Random Variable models) *This paper focuses on the development of robust estimators for step-stress ALTs under the CE model. However, many papers in the literature have adopted these two other important models, namely, the Tampered Failure Rate and Tampered Random Variable models, which offer different motivations for modelling the impact of stress changes on the lifetime distribution. But, no robust methods have been developed for either of these two life-stress models. Therefore, developing robust inference under these scenarios remains as an open problem.*

Open Problem 2 (Life-stress model selection) *The three life-stress models presented above have their own natural motivation for describing how products deteriorate over time and under increased stress levels. The three models coincide under the exponential distribution, but for any other lifetime distribution, they impose different restrictions on the distribution. If no prior knowledge is available about the deterioration of the devices when increasing the stress, one may wonder which model best fits step-stress lifetime data. Developing a life-stress model selection criterion for a given dataset remains as an important open problem.*

5 Inference under interval censoring

Let us consider a step-stress ALT with stress levels $x_1 < x_2 < \cdots < x_k$. We adopt the CE approach described earlier in Section 4.1 for relating the devices' lifetime distributions at different stress levels, and we assume a parametric form for the lifetime p.d.f. at constant stress x_i, $g(\cdot|\boldsymbol{\vartheta}_i)$, where the parameter $\boldsymbol{\vartheta}_i$ may depend on the stress level x_i. Observe that certain model parameters may be functions of x_i, whereas others are not.

Statistical models for analyzing ALT data require such relationships between the lifetime distribution of the devices and the stress factors to which they are exposed to, in order to extrapolate results to normal operating conditions following accurate inference. Typically, this relationship is expressed in terms of the failure rate, representing the expected number of failures per unit of time. However, a simple model, such as a linear model, may not be sufficient to accurately capture the complex connection between the failure rate and stress factors or the scale parameter of the distribution. Physical principles of the materials and stress factors utilized may provide insights into the functional form of this relationship.

In ALT literature, a log-linear relationship between the scale parameter, say λ of the distribution and the stress level is commonly used, as it fits some well-known physical models such as the Arrhenius model and the Inverse Power Law model. The log-linear relation is expressed as

$$\lambda_i = \exp(\theta_0 + \theta_1 x_i), \quad i = 1, ..., k, \tag{9}$$

for certain common parameters $(\theta_0, \theta_1) \in \mathbb{R}^2$ and scale parameter depending on the stress level x_i. Note that we have assumed that reliability deteriorates when the stress level is increased to a more "challenging" level. Therefore, if the stress level increases positively and this increase in the rate parameter monotonically affects the reliability, a constraint on θ_1 may need to be imposed. Furthermore, this functional relationship, which depends on a common parameter across all stress levels, links the lifetime distribution to the stress level and reduces thereby the number of parameters that need to be estimated. Naturally, the lifetime distribution may also depend on additional parameters. Let $\boldsymbol{\theta} = (\theta_0, \theta_1, \boldsymbol{\theta}_2)$ represent the vector of model parameters to be estimated, where (θ_0, θ_1) are the log-linear parameters in Equation (9), and $\boldsymbol{\theta}_2$ denotes a vector of additional distribution parameters held constant across stress levels.

Although the step-stress ALT models usually refer to continuous monitoring of the unit's lifetime, in some cases, experiments are not monitored continuously, but are inspected for failures at only particular time points. In such setups, the ALT response is interval censored and the information available is the number of failures falling between two successive inspection times. Consequently, the complete likelihood can not be applied and inferential methods adapted to such an interval-censored data need to be developed. Under interval monitoring, a frequent assumption for the inspection times on k-step stress ALTs is that they coincide with the times of stress change, $\tau_1, ..., \tau_{k-1}$ (see Gouno and Balakrishnan (2001a); Xiong and Ji (2004); Tsai et al. (2008); Wu et al. (2008)). However, that assumption is quite restrictive and unsuitable for simple step-stress tests. Seo and Yum (1991) considered additional intermediate inspection times, producing equally spaced or equal probability inspection intervals, and Lee and Pan (2012b) dealt with inspection intervals of fixed length. More work in this regard were developed by Bai et al. (1989); Bobotas and Kateri (2019); Han and Bai (2019, 2022) and Han and Bai (2025).

In this paper, we will consider L inspection times, including all times of stress change, and so the number of inspection times would be usually greater than the number of stress levels.

Let us now introduce some key notation. First, consider L inspection times satisfying

$$IT_0 = 0 < IT_1 < \cdots < IT_{L_1} = \tau_1 < IT_{L_1+1} < \cdots < IT_{L_{k-1}+1} < \cdots < IT_{L_k} = \tau_k,$$

where L_i denotes the number of inspection times fixed before the time of stress change τ_i, $i = 1, ..., k$, and $L \equiv L_k$. Denote by N the number of units under test.

Because the count of failures are grouped-sampled, the empirical observed data are counts of failures. Therefore, the proportion of failure will empirically estimate the probability of failure within each inspected interval and the incomplete likelihood must be defined in terms of these failure counts. Suppose n_j failures of test devices are observed in the interval $[IT_{j-1}, IT_j)$, $j = 1, .., L$, and for notational ease, let us denote n_{L+1} for the number of surviving devices at the end of the experiment. Table 6 then summarizes the data collected at different inspection times.

Given the true c.d.f. function of the devices' lifetime at constant stress level, x_i, belongs to a parametric family with c.d.f.'s $G_i(t|\boldsymbol{\vartheta}_i)$, $i = 1, ..., k$, the probability of failure of a device in the j-th interval is given by

$$\pi_j(\boldsymbol{\vartheta}_1, ..., \boldsymbol{\vartheta}_k) = G_i(IT_j + s_i|\boldsymbol{\vartheta}_i) - G_i(IT_{j-1} + s_i|\boldsymbol{\vartheta}_i) \quad \text{if} \quad \tau_{i-1} < IT_j \leq \tau_i, \quad (10)$$

with s_i being the shifting time determined by the CE model, for $i = 1, ..., k$, and the probability of survival at the end of the experiment is $\pi_{L+1} = 1 - G_k(\tau_k + s_k|\boldsymbol{\vartheta}_k)$.

Since the observed data are the number of failures per inspection interval, as well as the number of censored units, a multinomial distribution with $L + 1$ mutually exclusive events and event probabilities $\pi_j(\boldsymbol{\vartheta}_1, ..., \boldsymbol{\vartheta}_k)$, $j = 1, ..., L + 1$, describes such data. Therefore, parameter estimation based on the observed data will be carried out under a multinomial sampling model; hereafter, we refer to this specification as the multinomial model associated to the step-stress experiment. Consequently, the

Table 6 Form of observed data under step-stress ALTs with k stress levels and interval censoring

Stage	Inspection time	Number of failures	Stress level	Change time
1	IT_1	n_1	x_1	-
2	IT_2	n_2	x_1	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_1	IT_{L_1}	n_{L_1}	x_1	τ_1
L_1+1	IT_{L_1+1}	n_{L_1+1}	x_2	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_{k-1}	$IT_{L_{k-1}}$	$n_{L_{k-1}}$	x_{k-1}	τ_{k-1}
$L_{k-1}+1$	$IT_{L_{k-1}+1}$	$n_{L_{k-1}+1}$	x_k	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_k	IT_{L_k}	n_{L_k}	x_k	τ_k

incomplete likelihood is given by

$$\mathcal{L}(\boldsymbol{\vartheta}_1, ..., \boldsymbol{\vartheta}_k; n_1, .., n_{L+1}) = \frac{N!}{n_1! \cdots n_{L+1}!} \prod_{j=1}^{L+1} \pi_j(\boldsymbol{\vartheta}_1, ..., \boldsymbol{\vartheta}_k)^{n_j}. \tag{11}$$

From the above likelihood function, the MLE of $(\boldsymbol{\vartheta}_1, ..., \boldsymbol{\vartheta}_k)$ would then simply be

$$\left(\boldsymbol{\vartheta}_1{}^{MLE}, ..., \boldsymbol{\vartheta}_k{}^{MLE}\right) = \arg\max_{\boldsymbol{\theta}\in\Theta} \mathcal{L}(\boldsymbol{\vartheta}_1 ..., \boldsymbol{\vartheta}_k; n_1, .., n_{L+1}).$$

Remark 1 We could have alternatively derived the likelihood function of the model using binomial distribution for each interval, by using conditional probabilities of failure, given that the device did not fail in earlier time intervals. However, multinomial and conditional binomial modeling approaches yield the same likelihood function.

From the previous likelihood function, it can be easily seen that the MLE of the model parameter $\boldsymbol{\vartheta}_1$ exists only when there is at least one failure under the first stress level, the MLE of $\boldsymbol{\vartheta}_2$ exists only when there is at least one failure under the second stress level, and so on. These MLEs are then conditional MLEs. However, we could consider that restriction holds asymptotically and use the previous likelihood function. Moreover, under a log-linear link with the stress level, the model parameter vector reduces to $\boldsymbol{\theta} = (\theta_0, \theta_1, \boldsymbol{\theta}_2)$, where (θ_0, θ_1) denotes the log-linear coefficients and $\boldsymbol{\theta}_2$ collects the additional parameters that do not depend on the stress level. In this case, we may write the c.d.f. of the step-stress model (and the related quantities) in terms of $\boldsymbol{\theta}$.

6 Robust estimators

In this section, we discuss the estimation method for the MDPDE for step-stress ALTs under interval-censoring.

In real-life scenarios, it is quite common to encounter observations that do not follow the assumptions of the underlying model, often referred to as anomalous observations or outliers. Consequently, there has been a growing interest in developing statistical procedures that are not unduly influenced by these exceptional data points. Divergence measures have played a central role in developing robust statistics, providing a statistical tool for building robust estimators. In statistics, an outlier is defined as an observation that falls outside the general pattern of the bulk of the data. In particular, for grouped data from step-stress tests, outliers may appear as abnormal count of failures, rather than isolated outlying failures of devices. These abnormal counts may be caused by measurement errors or an external cause producing early failure of a set of devices.

Inference procedures based on the minimization of an appropriate statistical distance (or, more precisely, a divergence) derived from p.d.f.'s have emerged as a compelling research topic in recent years due to robust properties exhibited by many of these methods. Moreover, several of these procedures have demonstrated high asymptotic efficiency, making them competitive with classical likelihood-based estimators. The primary application of such statistical distances lies in parametric point estimation, where the parameter of interest is estimated by minimizing a suitable "distance" or similarity measure between the observed data and the assumed model. From the resulting estimators and their asymptotic distributions, approximate confidence intervals, point estimates of any function of the model parameters, and test statistics can also be developed, providing a complete framework of robust inferential techniques. The properties of the minimum distance estimators will depend on the measure of closeness used for its definition. For instance, the well-known Kullback-Leibler divergence yields the MLE from an information theoretic approach, while the DPD has been shown to produce robust estimators with only a small loss in efficiency in different statistical models under continuous and discrete data.

We keep the same step-stress design as in Section 5, with k stress levels, $x_1 < \cdots < x_k$ and L inspection times including all times of stress change, $\tau_1 < \cdots < \tau_{k-1}$ and the termination time, τ_k. A total of N units are placed under test, and the number of failures between two subsequent inspection times is recorded after each inspection. The number of surviving units is denoted by n_{L+1}. Furthermore, the CE model is assumed to relate the lifetime distribution under increasing stress levels, with a log-linear relationship being used to relate the scale parameter of the distribution to the stress level, as shown in Equation (9). The remaining distribution parameters are assumed to be independent of the stress level (homogeneity condition). The model parameter vector is thus $\boldsymbol{\theta} = (\theta_0, \theta_1, \boldsymbol{\theta}_2)$ and does not depend on the stress level.

As discussed in Section 5, a multinomial model can be used to fit the data with theoretical probabilities of failure within the j-th inspected interval given by

$$\pi_j(\boldsymbol{\theta}) = G_T(IT_j|\boldsymbol{\theta}) - G_T(IT_{j-1}|\boldsymbol{\theta}), \quad j = 1, ..., L,$$

and $\pi_{L+1}(\boldsymbol{\theta}) = 1 - G_T(IT_L|\boldsymbol{\theta})$. These probabilities would depend on the stress to which units are subjected between two subsequent inspection times, IT_{j-1} and IT_j. As all times of stress change are inspections times, we can assume that the stress level is constant within an interval.

Thus, the MLE of the step-stress ALT model under interval-censoring is simply given by

$$\widehat{\boldsymbol{\theta}}^{MLE} = \arg\max_{\boldsymbol{\theta}\in\Theta}\left[\sum_{j=1}^{L+1} n_j \log(\pi_j(\boldsymbol{\theta}))\right]$$

where the likelihood function is given in Eq. (11).

This MLE can be alternatively obtained from the Kullback-Leibler divergence as follows. Let $\widehat{\boldsymbol{p}} = (n_1/N, ..., n_{L+1}/N)^T$ be the empirical probability vector obtained from the observed data. Then, the Kullback-Leibler divergence between the empirical and theoretical probability vectors, $\widehat{\boldsymbol{p}}$ and

$$\boldsymbol{\pi}(\boldsymbol{\theta}) = (\pi_1(\boldsymbol{\theta}), \pi_2(\boldsymbol{\theta}), ..., \pi_{L+1}(\boldsymbol{\theta}))^T\,,$$

is given by

$$d_{KL}(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})) = \sum_{j=1}^{L+1} \widehat{p}_j \log\left(\frac{\widehat{p}_j}{\pi_j(\boldsymbol{\theta})}\right). \tag{12}$$

It is straightforward to see that the Kullback-Leibler divergence is related to the log-likelihood function in the form

$$d_{KL}(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})) = c - \frac{1}{N}\log \mathcal{L}(\boldsymbol{\theta}; n_1, .., n_{L+1}), \tag{13}$$

where the constant $c = \frac{1}{N}\left(\log(N!) - \sum_{j=1}^{L+1}\log(n_j!)\right) + \sum_{i=1}^{L+1}\widehat{p}_i\log(\widehat{p}_i)$ does not depend on the parameter $\boldsymbol{\theta}$. Hence, the MLE can equivalently be defined as

$$\widehat{\boldsymbol{\theta}}^{MLE} = \arg\min_{\boldsymbol{\theta}\in\Theta} d_{KL}(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})). \tag{14}$$

From an asymptotic point of view, it is well-known that the MLE is a BAN (Best Asymptotically Normal) estimator, and it has therefore been widely used for the step-stress ALT model. However, despite its high efficiency, the MLE is known to lack robustness as contaminated data could influence the parameter estimation considerably. Due to the limited information on lifetime behavior in interval-monitoring, even a small amount of contamination can significantly influence the parameter estimates and, consequently, the subsequent analysis. To address the lack of robustness issue of the MLE, the DPD robustly measures the closeness between the two probability vectors. In the following, we will develop robust estimators based on the DPD for step-stress ALTs under interval-censoring for a general lifetime distribution. Explicit expressions for some of the most prominent lifetime parametric distributions are then developed in Sect. 8.

For the considered multinomial model, the DPD between the theoretical and empirical probability vectors $\widehat{\boldsymbol{p}}$ and $\boldsymbol{\pi}(\boldsymbol{\theta})$, for $\beta > 0$, is given by

$$d_\beta(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})) = \sum_{j=1}^{L+1}\left(\pi_j(\boldsymbol{\theta})^{1+\beta} - \left(1+\frac{1}{\beta}\right)\widehat{p}_j\pi_j(\boldsymbol{\theta})^\beta + \frac{1}{\beta}\widehat{p}_j^{\beta+1}\right), \tag{15}$$

where $\widehat{p}_j = n_j/N$ and the MDPDE of the model parameter vector is then defined as

$$\widehat{\boldsymbol{\theta}}^\beta = \left(\widehat{\theta}_0^\beta, \widehat{\theta}_1^\beta, \widehat{\theta}_2^\beta\right)^T = \arg\min_{\boldsymbol{\theta}\in\Theta} d_\beta(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta})). \tag{16}$$

The first important observation of the above definition is that the last term in (15) does not depend on the model parameter and so can be left out of the minimization. Also, the DPD depends on a tuning parameter β which will control the trade-off between efficiency and robustness of the resulting method. Indeed, for $\beta = 0$, the DPD can be extended by taking continuous limits, leading to the Kullback-Leibler divergence defining the MLE, which produces the most efficient but least robust estimator within the family. Thus, the MDPDE family extends the likelihood procedure to a wide class of estimators indexed by a tuning parameter β controlling the compromise between efficiency and robustness; large β produces more robust estimators although less efficient.

6.1 Properties of the robust estimators

Because the MDPDE is defined as the minimizer of a differentiable function, it must be a critical point of the DPD. In other words, the MDPDE should annul the first derivatives of the DPD function, resulting in the MDPDE estimating equations. The following results present the estimating equations and asymptotic distribution of the MDPDE with tuning paramter β.

Theorem 1 *The estimating equations of the MDPDE, for step-stress ALTs under interval-censoring assuming a log-linear relation in (9), are given by*

$$\boldsymbol{W}(\boldsymbol{\theta})^T \boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}^{\beta-1}(\widehat{\boldsymbol{p}} - \boldsymbol{\pi}(\boldsymbol{\theta})) = \boldsymbol{0}_p,$$

where $\boldsymbol{0}_p$ *is the p-dimensional null vector,* $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ *denotes a* $(L+1)\times(L+1)$ *diagonal matrix with diagonal entries* $\pi_j(\boldsymbol{\theta}) = G_T(IT_j) - G_T(IT_{j-1})$, $j = 1, ..., L$, *and* $\pi_{L+1}(\boldsymbol{\theta}) = 1 - G_T(IT_L)$, *and* $\boldsymbol{W}(\boldsymbol{\theta})$ *is a* $(L+1)\times p$ *matrix with rows* $\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}$, *where* $\boldsymbol{z}_j = \frac{\partial G_T(IT_j)}{\partial \boldsymbol{\theta}}$
for any $j = 1, ..., L$ *and* $\boldsymbol{z}_{-1} = \boldsymbol{z}_{L+1} = \boldsymbol{0}$.

Note that the estimating equations of the MLE are obtained readily by evaluating the previous expression at $\beta = 0$, resulting in

$$\boldsymbol{W}(\boldsymbol{\theta})^T \boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}^{-1}(\widehat{\boldsymbol{p}} - \boldsymbol{\pi}(\boldsymbol{\theta})) = \boldsymbol{0}_p.$$

Theorem 2 *Let $\boldsymbol{\theta}^*$ be the true value of the parameter $\boldsymbol{\theta}$. Then, the asymptotic distribution of the MDPDE with tuning parameter β, for the step-stress ALT model under interval censoring, $\widehat{\boldsymbol{\theta}}^\beta$, is given by*

$$\sqrt{N}\left(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta}^*\right) \xrightarrow[N\to\infty]{L} \mathcal{N}\left(\mathbf{0}, \boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*)\boldsymbol{K}_\beta(\boldsymbol{\theta}^*)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*)\right),$$

where

$$\begin{aligned}\boldsymbol{J}_\beta(\boldsymbol{\theta}^*) &= \boldsymbol{W}(\boldsymbol{\theta}^*)^T \boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta}^*)}^{\beta-1}\boldsymbol{W}(\boldsymbol{\theta}^*),\\ \boldsymbol{K}_\beta(\boldsymbol{\theta}^*) &= \boldsymbol{W}(\boldsymbol{\theta}^*)^T\left(\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta}^*)}^{2\beta-1} - \boldsymbol{\pi}(\boldsymbol{\theta}^*)^\beta\boldsymbol{\pi}(\boldsymbol{\theta}^*)^{\beta T}\right)\boldsymbol{W}(\boldsymbol{\theta}^*),\end{aligned} \tag{17}$$

and matrices $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$, $\boldsymbol{\pi}(\boldsymbol{\theta})$, and $\boldsymbol{W}(\boldsymbol{\theta})$ are all as defined in Theorem 1.

The proofs of the Theorems proceed along lines similar to those presented in Balakrishnan et al. (2023a).

Remark 2 The Fisher information matrix associated with the step-stress ALT model is given by $\boldsymbol{I}_F(\boldsymbol{\theta}^*) = \boldsymbol{W}(\boldsymbol{\theta}^*)^T\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta}^*)}^{-1}\boldsymbol{W}(\boldsymbol{\theta}^*)$, where the matrices $\boldsymbol{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}(\boldsymbol{\theta})$ are as defined in Theorem 1 and so the convergence of the MLE is obtained as a particular case at $\beta = 0$.

Naturally, the explicit expressions for the estimating equations and the asymptotic distribution will depend on the assumed lifetime distribution under constant stress. Consequently, the properties of the resulting estimators, such as the asymptotic variance-covariance matrix, will also depend on these expressions.

Remark 3 As $\widehat{\boldsymbol{\theta}}^\beta$ is a consistent estimator of $\boldsymbol{\theta}^*$, the asymptotic variances of $\widehat{\theta}_0^\beta$ and $\widehat{\theta}_1^\beta$, for $\beta > 0$, denoted by $\sigma^2(\theta_0^\beta)$ and $\sigma^2(\theta_1^\beta)$, respectively, can be estimated by the diagonal entries of $\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{K}_\beta(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)$. Therefore, asymptotic confidence intervals for θ_0 and θ_1, at confidence level $100(1-\alpha)\%$, are given by

$$\widehat{\theta}_i^\beta \pm \frac{\widehat{\sigma}(\theta_i^\beta)}{\sqrt{N}} z_{\alpha/2}, \quad i = 0, 1, \tag{18}$$

with $z_{\alpha/2}$ being the upper $\alpha/2$-quantile of a standard normal distribution. Moreover, denoting r for the dimension of the parameter space, we have

$$N(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta})^T(\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*)\boldsymbol{K}_\beta(\boldsymbol{\theta}^*)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*))^{-1}(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta}) \xrightarrow[N\to\infty]{L} \chi_r^2;$$

consequently, an associated ellipsoidal confidence region for $\boldsymbol{\theta} = (\theta_0, \theta_1, \boldsymbol{\theta}_2)$ is given by

$$C_{N,\beta}^\alpha = \{\boldsymbol{\theta} | N(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta})^T(\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*)\boldsymbol{K}_\beta(\boldsymbol{\theta}^*)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}^*))^{-1}(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta}) \leq c\}.$$

Choosing $c = \chi_{r,\alpha}^2$, the $100(1-\alpha)\%$-percentile of the chi-square distribution with r degrees of freedom, we have $\mathbb{P}_{\boldsymbol{\theta}}(C_{N,\beta}^\alpha)$ tending to $1-\alpha$ as $N \to \infty$. Hence, $C_{N,\beta}^\alpha$

represents an ellipsoidal confidence region for $\boldsymbol{\theta}$ having limiting confidence coefficient $1-\alpha$ as $N \to \infty$ (see Serfling (2009) for more details). The volume of the ellipsoidal region $C^{\alpha}_{N,\beta}$, based on Cramer (1946), is given by

$$\chi^2_{r,\alpha}\pi \det\left(\left(\boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta})\right)^{-1}\right)^{1/2}.$$

Thus, a measure of the asymptotic relative efficiency of $\widehat{\boldsymbol{\theta}}^{\beta}$, for $\beta > 0$, with respect to the MLE, $\widehat{\boldsymbol{\theta}}^{\beta=0}$, is given by

$$\left(\frac{\det\left(\boldsymbol{I}_F(\widehat{\boldsymbol{\theta}}^{\beta=0})^{-1}\right)}{\det(\boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta}))}\right)^{1/2},$$

where $\boldsymbol{I}_F(\cdot)$ is the Fisher information matrix associated with the discrete model under consideration.

We will discuss at length the lack of robustness drawback of the MLE, illustrating theoretically and empirically the influence that an outlying observation may have on the estimation process. We will examine the explicit expressions of the MLE under different lifetime distributions modelling the lifetime data. The performance of the MLE would then be compared with the robust proposal in many different scenarios, including different forms of contamination.

6.2 Robustness of the MDPDE

The robustness of an estimator can be theoretically evaluated through its influence function (IF), as introduced in Hampel et al. (1986). Intuitively, it can be visualized as a measure of the influence that a single data point exerts on the estimator. Thus, robust estimators are expected to have a bounded influence of outlying points, meaning that the IF should be bounded. In this Section, we derive the IF of the MDPDEs and discuss the boundedness of the function depending on the outlying variable.

The IF of an estimator is defined in terms of its statistical functional, defining the estimator for a general underlying true distribution, G_T. Let us consider $\widehat{\boldsymbol{\theta}}^{\beta}$ the MDPDE with tuning parameter β and $\boldsymbol{T}_{\beta}(G_T)$ its associated statistical functional. Notably, the MDPDE from interval-censored data is computed using the associated multinomial model derived from the grouped data, that is, $\widehat{\boldsymbol{\theta}}^{\beta} = \boldsymbol{T}_{\beta}(\widehat{\boldsymbol{p}})$.

Further, let us consider a contaminated lifetime distribution at a point perturbation t_0, $G_{\varepsilon} = (1-\varepsilon)G_T + \varepsilon\Delta_{t_0}$, where ε denotes the contamination proportion and Δ is a degenerate distribution at t_0. Then, the IF of the MDPDE at a point perturbation t_0 and the theoretical lifetime $G_T(\cdot|\boldsymbol{\theta})$ belonging to the assumed distribution, is given by

$$\mathrm{IF}\left(t_0, \boldsymbol{T}_{\beta}, G_T(\cdot|\boldsymbol{\theta})\right) = \boldsymbol{J}_{\beta}^{-1}(\boldsymbol{\theta})\boldsymbol{W}^T\boldsymbol{D}^{\beta-1}_{\boldsymbol{\pi}(\boldsymbol{\theta})}\left(-\boldsymbol{\pi}(\boldsymbol{\theta}) + \boldsymbol{e}_{t_0}\right), \tag{19}$$

where $\boldsymbol{J}_\beta(\boldsymbol{\theta})$ is as defined in Theorem 2, $\boldsymbol{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ are as defined in Theorem 1, $\boldsymbol{\pi}(\boldsymbol{\theta})$ is the probability vector of the multinomial model with probabilities given in (10) and $\boldsymbol{e}_{t_0}$ is an $L+1$-dimensional degenerate probability vector with probability 1 at the inspection interval containing the perturbation point t_0, say i_0; See Balakrishnan et al. (2023c) and Balakrishnan et al. (2024a) for a deeper discussion and derivation of the IF. Indeed, if a continuous c.d.f. is contaminated by a point perturbation at t_0, the associated probability vector of the multinomial distribution places all probability on a single cell, here denoted by i_0, with the remaining components equal to zero; in this case, the multinomial distribution is degenerate.

For the step-stress ALT model with interval-censoring, we should consider two forms of contamination: contamination in the covariates, including inspection times and stress levels, and outliers in the response variable given by the failure counts of each inspected cell. As both covariates are continuous, we should examine the boundedness of the IF in (19) under increasing inspection times and stress levels, always taking into account that both grids are ordered. The boundedness (and maximum influence) will depend on the assumed lifetime distribution and must be studied on a case-by-case basis. For example, Balakrishnan et al. (2023a) established the boundedness of the IF of the MDPDE for the step-stress ALT model with interval-censoring and exponential lifetimes for contamination in both covariates, and found that the IF is bounded only for any positive value of β. That is, the only non-robust estimator in the DPD family is the MLE.

On the other hand, since the multinomial model has discrete support, the IF of the MDPDE at the outlier cells is bounded for any $\beta \geq 0$, including the MLE. For discrete models where the IF is always bounded, the gross error sensitivity can be used for comparing the robustness of different estimators, defined as the maximum perturbation that an outlying cell can exert. We define the gross error sensitivity of the functional $\boldsymbol{T}_\beta$ considering the contamination point at the i_0-cell as

$$\gamma(i_0, \boldsymbol{T}_\beta, G_T) = \sup_{\{i_0=1,\ldots,L+1\}} \parallel IF\left(t_0, \boldsymbol{T}_\beta, G_T\right) \parallel. \tag{20}$$

This maximum value, which describes the maximum bias on the MDPDE over the neighborhood of the assumed model distribution, can be very large depending on the choice of the underlying distribution parameters, particularly for scale families. It is natural to postulate that a measure of the robustness of parameter estimators should be invariant to scale transformations of individual parameter components at least, but the sensitivity defined above is not. To overcome this defect, Hampel et al. (1986) measured the IF, which is an asymptotic bias, in the metric given by the asymptotic covariance matrix of the estimator. That is, given the asymptotic variance-covariance matrix of the MDPDE defined in Theorem 2, $\boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}) = \boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta})\boldsymbol{K}_\beta(\boldsymbol{\theta})\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta})$, the self-standardized sensitivity is defined by

$$\gamma^*(i_0, \boldsymbol{T}_\beta, G_T) = \sup_{\{i_0=1,\ldots,L+1\}} \parallel IF\left(t_0, \boldsymbol{T}_\beta, G_T\right)^T \boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\theta}) IF\left(t_0, \boldsymbol{T}_\beta, G_T\right) \parallel. \tag{21}$$

Again, this maximum value will depend on the underlying distribution $G_T(\cdot|\boldsymbol{\theta})$ assumed, and so need to be studied case-by-case. For example, Balakrishnan et al. (2024b) studied the self-standardized sensitivity of the MDPDE under exponential lifetimes and competing risks, and demonstrated how sensitivity to outliers decreases as β increases, indicating that robustness of the corresponding estimators improve with increasing β.

7 Point estimation of some lifetime characteristics

Engineering applications often require estimates of important lifetime characteristics, such as the reliability function, distribution quantiles, and MTTF of a device under normal operating conditions. Indeed, these characteristics are often the primary focus of the analysis, rather than the complete lifetime distribution. If these characteristics can be expressed as functions of the model parameters, say $h(\boldsymbol{\theta})$, then they can be easily estimated by plugging-in the MDPDEs into the corresponding function and computing the relevant quantities. However, some characteristics of interest may not have an explicit expression in terms of the lifetime parameters, as is the case with the quantiles of the gamma distribution. In this section, we develop robust point estimates and approximate confidence intervals for any differentiable function of the model parameters, as well as some transformed confidence intervals for the reliability at certain mission time, distribution quantiles, and MTTF, assuming that they can be explicity expressed as a function $h(\boldsymbol{\theta})$.

Given the functional relationship $h(\boldsymbol{\theta})$, the MDPDE with tunning parameter $\beta \geq 0$ is given by $h(\widehat{\boldsymbol{\theta}}^{\beta})$. Moreover, applying the delta-method, the standard error of the estimate can be obtained as

$$\sigma_{\beta}(h(\widehat{\boldsymbol{\theta}}^{\beta})) = \left[\nabla h(\boldsymbol{\theta}^*)^T \boldsymbol{J}_{\beta}^{-1}(\boldsymbol{\theta}^*)\boldsymbol{K}_{\beta}(\boldsymbol{\theta}^*)\boldsymbol{J}_{\beta}^{-1}(\boldsymbol{\theta}^*)\nabla h(\boldsymbol{\theta}^*)\right]^{1/2},$$

where $\boldsymbol{\theta}^*$ denotes the true value of the model parameter vector. These expressions depend on the tuning parameter β of the MDPDE they are based on. Indeed, the MDPDE with tuning parameter β will transfer its efficiency and robustness properties to the reliability and mean lifetime estimators.

On the other hand, due to the consistency of the MDPDE, the standard error can be approximated by

$$\widehat{\sigma}_{\beta}(h(\widehat{\boldsymbol{\theta}}^{\beta})) = \left[\nabla h(\widehat{\boldsymbol{\theta}}^{\beta})^T \boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\boldsymbol{\theta}}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\boldsymbol{\theta}}^{\beta})\nabla h(\widehat{\boldsymbol{\theta}}^{\beta})\right]^{1/2},$$

and once the corresponding standard error is computed, approximate $100(1-\alpha)\%$ CI for $h(\boldsymbol{\theta}^{\beta})$ can be given as

$$\left[h(\widehat{\boldsymbol{\theta}}^{\beta}) \pm z_{\alpha/2}\frac{\widehat{\sigma}(h(\widehat{\boldsymbol{\theta}}^{\beta}))}{\sqrt{N}}\right]$$

where z_{α} denotes the upper α-quantile of a standard normal distribution.

The above asymptotic confidence intervals are based on the asymptotic properties of the MDPDEs and so they may be satisfactory only for large sample sizes. However, some lifetime characteristics satisfy some natural restrictions and consequently the previous confidence intervals may have to be truncated. In this regard, Viveros and Balakrishnan (1993) employed a reparametrization of the reliability at a certain mission time, distribution quantiles, and MTTF to ensure that the natural restrictions of these characteristics are fulfilled by the corresponding confidence intervals as well. Specifically, they applied a logarithmic transformation to the mean lifetime and distribution quantiles, and a logit transformation to the reliability, to obtain more accurate confidence intervals based on the MLEs. These transformations can also be applied to the robust estimators presented here, producing a similar effect on the robust confidence intervals. The transformation approach has been shown to improve the coverage of approximate intervals in case of small samples. Given $R(\boldsymbol{\theta})$ a differentiable function defining the reliability at a mission time of the devices, $Q_{1-\gamma}(\boldsymbol{\theta})$ a differentiable function for the $(1-\gamma)-$upper quantile and $E(\boldsymbol{\theta})$ a function for the MTTF, the transformed CIs are as follows:

$$
\begin{aligned}
IC_\alpha(E(\widehat{\boldsymbol{\theta}}^\beta)) &= \left[E(\widehat{\boldsymbol{\theta}}^\beta)\exp\left(-\frac{z_{\alpha/2}}{\sqrt{N}}\frac{\widehat{\sigma}(E(\widehat{\boldsymbol{\theta}}^\beta))}{E(\widehat{\boldsymbol{\theta}}^\beta)}\right), E(\widehat{\boldsymbol{\theta}}^\beta)\exp\left(\frac{z_{\alpha/2}}{\sqrt{N}}\frac{\widehat{\sigma}(E(R(\widehat{\boldsymbol{\theta}}^\beta)))}{E(\widehat{\boldsymbol{\theta}}^\beta)}\right)\right],\\
IC_\alpha(R(\widehat{\boldsymbol{\theta}}^\beta)) &= \left[\frac{R(\widehat{\boldsymbol{\theta}}^\beta)}{R(\widehat{\boldsymbol{\theta}}^\beta)+\left(1-R(\widehat{\boldsymbol{\theta}}^\beta)\right)S}, \frac{R(\widehat{\boldsymbol{\theta}}^\beta)}{R(\widehat{\boldsymbol{\theta}}^\beta)+\left(1-R(\widehat{\boldsymbol{\theta}}^\beta)\right)/S}\right],\\
IC_\alpha(Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta)) &= \left[Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta)\exp\left(-\frac{z_{\alpha/2}}{\sqrt{N}}C\right), Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta)\exp\left(\frac{z_{\alpha/2}}{\sqrt{N}}C\right)\right],
\end{aligned}
\tag{22}
$$

where $S = \exp\left(\frac{z_{\alpha/2}}{\sqrt{N}}\frac{\widehat{\sigma}(R(\widehat{\boldsymbol{\theta}}^\beta))}{R(\widehat{\boldsymbol{\theta}}^\beta)(1-R(\widehat{\boldsymbol{\theta}}^\beta))}\right)$, $C = \frac{\widehat{\sigma}(Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta))}{Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta)}$, z_α denotes the upper α quantile of a standard normal distribution, $\sigma(E(\widehat{\boldsymbol{\theta}}^\beta))$, $\sigma(R(\widehat{\boldsymbol{\theta}}^\beta))$ and $\sigma(Q_{1-\gamma}(\widehat{\boldsymbol{\theta}}^\beta))$ are the standard errors of the mean lifetime, reliability and $1-\gamma$ quantile, respectively.

8 Prominent lifetime distributions

We now discuss the explicit expressions under four important lifetime distributions: exponential, Weibull, gamma, and lognormal. These four distributions have been widely studied in the literature as they are well-suited for fitting many lifetime data.

8.1 Exponential distribution

The exponential distribution is one of the most popular lifetime distributions due to its mathematical simplicity, tractable form and its adequacy for modelling lifetime events; see, for example, Johnson et al. (1994) and Balakrishnan and Basu (1995) for developments on various methods and applications of the exponential distribution. In the context of ALTs, Rodrigues et al. (1993) presented two approaches based on the likelihood ratio statistics for comparing exponential accelerated life models. Inference under the CE model under exponential distributions were discussed by Xiong (1998) and Xiong and Milliken (1999). Exact conditional distributions of the MLEs under the assumption of exponential failures and Type-II censoring was derived in Balakrishnan et al. (2007), and optimal step-stress tests for progressively Type-I censored data from exponential distributions was presented in Balakrishnan and Han (2009). A complete review on step-stress inferential methods under exponential lifetimes can be found in Balakrishnan (2009); Kundu and Ganguly (2017). In this section, we present explicit expression for the interval-censored ALTs under the exponential distribution with a single stress factor.

Let us consider a random variable T following an exponential distribution with scale parameter λ. The c.d.f. and p.d.f. of the exponential distribution are given by

$$F_T(t;\lambda) = 1 - \exp\left(-\frac{t}{\lambda}\right), t > 0,$$

and

$$f_T(t;\lambda) = \frac{\partial F_T(t;\lambda)}{\partial t} = \frac{1}{\lambda}\exp\left(-\frac{t}{\lambda}\right), t > 0.$$

The scale parameter λ may depend on some stress factor influencing the reliability of the product. This relationship is stated through the log-link function as given in (9). Now, assuming the cumulative exposure model as in Section 4.1, the lifetime c.d.f. through the step-stress experiment is given by

$$G_T(t) = \begin{cases} G_1(t) = 1 - \exp\left(-\frac{t}{\lambda_1}\right), & 0 < t < \tau_1, \\ G_2(t+s_1) = 1 - \exp\left(-\frac{t+s_2}{\lambda_2}\right), & \tau_1 \leq t < \tau_2, \\ \vdots & \vdots \\ G_k(t+s_{k-1}) = 1 - \exp\left(-\frac{t+s_k}{\lambda_k}\right), & \tau_{k-1} \leq t < \infty, \end{cases} \tag{23}$$

with the shifting time s_i being as given in Equation (3), for $i = 1, ..., k$. Therefore, the probability of failure within the $j-$inspected interval, under exponential times, is given by

$$\pi_j(\boldsymbol{\theta}) = -\exp\{-\exp(-\theta_0 - \theta_1 x_i)(IT_j + s_i)\} + \exp\{-\exp(-\theta_0 - \theta_1 x_i)(IT_{j-1} + s_i)\}, \tag{24}$$

$j = 1, ..., L$, and i denotes the stress level index at which units are subjected to before inspection time IT_j and $\pi_{L+1}(\boldsymbol{\theta}) = \exp\{-\exp(-\theta_0 - \theta_1 x_k)(\tau_k + s_k)\}$.

Theorem 3 *For step-stress ALT model under interval censoring and exponential lifetimes, matrices $\mathbf{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ required for the estimating equations and the asymptotic distribution stated in Theorems 1 and 2, respectively, are as follows:*

- $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ *is a $(L+1)\times(L+1)$ diagonal matrix with diagonal entries given by the probabilities of failure within each inspected interval, as derived in (24),*

$$\begin{aligned}\pi_j(\boldsymbol{\theta}) = &-\exp\{-\exp(-\theta_0 - \theta_1 x_i)(IT_j + s_i)\} \\ &+ \exp\{-\exp(-\theta_0 - \theta_1 x_i)(IT_{j-1} + s_i)\},\end{aligned}$$

  $j = 1, ..., L$, *where i denotes the stress level index at which units are subjected to before inspection time IT_j, $\pi_{L+1}(\boldsymbol{\theta}) = \exp\{-\exp(-\theta_0 - \theta_1 x_k)(\tau_k + s_k)\}$;*

- *$\boldsymbol{W}(\boldsymbol{\theta})$ is a $(L+1)\times 2$ matrix with rows $\mathbf{w}_j(\boldsymbol{\theta}) = z_j - z_{j-1}$, where*

$$z_j = g_T(IT_j)\begin{pmatrix} -(IT_j + s_i) \\ -(IT_j + s_i)x_i + s_i^* \end{pmatrix}, \quad j = 1, ..., L, \tag{25}$$

$$s_i = \lambda_i \sum_{l=1}^{i-1}\left(-\frac{1}{\lambda_{l+1}} + \frac{1}{\lambda_l}\right)\tau_l,\ i = 2, ..., k.,$$

$$s_i^* = s_i x_i + \lambda_i \sum_{l=1}^{i-1}\left(\frac{x_{l+1}}{\lambda_{l+1}} - \frac{x_l}{\lambda_l}\right)\tau_l, \quad i = 2, .., k, \tag{26}$$

$z_{-1} = z_{L+1} = \mathbf{0}$ and i is the stress level at which the units are tested after the $j-$th inspection time.

8.2 Weibull distribution

Weibull distribution is a commonly used lifetime model in engineering, physical and biomedical sciences, among others. For example, in reliability engineering, it is frequently used for analyzing the reliability and maintainability of products or systems. It is a simple distribution, but can capture different types of failure behaviours such as constant, increasing or decreasing failure rates. In the context of ALTs, Khamis (1997) compared constant and step-stress tests for Weibull models and Bai and Kim (1993) discussed optimum simple step-stress ALTs for Weibull distributions and type I censoring. Zhao and Elsayed (2005) developed inference for step-stress accelerated models using both Weibull and lognormal distributions. A Bayesian approach for estimating under Weibull lifetimes was proposed in Ganguly et al. (2014). More recently, Samanta et al. (2019) have analyzed step-stress model in the presence of competing risks under Weibull lifetimes.

Let us consider a Weibull random variable T with scale parameter λ and shape parameter η. The c.d.f. under Weibull distribution is given by

$$F_T(t) = 1 - \exp\left(-\left(\frac{t}{\lambda}\right)^{\eta}\right), \quad t \geq 0.$$

As the lifetime distribution depends on both scale and shape parameters, the homogeneity condition of the CE implies that the shape parameter η remains constant across all stress levels, while the scale parameter depends on the stress level through a log-linear relationship, as shown in Equation (9). Therefore, under the CE model, the Weibull c.d.f. of the lifetime, T, is given by

$$G_T(t) = G_i(t + s_i) = 1 - \exp\left(-\left(\frac{t + s_i}{\lambda_i}\right)^{\eta}\right), \quad \tau_{i-1} \leq t < \tau_i, \quad i = 1, ..., k, \tag{27}$$

where s_i denotes the shifting time at time of stress change τ_i as given in Equation (3).

With the previous notation, the probability of failure within the j-th interval, $j = 1, ..., L$, under Weibull lifetimes is given by

$$\pi_j(\boldsymbol{\theta}) = G_T(IT_j|\boldsymbol{\theta}) - G_T(IT_{j-1}|\boldsymbol{\theta}) = -\exp\left[-\left(\frac{IT_j + s_i}{\lambda_i}\right)^{\eta}\right] + \exp\left[-\left(\frac{IT_{j-1} + s_i}{\lambda_i}\right)^{\eta}\right], \tag{28}$$

where $\boldsymbol{\theta} = (\theta_0, \theta_1, \eta)^T$, i is the stress level at which units are exposed just before IT_j, and s_i is the shifting time as defined in Equation (3). The probability of survival at the end of the experiment is given by

$$\pi_{L+1}(\boldsymbol{\theta}) = 1 - G_T(\tau_k|\boldsymbol{\theta}) = \exp\left(-\left(\frac{\tau_k + s_k}{\lambda_k}\right)^{\eta}\right).$$

Theorem 4 *For step-stress ALT model under interval censoring and Weibull lifetimes, matrices $\boldsymbol{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ required for the estimating equations and the asymptotic distribution stated in Theorems 1 and the 2, respectively, are as follows:*

- $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ *denotes a $(L+1) \times (L+1)$ diagonal matrix with diagonal entries $\pi_j(\boldsymbol{\theta})$, $j = 1, ..., L+1$, defined as in Equation (28),*

$$\pi_j(\boldsymbol{\theta}) = -\exp\left[-\left(\frac{IT_j + s_i}{\lambda_i}\right)^{\eta}\right] + \exp\left[-\left(\frac{IT_{j-1} + s_i}{\lambda_i}\right)^{\eta}\right],$$

where i is the stress level at which units are exposed just before IT_j and

$$\pi_{L+1}(\boldsymbol{\theta}) = \exp\left(-\left(\frac{\tau_k + s_k}{\lambda_k}\right)^{\eta}\right);$$

- $\boldsymbol{W}(\boldsymbol{\theta})$ *is a $(L+1) \times 3$ matrix with rows $\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}$, where*

$$\boldsymbol{z}_j = g_T(IT_j)\begin{pmatrix} -(IT_j + s_i) \\ -(IT_j + s_i)x_i + s_i^* \\ \log\left(\frac{IT_j + s_i}{\lambda_i}\right)\frac{IT_j + s_i}{\eta} \end{pmatrix}, \tag{29}$$

$$s_i^* = s_i x_{i+1} + \lambda_i \sum_{l=1}^{i-1}\left(\frac{x_{l+1}}{\lambda_{l+1}} - \frac{x_l}{\lambda_l}\right)\tau_l, \tag{30}$$

for any $j = 1, ..., L$ and $i = 1, ..., k-1$ with $\boldsymbol{z}_{-1} = \boldsymbol{z}_{L+1} = \boldsymbol{0}$ and i being the stress level at which the units are tested before the $j-$th inspection time.

8.3 Gamma distribution

The gamma distribution is a flexible distribution that can model a wide range of shapes for the failure time distribution, when the failure times are continuous and positively skewed. Moreover, it can be used in reliability growth models to characterize the improvement in reliability over time, since it allows for the modelling of a system's increasing reliability as it undergoes modifications or improvements.

The gamma distribution has not received great attention as the exponential and Weibull distributions have received in step-stress testing, although it is an important distribution for modelling lifetime data. Alkhalfan (2012) developed in detail inference for step-stress tests under censoring with gamma lifetimes.

Let T be a gamma random variable with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$. Then, the p.d.f. of T is given by

$$f_T(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha; 0)\lambda^{\alpha}} \exp\left(-\frac{t}{\lambda}\right), \quad t > 0,$$

where $\Gamma(\alpha; t)$ denotes the upper incomplete gamma function, $\Gamma(\alpha; t) = \int_t^{\infty} x^{\alpha-1} \exp(-x)dx$, for any $t > 0$, and its c.d.f. is given by $F_T(t) = P\left(\alpha; \frac{t}{\lambda}\right)$, where $P(\alpha; t)$ is the standardized lower incomplete gamma function defined as

$$P(\alpha; t) = \frac{\Gamma(\alpha; 0) - \Gamma(\alpha; t)}{\Gamma(\alpha; 0)} = \int_0^t \frac{1}{\Gamma(\alpha; 0)} x^{\alpha-1} \exp(-x)dx, \quad t > 0, \ \alpha > 0.$$

This ratio depends on the upper integral bound t and the shape parameter α. The shape of the p.d.f. in the gamma distribution is influenced by parameter α, with the distribution's skewness being inversely proportional to $\sqrt{\alpha}$. Meanwhile, the rate of decrease in the distribution is governed by the scale parameter λ, where a higher λ leads to a faster decrease in the tail of the distribution.

Let us consider, as in the previous sections, a k-step stress experiment under the CE model with N devices under test and L inspection times, including all the times of stress change. We now assume that the lifetimes of the devices under constant stress follow a gamma distribution with common shape parameter α and scale parameter depending on the stress level through a log-linear relationship as in Equation (9).

Therefore, the vector of model parameter under gamma lifetime distributions is referred to as $\boldsymbol{\theta} = (\theta_0, \theta_1, \alpha)^T$. The MTTF of a device at constant stress x_i is known to be $E[T] = \alpha\lambda_i$. This implies that larger scale parameters produce more reliable products. As the probability of failure is to be expected to increase when the stress level increases, for positive and increasing stress levels, the parameter θ_1 in Equation (9) should be negative and the parameter space then gets reduced to $\Theta = \mathbb{R} \times \mathbb{R}^{-} \times \mathbb{R}^{+}$. Note that this assumption can always be made by transforming the stress levels suitably. Moreover, as discussed in the previous Sections, assuming a link function relating the scale parameter with the stress avoids the estimation of different scale parameters, and it suffices to estimate only two parameters, θ_0 and θ_1, to determine the scale parameter at any step, regardless of the number of stress changes considered in the experiment. Reducing the parameters in gamma models is appealing given the relatively high complexity of the equations, which depend on the incomplete gamma function.

Under the CE model, and assuming the homogeneity condition for the shape parameter α, the lifetime distribution of a device under gamma lifetime distribution with common shape parameter $\alpha > 0$ and scale parameter $\lambda_i > 0$ depending on the stress level x_i, is given by

$$G_T(t) = P\left(\alpha; \frac{t+s_i}{\lambda_i}\right), \; \tau_{i-1} \leq t < \tau_i, \; i = 1, ..., k,$$

with the shifting time s_i being as in Equation (3). Accordingly, the p.d.f. of the lifetime is given by the composite p.d.f.

$$g_T(t) = g_i(t+s_i) = \frac{(t+s_i)^{\alpha-1}}{\Gamma(\alpha;0)} \frac{1}{\lambda_i^{\alpha}} \exp\left(-\frac{t+s_i}{\lambda}\right), \; \tau_{i-1} \leq t < \tau_i, \; i = 1, ..., k.$$

It is useful to note that the above p.d.f. is not continuous at the times of stress change, and so neither is the hazard function. With the previous notation, if we assume that the lifetime of the devices follow a gamma distribution, the probability of failure within the j-th interval is given by

$$\pi_j(\boldsymbol{\theta}) = P\left(\alpha; \frac{IT_j + s_i}{\lambda_i}\right) - P\left(\alpha; \frac{IT_{j-1} + s_i}{\lambda_i}\right), \tag{31}$$

where $\boldsymbol{\theta} = (\theta_0, \theta_1, \alpha)^T$ and the index i denotes the stress level at which the units are tested before the $j-$th inspection time. Similarly, the probability of survival at the end of the experiment is given by

$$\pi_{L+1}(\boldsymbol{\theta}) = 1 - P\left(\alpha; \frac{\tau_k + s_k}{\lambda_k}\right).$$

Theorem 5 *For step-stress ALT model under interval censoring and gamma lifetimes, matrices $\boldsymbol{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ required for the estimating equations and the asymptotic distribution stated in Theorems 1 and 2, respectively, are as follows:*

- *The matrix $\boldsymbol{W}(\boldsymbol{\theta})$ is a $(L+1) \times 3$ matrix with rows*

$$\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}, \; j = 1, ..., L+1, \tag{32}$$

where

$$\boldsymbol{z}_j = \begin{bmatrix} -g_T(IT_j)(IT_j + s_i) \\ g_T(IT_j)\left[-(IT_j + s_i)x_i + s_i^*\right] \\ P\left(\alpha; \frac{IT_j+s_i}{\lambda_i}\right)\left[\log\left(\frac{IT_j+s_i}{\lambda_i}\right) - \psi(\alpha)\right] - \left(\frac{IT_j+s_i}{\lambda_i}\right)^{\alpha} \frac{{}_2F_2\left(\alpha,\alpha;\alpha+1,\alpha+1;-\frac{IT_j+s_i}{\lambda_i}\right)}{\alpha^2\Gamma(\alpha)} \end{bmatrix}$$

for $j = 1, ..., L$, and $\boldsymbol{z}_{L+1} = 0$. The shifting times s_i are as given in Equation (3) and

$$s_i^* = s_i x_i + \lambda_i \sum_{l=1}^{i-1} \left(\frac{x_{l+1}}{\lambda_{l+1}} - \frac{x_l}{\lambda_l}\right) \tau_l, \quad i = 1, .., k, \tag{33}$$

and function ψ denotes the digamma function. On the last element, ${}_2F_2(\cdot)$ denotes the Gaussian hypergeometric function,

$$
{}_pF_q(a_1, a_2, .., a_p; b_1, b_2, ..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!},
$$

with the Pochhammer symbol being $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ for all $\lambda \in \mathbb{C}$ (see Seaborn (1991));

- $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ *denotes a $(L+1) \times (L+1)$ diagonal matrix with diagonal entries as in Equation (31),*

$$
\pi_j(\boldsymbol{\theta}) = P\left(\alpha; \frac{IT_j + s_i}{\lambda_i}\right) - P\left(\alpha; \frac{IT_{j-1} + s_i}{\lambda_i}\right).
$$

See Balakrishnan and Ling (2014) for more details on the derivation of these expressions.

8.4 Lognormal distribution

The lognormal distribution is an important lifetime distribution which has been extensively used for analyzing data from reliability and survival studies. For example, Meeker et al. (1998) compared ALTs for Weibull and lognormal distributions under Type-I censoring. Ng et al. (2002) and Basak et al. (2009) developed EM algorithms for estimating the parameters of two-parameter lognormal and three-parameter lognormal distributions, respectively, based on progressively censored data. Balakrishnan and Mitra (2011) developed likelihood inference for lognormal data with left truncation and right censoring based on the EM algorithm, while Nagatsuka and Balakrishnan (2012, 2013, 2016) proposed likelihood-based estimation methods for the three parameter lognormal distribution. Calviño (2020) reviewed inferential procedures for a step-stress model with type-II and progressive type-II censoring and lognormally distributed lifetimes. Likelihood inference procedures were extended to one-shot device test data under log-normal lifetimes and constant stress ALTs by Balakrishnan and Castilla (2021).

Let T be a lognormal lifetime random variable with parameters μ and $\sigma > 0$. The p.d.f. of T is given by

$$
f_T(t) = \frac{1}{\sqrt{2\pi}\sigma t} \exp\left(-\frac{(\log(t) - \mu)^2}{2\sigma^2}\right), \quad t > 0,
$$

and the corresponding c.d.f. is given by

$$
F_T(t) = \Phi\left(\frac{\log(t) - \mu}{\sigma}\right), \quad t > 0,
$$

where $\Phi(\cdot)$ denotes the c.d.f. of the standard normal distribution. Here, σ is a shape parameter and $m = \exp(\mu)$ is a scale parameter. We could equivalently use the parametrization of the p.d.f. in terms of shape and scale (σ, m), but we prefer to keep the above parametrization as it is the most common form of the lognormal family in the literature.

Let us consider a multiple step-stress ALT with k ordered stress levels $x_1 < x_2 < \cdots < x_k$ and their corresponding change times $\tau_1 < \tau_2 < \cdots < \tau_k$. Let G_i denote the lognormal lifetime distribution at a constant stress level x_i, and G_T denote the lifetime distribution of a device during the step-stress experiment. Under the CE model, the c.d.f. of the devices' lifetime is given by the composite function

$$G_T(t) = \Phi\left(\frac{\log(t+s_i) - \mu_i}{\sigma}\right), \text{ if } \tau_{i-1} \leq t < \tau_i, \quad i = 1, ..., k, \tag{34}$$

where $\tau_0 = 0$ and the last part of the function can be extended up to ∞, and the corresponding p.d.f. is given by

$$g_T(t) = \begin{cases} g_1(t) = \frac{1}{\sqrt{2\pi}\sigma t}\exp\left(-\frac{(\log(t)-\mu_1)^2}{2\sigma^2}\right), & 0 \leq t < \tau_1, \\ g_2(t+s_2) = \frac{1}{\sqrt{2\pi}\sigma(t+s_2)}\exp\left(-\frac{(\log(t+s_2)-\mu_2)^2}{2\sigma^2}\right), & \tau_1 \leq t < \tau_2, \\ \vdots & \vdots \\ g_k(t+s_k) = \frac{1}{\sqrt{2\pi}\sigma(t+s_k)}\exp\left(-\frac{(\log(t+s_k)-\mu_k)^2}{2\sigma^2}\right), & \tau_{k-1} \leq t < \infty, \end{cases}$$

with s_i being defined as

$$s_i = \exp(\mu_i)\sum_{l=1}^{i-1}\left(\exp(-\mu_{l+1}) - \exp(-\mu_l)\right)\tau_l, \text{ for } i = 2, ..., k, \tag{35}$$

and index i indicates the stress level step.

For log-normal distributions, the log-linear relationship given in Equation (9) can be rewritten as

$$\mu_i(\boldsymbol{\theta}) = \theta_0 + \theta_1 x_i, \quad i = 1, ..., k, \tag{36}$$

where the scale parameter $\exp(\mu_i)$ depends on the stress level through a log-linear relationship. Further, since the increased stress is expected to shorten the lifetime to failure of the devices, under positive stress levels θ_1 must be negative and the space parameter thus gets reduced to $\Theta = \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}^+$.

In many practical applications, the lognormally distributed lifetimes are transformed by $W = \log(T)$, as the resulting model then belongs to a location-scale normal family of distributions. In this case, the location parameter would depend on the stress factors, but the scale parameter would remain constant.

Under log-normal lifetimes, the probability of failure between two subsequent inspection times IT_j and IT_{j-1} is given by

$$\pi_j(\boldsymbol{\theta}) = \Phi\left(\frac{\log(IT_j + s_i) - \mu_i}{\sigma}\right) - \Phi\left(\frac{\log(IT_{j-1} + s_i) - \mu_i}{\sigma}\right),$$

where $\boldsymbol{\theta} = (\theta_0, \theta_1, \sigma)^T$ is the model parameter vector, IT_j is the current inspection time and i denotes the stress level at which units are tested before the j-th inspection time, $j = 1, ..., L$.. The survival probability is given by

$$\pi_{L+1}(\boldsymbol{\theta}) = 1 - \Phi\left(\frac{\log(\tau_k + s_k) - \mu_k}{\sigma}\right).$$

Theorem 6 *For step-stress ALT model under interval censoring and lognormal lifetimes, matrices $\boldsymbol{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ required for the estimating equations and the asymptotic distribution stated in Theorems 1 and 2, respectively, are as follows:*

- *$\boldsymbol{W}(\boldsymbol{\theta})$ is a $(L+1) \times 3$ matrix with rows*

$$\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}, \text{ for } j = 1, ..., L+1, \tag{37}$$

where

$$\boldsymbol{z}_j = -\frac{1}{\sigma}\phi\left(\frac{\log(IT_j + s_i) - \mu_i}{\sigma}\right)\begin{bmatrix} 1 \\ \frac{-s_i^*}{IT_j + s_i} + x_i \\ \frac{\log(IT_j + s_i) - \mu_i}{\sigma} \end{bmatrix}$$

with s_i being as defined in Equation (35) and

$$s_i^* = s_i x_i + \exp(\mu_i)\sum_{l=1}^{i-1}\left(\frac{-x_l}{\exp(\mu_l)} + \frac{x_{l+1}}{\exp(\mu_{l+1})}\right)\tau_l, \text{ for } i = 2, ..., k, \tag{38}$$

i denotes the stress level at which units are tested after the j-th inspection time, t_j, and $\boldsymbol{z}_{L+1} = \boldsymbol{0}$;

- *$\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ denotes a $(L+1) \times (L+1)$ diagonal matrix with diagonal entries $\pi_j(\boldsymbol{\theta})$, defined in (37),*

$$\pi_j(\boldsymbol{\theta}) = \Phi\left(\frac{\log(IT_j + s_i) - \mu_i}{\sigma}\right) - \Phi\left(\frac{\log(IT_{j-1} + s_i) - \mu_i}{\sigma}\right) \quad j = 1, ..., L,$$

and $\pi_{L+1}(\boldsymbol{\theta}) = 1 - \Phi\left(\frac{\log(\tau_k + s_k) - \mu_k}{\sigma}\right)$.

While acknowledging that a rigorous proof of existence and uniqueness of the MDPDEs (including the MLE) for the step-stress ALT under lognormal lifetime distributions remains an open problem, we proceed with the assumption that the solution of the system of equations in Theorem 1 does exist. In that case, the MDPDE should be computed numerically using any standard optimization algorithm such as Newton-Raphson method, or BFGS (Broyden-Fletcher-Goldfarb-Shanno) method.

Open Problem 3 (Generalized Gamma Distribution) *We have explored four prominent distributions for fitting the lifetime data. However, these are all particular members of the generalized gamma family of distributions; see Balakrishnan and Peng (2006) . It will therefore be of great interest to develop the corresponding inferential results for this general family of lifetime distributions, and then study the associated model selection/discrimination problems within this flexible family.*

Open Problem 4 (Birnbaum-Saunders distribution)

In addition to parametric families belonging to the generalized gamma family, other important lifetime distributions that fit failure data well need to be studied. In particular, the Birnbaum-Saunders distribution (also known as the fatigue life distribution) and its generalized version may be considered. This family of distributions was developed to model failures due to cracks and depends on two parameters: a shape parameter α and a scale parameter β. The c.d.f. of a random variable T under Birnbaum-Saunders distribution is given by

$$F_T(t; \alpha, \beta) = \Phi\left(\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right),$$

where Φ denotes the standard normal distribution function.

Note that the parameter α is related to the standard deviation of the normal distribution, and a transformation

$$T^* = \left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{T}\right)^{\frac{1}{2}}$$

is normally distributed with zero mean and standard deviation $\sqrt{\alpha}$. Therefore, the shape parameter α can be estimated using the normal distribution, given a known scale parameter β. Estimation can be performed iteratively by alternately updating α and β, and this same idea can be applied in the process of robust estimation as well.

9 Competing risks model

In lifetime data analysis, it is common for products under study to experience different types of failure. In the context of survival analysis, multiple types of failure (e.g., death, relapse, opportunistic infection) may be of interest, leading to a "competing risks" scenario. Competing risks arise in experiments in which units are subject to several potential failure causes and the occurrence of one event might impede the occurrence of other events. For instance, in a study examining time to death due to cardiovascular causes, death from non-cardiovascular causes would be considered a competing risk. In engineering experiments, when investigating the lifetime to failure of a mechanical component like a gearbox, competing risks can emerge. For instance, the gearbox may have two competing failure modes: bearing failure and gear tooth failure. The occurrence of one failure mode can prevent the other from happening. Analyzing the competing risks helps engineers understand the reliability characteristics of the component so as to optimize its design and operating conditions.

Crowder (2006) presented a review of the competing risks problem, which involves estimating the failure rates for each cause. Balakrishnan and Han (2008) developed exact inference for a simple step-stress model with competing risks for exponential lifetimes under Type-II censoring. Along similar lines, Han and Balakrishnan (2010) developed exact inference for a simple step-stress model for the case of exponential lifetime distribution, but under time constraints. Han and Kundu (2014) developed inferential methods for a step-stress model with competing risks assuming generalized exponential distributions under Type-I censoring. More recently, Baghel and (2024) proposed divergence-based estimation methods for dependant competing risks under interval monitoring for reliability tests under normal operating conditions. For extreme interval-censoring schemes, Balakrishnan et al. (2015) studied MLEs for one-shot device testing with competing risks under exponential distribution.

In this section, we develop robust estimators under the step-stress model with independent competing risks and exponential distributions. For mathematical simplicity, in this section, we will develop the theory for simple step-stress ALTs, but all arguments discussed here can be easily extended to multiple step-stress set-up.

Let us consider a simple step-stress ALT with two stress levels x_1 and x_2, R independent competing risks causing failure and N units under test. We denote by τ_1 the time of stress change from x_1 to x_2 and by τ_2 the experiment termination time. Moreover, we consider L pre-fixed inspection times where failure counts are recorded, and we assume as usual that those inspection times include the time of stress change τ_1. Assuming exponential distributions for the lifetimes of devices for each competing risk T_j, where $j = 1, ..., R$, at a constant stress level x_i, for $i = 1, 2$, with scale parameter $\theta_{ij} > 0$, the c.d.f. of the lifetime attributed to cause j can be expressed as

$$F_j(t) = \begin{cases} 1 - \exp\left(-\frac{t+s_j^{(1)}}{\lambda_{1j}}\right), & 0 < t < \tau_1, \\ 1 - \exp\left(-\frac{t+s_j^{(2)}}{\lambda_{2j}}\right), & \tau_1 \le t < \infty, \end{cases}$$

with

$$s_j^{(1)} = 0 \quad \text{and} \quad s_j^{(2)} = \frac{\lambda_{2j}}{\lambda_{1j}}\tau_1 - \tau_1, \tag{39}$$

for all $j = 1, ..., R$. Here, we have included the shifting time at the first step for notational simplicity in the following expressions. The corresponding p.d.f. of T_j is given by

$$f_j(t) = \frac{1}{\lambda_{ij}} \exp\left(-\frac{t + s_j^{(i)}}{\lambda_{ij}}\right) \text{ if } \tau_{i-1} \le t < \tau_i,$$

with $\tau_0 = 0$ and τ_2 can be extended to ∞. Note that the shifting times are different for each competing risk, and depend on the distribution parameters of each lifetime T_j.

Further, we will assume the scale parameter of each marginal distribution is related to the stress level through a log-linear relationship. That is, for any competing risk $j = 1, ..., R$, and stress level x_i, $i = 1, 2$, we have

$$\lambda_{ij} = \exp(\theta_{0j} + \theta_{1j} x_i),$$

and so the model can be re-parametrized in terms of the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_j)_{j=1,...,R}$ with $\boldsymbol{\theta}_j = (\theta_{0j}, \theta_{1j})$.

As the risks are mutually exclusive, we will observe only the smaller time of failure $T = \min_{j=1,...,R}(T_j)$. Then, the c.d.f. of the overall failure time of a test unit is readily obtained as

$$F_T(t) = 1 - \prod_{j=1}^{R}(1 - F_j(t)) = 1 - \exp\left(-\sum_{j=1}^{R} \frac{t + s_j^{(i)}}{\lambda_{ij}}\right), \quad \tau_{i-1} \leq t < \tau_i, \quad (40)$$

with $s_j^{(i)}$ being as defined in Equation (39), for any $i = 1, 2$ and $j = 1, ..., R$. The corresponding p.d.f. is consequently given by

$$f_T(t) = \left[\sum_{j=1}^{R} \frac{1}{\lambda_{ij}}\right] \exp\left(-\sum_{j=1}^{R} \frac{t + s_j^{(i)}}{\lambda_{ij}}\right), \quad \tau_{i-1} \leq t < \tau_i. \quad (41)$$

Furthermore, let C denote the indicator for the cause of failure. Then, the joint p.d.f. of the failure time T and the indicator failure mode C can be expressed using the marginal distributions as

$$f_{(T,C)}(t, j) = f_j(t) \prod_{\substack{j^*=1 \\ j^* \neq j}}^{R} (1 - F_{j}^{*}(t)) = \frac{1}{\lambda_{ij}} \exp\left(-\sum_{j=1}^{R} \frac{t + s_j^{(i)}}{\lambda_{ij}}\right), \quad \tau_{i-1} \leq t < \tau_i. \quad (42)$$

From the above joint p.d.f., we can compute the conditional probabilities, such as the relative risk imposed on a test unit before the time of stress change τ_1 due to the risk factor j, as

$$\pi_{1j}(\boldsymbol{\theta}) = P(C = j | 0 < T < \tau_1) = \frac{\lambda_{1j}^{-1}}{\sum_{j=1}^{R} \lambda_{1j}^{-1}} \quad (43)$$

and similarly, the relative risk after τ_1 due to the factor j,

$$\pi_{2j}(\boldsymbol{\theta}) = P(C = j | \tau_1 < T < \infty) = \frac{\lambda_{2j}^{-1}}{\sum_{j=1}^{R} \lambda_{2j}^{-1}}. \quad (44)$$

Both relative risks are the proportion of failure rates in the given time frame.

As we are dealing with interval-monitoring experiments, the resulting data would be failure counts within each inspected interval, but now we will jointly record the cause of failure. From the p.d.f. in (42), we can compute the theoretical probability of failure within an interval due to each competing risk as

$$\begin{aligned} p_{lj}(\boldsymbol{\theta}) &= P\left(IT_{l-1} < t < IT_l, C = j\right) = \int_{IT_{l-1}}^{ITl} f_{(T,C)}(t, C = j)dt \\ &= \pi_{ij}(\boldsymbol{\theta}) \left(\exp\left(-\sum_{j=1}^{R} \frac{IT_{l-1} + s_j^{(i)}}{\lambda_{ij}} \right) - \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\lambda_{ij}} \right) \right), \\ &\quad \text{for } \tau_{i-1} \leq IT_l < \tau_i. \end{aligned} \tag{45}$$

Note that i corresponds to the stress level at which the units are subjected during the period from IT_{i-1} to IT_i and $\pi_{ij}(\boldsymbol{\theta})$ is the relative risk imposed on a test unit before the time of stress change due to the risk factor j, as defined in Equations (43) and (44), for $i = 1$ and $i = 2$, respectively. On the other hand, the probability of survival is given by

$$p_0(\boldsymbol{\theta}) = 1 - F_T(\tau_2) = \exp\left(-\sum_{j=1}^{R} \frac{\tau_2 + s_j^{(2)}}{\lambda_{2j}} \right).$$

Because we have considered all exclusive failure events, all defined probabilities sum up to one, i.e.,

$$\sum_{j=1}^{R} \sum_{l=1}^{L} p_{lj}(\boldsymbol{\theta}) + p_0(\boldsymbol{\theta}) = 1.$$

Given that the observed data consist of counts of events, the above equation provides an understanding of considering a multinomial model with N trials (devices) and $R \cdot L + 1$ events; the first $R \cdot L$ events correspond to failures attributed to each competing risk at each inspection interval, while the last event represents the survival beyond the experiment's termination. The probability vector of the multinomial model is then given by

$$\boldsymbol{p}(\boldsymbol{\theta}) = (p_{11}(\boldsymbol{\theta}), ..., p_{1R}(\boldsymbol{\theta}), p_{21}(\boldsymbol{\theta}), ..., p_{2R}(\boldsymbol{\theta}), ..., p_{L1}(\boldsymbol{\theta}), p_{LR}(\boldsymbol{\theta}), p_0(\boldsymbol{\theta})).$$

In terms of the regression parameter $\boldsymbol{\theta}$, we can rewrite the probability of failure at the l-th interval due to risk j as

$$\begin{aligned} p_{lj}(\boldsymbol{\theta}) = \frac{\exp(-\theta_{0j} - \theta_{1j}x_i)}{\sum_{j=1}^{R} \exp(-\theta_{0j} - \theta_{1j}x_i)} &\left(\exp\left(-\sum_{j=1}^{R} \frac{IT_{l-1} + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)} \right) \right. \\ &\left. - \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)} \right) \right), \quad \tau_{i-1} \leq IT_l < \tau_i, \end{aligned} \tag{46}$$

and the probability of survival as

$$p_0(\boldsymbol{\theta}) = \exp\left(-\sum_{j=1}^{R} \frac{\tau_2 + s_j^{(2)}}{\exp(\theta_{0j} + \theta_{1j}x_2)}\right),$$

with

$$s_j^{(1)} = 0 \quad \text{and} \quad s_j^{(2)} = \tau_1\left(\frac{\exp(\theta_{0j} + \theta_{1j}x_2)}{\exp(\theta_{0j} + \theta_{1j}x_1)} - 1\right). \tag{47}$$

We denote by n_{lj} the number of failures recorded at the l-inspection time due to risk j, $l = 1, ..., L$ and $j = 1, ..., R$. The main difference compared to the model with a single cause of failure is that we now need to consider each competing risk separately. However, all the theory presented earlier in Section 5 for the multinomial model with a single cause of failure, including the estimating equations and the asymptotic distribution, remains valid for the competing risk model, with some adjustments to account for all failure causes.

Note that the (incomplete) log-likelihood function as defined in (11) is not well-defined when the number of failed units due to cause j, with $j = 1, ..., R$, at any of the step-stress, is zero or N. That is, for estimating the MLE, we need at least one failure due to each competing risk and under each of the stress levels. For exact inference of the MLE conditional on that there is at least one failure observed due to each competing risk and under each of the stress levels, we refer to Balakrishnan et al. (2007). However, if the MLE is well-defined, then it will satisfy the asymptotic properties, and particularly the asymptotic distribution, of the (unconditional) MLE. In what follows, we will now assume that there is at least one observation for each combination of stress level and cause of failure. The following result provides explicit expressions for the matrices $\boldsymbol{W}$ and $\boldsymbol{D}_{\boldsymbol{p}(\boldsymbol{\theta})}$ required for Theorems 1 and 2 under competing risks.

Theorem 7 *For step-stress ALT model under interval censoring and exponential lifetimes and competing risks, matrices $\boldsymbol{W}(\boldsymbol{\theta})$ and the $\boldsymbol{D}_{p(\boldsymbol{\theta})}$ required for the estimating equations and the asymptotic distribution stated in Theorems 1 and 2, respectively, are as follows:*

- *The matrix $\boldsymbol{W}(\boldsymbol{\theta})$ is a $(LR+1) \times 2R$ matrix consisting of R columns blocks of dimension $(LR+1) \times 2$ defined as $\boldsymbol{w}^k = \left[(z_{l-1,j}^k - z_{lj}^k)^T_{j=1,..R,l=1,...,L}, z_0^k\right]$, for $k = 1, ..., R$ with $z_{l,j}^k = ((z_{l,j})_{1k}, (z_{lj})_{2k})$,*

$$(z_{lj})_{1k} = \begin{cases} \pi_{ij} \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right)\left(\pi_{ik} + \frac{IT_l + s_k^{(i)}}{\exp(a_{0k} + a_{1k}x_i)}\right), & j \neq k, \\ \pi_{ij} \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right)\left(\pi_{ij} - 1 + \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right), & j = k, \end{cases}$$

$$(z_{lj})_{2k} = \begin{cases} \pi_{ij} \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right)\left(\pi_{ik}x_i + \frac{-s_k^{*(i)} + (IT_l + s_k^{(i)})x_i}{\exp(a_{0k} + a_{1k}x_i)}\right), & j \neq k, \\ \pi_{ij} \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right)\left(\pi_{ij}x_i - x_i + \frac{-s_j^{*(i)} + (IT_l + s_j^{(i)})x_i}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right), & j = k, \end{cases}$$

where i is chosen such that $\tau_{i-1} \leq IT_l < \tau_i$, $s_j^{(i)}$ is as defined in (47) and

$$s_j^{*(1)} = 0 \quad and \quad s_j^{*(2)} = \tau_1 \frac{\exp(\theta_{0j} + \theta_{1j}x_2)}{\exp(\theta_{0j} + \theta_{1j}x_1)}(x_2 - x_1). \tag{48}$$

For the last column of the block $\mathbf{w}^k$, the entries of the matrix are defined as

$$(z_0)_{1k} = \exp\left(-\sum_{j=1}^{R} \frac{\tau_2 + s_j^{(2)}}{\exp(\theta_{0j} + \theta_{1j}x_2)}\right)\left(\frac{\tau_2 + s_j^{(2)}}{\exp(\theta_{0k} + \theta_{1k}x_2)}\right),$$

$$(z_0)_{2k} = \exp\left(-\sum_{j=1}^{R} \frac{\tau_2 + s_j^{(2)}}{\exp(\theta_{0j} + \theta_{1j}x_2)}\right)\left(\frac{-s_k^{*(2)} + (\tau_2 + s_k^{(2)})x_2}{\exp(a_{0k} + a_{1k}x_2)}\right);$$

- *The matrix $\boldsymbol{D}_{\boldsymbol{p}(\boldsymbol{\theta})}$ denotes a $(RL+1) \times (RL+1)$ diagonal matrix with diagonal entries $p_{lj}(\boldsymbol{\theta})$, $l = 1, ..., L$, $j = 1, ..., R$, and $p_0(\boldsymbol{\theta})$ as defined in (46),*

$$p_{lj}(\boldsymbol{\theta}) = \frac{\exp(-\theta_{0j} - \theta_{1j}x_i)}{\sum_{j=1}^{R}\exp(-\theta_{0j} - \theta_{1j}x_i)}\left(\exp\left(-\sum_{j=1}^{R} \frac{IT_{l-1} + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right) - \exp\left(-\sum_{j=1}^{R} \frac{IT_l + s_j^{(i)}}{\exp(\theta_{0j} + \theta_{1j}x_i)}\right)\right), \quad \tau_{i-1} \leq IT_l < \tau_i,$$

$$p_0(\boldsymbol{\theta}) = \exp\left(-\sum_{j=1}^{R} \frac{\tau_2 + s_j^{(2)}}{\exp(\theta_{0j} + \theta_{1j}x_2)}\right).$$

Open Problem 5 (Robust Inference with Competing Risks under Weibull, Gamma and Lognormal Distributions) *In the above discussion, robust inference for interval-censored step-stress ALT under competing risks is developed under the assumption of exponential lifetime distributions. However, the competing risks model has been also considered in reliability literature under other distributions. For example, Samanta et al. (2019) considered the estimation of model parameters with Weibull lifetime distribution. So, the development of robust inferential methods with competing risks under Weibull, gamma and lognormal distributions will be of great interest and remains as an open problem.*

Open Problem 6 (Robust Inference for Dependent Competing Risks) *The assumption of independent competing risks may not be valid in practical applications. Baghel and (2024) derived robust point estimation and hypothesis testing for dependent competing* *risks under interval-censoring for constant ALTs, and determined the optimal set of inspection times based on some predefined objectives. Such a theory should be properly extended to step-stress ALT designs, and this remains as an open problem.*

Open Problem 7 (Robust Test Statistics under Competing Risks) *A robust generalization of the Wald, Rao and likelihood test statistics have been discussed in Balakrishnan et al. (2023a, b) for exponential distribution and an unique cause of failure. The theory could be extended to more general lifetime assumptions such as more causes of failure. To the best of our knowledge, hypothesis testing under competing risks remains as an open research area for classical inferential methods relying on the MLE as well as for robust tests based on divergence measures.*

10 Proportional hazards model

Several known distributions have been explored so far to describe the lifetimes of tested products. However, the true underlying distribution for the data is generally unknown and so, none of the previously discussed parametric distributions may be adequate to fit the observed data. Alternatively, proportional hazards (PH) model offers a semi-parametric multiple regression approach for reliability estimation (see Cox (1972)), in which the baseline hazard function is multiplicatively affected by the applied stresses. That is, if we denote by $h(t, x)$ the hazard rate function of a product lifetime under a constant stress x, then we can multiplicatively separate the effects of the natural product reliability and the stress level experienced as

$$h(t, x) = h_0(t)\lambda(x; \boldsymbol{\theta}), \tag{49}$$

where $\lambda(x; \boldsymbol{\theta})$ is the log-linear relation stated in Equation (9). The PH model assumes that the ratio of hazard rates between two stress levels is constant over time, i.e.,

$$\frac{h(t, x_1)}{h(t, x_2)} = \frac{\lambda(x_1; \boldsymbol{\theta})}{\lambda(x_2; \boldsymbol{\theta})}$$

for any two stress levels x_1 and x_2. The PH model is distribution-free and relies on the assumption that the hazard rate ratio between two stress levels remains constant over time. This model can be particularly useful when no expert knowledge may be available to select a specific parametric lifetime distribution family, but certain natural constraints, such as the constant hazard rate ratio, can be made.

Note that, from the hazard function $h(t, x)$, the cumulative hazard, p.d.f.'s and c.d.f.'s can all be obtained. The CE assumption can alternatively be as stated in terms of the cumulative hazard function at constant stress level x_i, denoted in the following by $H_i(\cdot)$, $i = 1, ...k$, as follows:

$$H_i(\tau_i + s_i) = H_{i+1}(\tau_i + s_{i+1}), i = 1, ..., k,$$

where $H_i(t) = -\log(1 - F_{i+1}(t))$, $s_1 = 0$ and s_{i+1} is the shifting time at time τ_i.

Bringing the PH and the cumulative exposure assumptions together, the shifting time s must satisfy the following relationship:

$$H_i(\tau_i+s_i)=\lambda(x_i,\boldsymbol{\theta})\int_0^{\tau_i+s_i} h_0(t)dt=\lambda(x_{i+1},\boldsymbol{\theta})\int_0^{\tau_i+s_{i+1}} h_0(t)dt=H_{i+1}(\tau_i+s_{i+1}). \tag{50}$$

From the above formula, the shifting time $s_{i+1} = H_{i+1}^{-1}(H_i(\tau_i + s_i)) - \tau_i$. In practice, the shifting time can be estimated either non-parametrically or by assuming a specific form for the baseline hazard. However, given that the observed data are censored, non-parametric estimation may be inefficient. Therefore, we will assume a linear form for the baseline hazard

$$h_0(t) = \gamma_0 + \gamma_1 t, \quad t > 0, \tag{51}$$

where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1)$ are the linear relationship coefficients. While higher-order polynomial hazard functions could be considered, we maintain a linear assumption for the sake of simplicity and also to reduce complexity in ensuring estimation process. The assumption of a polynomial hazard rate is equivalent to the assumption of polynomial exponential transformation of the lifetime adopted in Krane (1963) and, as pointed out therein, it is suitable for industrial survival data. A reasonable justification is derived from the fact that some hazard functions from known distributions may be expanded in a converging power series. Therefore, the linear form can be seen as first-order Taylor approximation of any underlying general hazard function.

Therefore, the hazard rate function of a device lifetime subjected to constant stress x under PH model is given by

$$h(t, x) = (\gamma_0 + \gamma_1 t) \exp(\theta_1 x), \tag{52}$$

where $\theta_1 \in \mathbb{R}$ is the coefficient of the log-linear relationship in Equation (9) relating the stress level to the lifetimes of devices. The corresponding cumulative hazard at constant stress x is given, respectively, by

$$H_{\boldsymbol{\theta}}(t, x) = \left(\gamma_0 t + \gamma_1 \frac{t^2}{2}\right) \exp(\theta_1 x).$$

Note that the intercept θ_0 is omitted, as it becomes superfluous in Eq. (52) under linear baseline assumption. Its effect can be absorbed within the baseline hazard parameters due to its purely multiplicative role. Therefore, we would need to estimate both relationships coefficients in (52) resulting in a 3-dimensional model parameter $\boldsymbol{\theta} = (\boldsymbol{\gamma}, \theta_1)$. Additionally, the parameter space should satisfy some natural restrictions. Firstly, because the hazard function is positive at any time $t > 0$, both coefficients γ_0 and γ_1 in (51) must be positive. Moreover, under positive and increasing stress levels, the reliability of the devices should decrease for increasing stress levels. So, the regression coefficient in (9), θ_1, should also be positive. That is, assuming increasing and positive stress levels, the parameter space now becomes $\Theta = \mathbb{R}^{+3}$.

Under the cumulative exposure assumption and linear baseline hazards, the shifting time under stress level x_{i+1} can be recursively calculated as the solution of the second-order equation

$$\frac{\gamma_1}{2}(\tau_i+s_{i+1})^2+\gamma_0(\tau_i+s_{i+1})-\exp\left(\theta_1(x_i-x_{i+1})\right)\left(\gamma_0(\tau_i+s_i)+\frac{\gamma_1}{2}(\tau_i+s_i)^2\right)=0. \tag{53}$$

Let us discuss the existence of a real root of the previous equation such that $\tau_i+s_{i+1}>0$ and $s_{i+1}<0$. To simplify the notation, denote $y=\tau_i+s_{i+1}$ and

$$\delta_i = \exp\big(\theta_1(x_i-x_{i+1})\big)\left(\gamma_0(\tau_i+s_i)+\tfrac{\gamma_1}{2}(\tau_i+s_i)^2\right).$$

Then, the discriminant of the second-order equation in $y=\tau_i+s_{i+1}$ is strictly positive under the positivity constraints $\gamma_0>0$ and $\gamma_1>0$. Hence, the quadratic has exactly two real roots; denote them by $y^{(1)}$ and $y^{(2)}$. Moreover, by Vieta's formula,

$$y^{(1)}\times y^{(2)}=-\frac{2}{\gamma_1}\delta_i<0,$$

so one root is positive (say $y^{(1)}>0$) and the other is negative ($y^{(2)}<0$).

Reversing the reparametrization $y=\tau_i+s_{i+1}$ with known $\tau_i>0$, the corresponding solutions in s_{i+1} are

$$s_{i+1}^{(1)}=y^{(1)}-\tau_i,\qquad s_{i+1}^{(2)}=y^{(2)}-\tau_i,$$

so we immediately have $s_{i+1}^{(2)}<0$. Now, applying the same reasoning to the quadratic written directly in s_{i+1}, the roots product is

$$s_{i+1}^{(1)}\times s_{i+1}^{(2)}=\frac{-2}{\gamma_1}(\delta_i-\frac{\gamma_1}{2}\tau_i^2-\gamma_0\tau_i).$$

A straightforward calculation shows that $\delta_i-\frac{\gamma_1}{2}\tau_i^2-\gamma_0\tau_i<0$ (since $\tau_i+s_i<\tau_i$), and therefore both roots are negative. Hence, we will choose the greater negative solution of the above equation to ensure that $s_{i+1}<0$ and $\tau_i+s_{i+1}>0$.

The probability of failure within an interval $[IT_{j-1}, IT_j)$ where devices are tested under stress level x_i can be determined by evaluating the reliability of the lifetime as

$$\begin{aligned}\pi_j(\boldsymbol{\theta}) &= R_{\boldsymbol{\theta}}(IT_{j-1})-R_{\boldsymbol{\theta}}(IT_j),\\ &=\exp\left(-H_{\boldsymbol{\theta}}(IT_{j-1})\right)-\exp\left(-H_{\boldsymbol{\theta}}(IT_j)\right),\\ &=\exp\left(-\exp(\theta_1x_i)\left(\gamma_0(IT_{j-1}+s_i)+\frac{\gamma_1}{2}(IT_{j-1}+s_i)^2\right)\right)\\ &\quad-\exp\left(-\exp(\theta_1x_i)\left(\gamma_0(IT_j+s_i)+\frac{\gamma_1}{2}(IT_j+s_i)^2\right)\right),\end{aligned} \tag{54}$$

and the probability of survival at termination time IT_L is given by

$$\begin{aligned}\pi_{L+1}(\boldsymbol{\theta}) &= R_{\boldsymbol{\theta}}(IT_L) = \exp\left(-H_{\boldsymbol{\theta}}(t_L)\right)\\ &= \exp\left(-\exp\left(\theta_1 x_k\right)\left(\gamma_0(IT_L + s_k) + \frac{\gamma_1}{2}(IT_L + s_k)^2\right)\right).\end{aligned}$$

We discuss the asymptotic properties of the MDPDEs for interval-monitored step-stress ALTs under linear baseline PH. For mathematical simplicity, we will derive the formula for the simple step-stress model, but expressions for the multiple step-tress case can be derived in a very similar manner. For more details, please refer to Balakrishnan et al. 2022b.

Theorem 8 *For step-stress ALT model under interval censoring and proportional hazards, matrices $\mathbf{W}(\boldsymbol{\theta})$ and $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ required for the estimating equations and asymptotic distribution stated in Theorems 1 and 2 are as follows:*

- *The matrix $\mathbf{W}(\boldsymbol{\theta})$ is the $4 \times (L+1)$ Jacobian matrix of the theoretical probability of failure $\boldsymbol{\pi}(\boldsymbol{\theta})$ as defined in (54), with columns given by*

$$\boldsymbol{w}_j = \begin{cases} \frac{\partial R_{\boldsymbol{\theta}}(IT_{j-1}, x_1)}{\partial \boldsymbol{\theta}} - \frac{\partial R_{\boldsymbol{\theta}}(IT_j, x_1)}{\partial \boldsymbol{\theta}}, & IT_{j-1} < \tau_1, \\ \frac{\partial R_{\boldsymbol{\theta}}(IT_{j-1}, x_2)}{\partial \boldsymbol{\theta}} - \frac{\partial R_{\boldsymbol{\theta}}(IT_j, x_2)}{\partial \boldsymbol{\theta}}, & \tau_1 \leq IT_{j-1}, \end{cases}$$

where the four components of the gradient vector of the reliability function,

$$\frac{\partial R_{\boldsymbol{\theta}}(t,x)}{\partial \boldsymbol{\theta}} = \left(\frac{\partial R_{\boldsymbol{\theta}}(t,x)}{\partial \gamma_0}, \frac{\partial R_{\boldsymbol{\theta}}(t,x)}{\partial \gamma_1}, \frac{\partial R_{\boldsymbol{\theta}}(t,x)}{\partial \theta_1}\right)^T$$

are given, under the first stress level x_1, by

$$\begin{aligned}\frac{\partial R_{\boldsymbol{\theta}}(t,x_1)}{\partial \gamma_0} &= -R_{\boldsymbol{\theta}}(t,x_1)\exp(\theta_1 x_1) t,\\ \frac{\partial R_{\boldsymbol{\theta}}(t,x_1)}{\partial \gamma_1} &= -R_{\boldsymbol{\theta}}(t,x_1)\exp(\theta_1 x_1)\frac{t^2}{2},\\ \frac{\partial R_{\boldsymbol{\theta}}(t,x_1)}{\partial \theta_1} &= -R_{\boldsymbol{\theta}}(t,x_1)\exp(\theta_1 x_1) t k_2(t) x_2,\end{aligned}$$

and under the increased stress level x_2 by

$$
\begin{aligned}
\frac{\partial R_{\boldsymbol{\theta}}(t, x_2)}{\partial \gamma_0} &= -R_{\boldsymbol{\theta}}(t, x_2)\exp(\theta_1 x_2)\left((t_1 + s_1) + \frac{\gamma_1}{2}\frac{(\tau_1 + s_1)s}{k_2(\tau_1)}\frac{k_1(t + s_1)}{k_1(\tau_1 + s_1)}\right),\\
\frac{\partial R_{\boldsymbol{\theta}}(t, x_2)}{\partial \gamma_1} &= -R_{\boldsymbol{\theta}}(t, x_2)\exp(\theta_1 x_2)\left(\frac{(t + s_1)^2}{2} - \frac{\gamma_0}{2}\frac{(\tau_1 + s_1)s}{k_2(\tau_1)}\frac{k_1(t + s_1)}{k_1(\tau_1 + s_1)}\right),\\
\frac{\partial R_{\boldsymbol{\theta}}(t, x_2)}{\partial \theta_1} &= -R_{\boldsymbol{\theta}}(t, x_2)\exp(\theta_1 x_2)\\
&\quad\times\left[(t + s_1)k_2(t + s_1)x_2 + \frac{k_1(t + s_1)}{k_1(\tau_1 + s_1)}(\tau_1 + s_1)k_2(\tau_1 + s_1)(x_1 - x_2)\right],
\end{aligned}
$$

and the functions $k_1(t)$ and $k_2(t)$ are

$$
k_1(t) = \gamma_1 t + \gamma_0 \qquad \textit{and} \qquad k_2(t) = \frac{\gamma_1}{2}t + \gamma_0;
$$

- *The matrix $\boldsymbol{D}_{\boldsymbol{\pi}(\boldsymbol{\theta})}$ denotes a $(L + 1) \times (L + 1)$ diagonal matrix with diagonal entries $\pi_j(\boldsymbol{\theta})$, $j = 1, ..., L + 1$, as defined in (54).*

11 Model selection and validation

So far, we have described robust methods for estimating reliability under parametric lifetime assumptions, as well as a semi-parametric approach based on the PH assumption. Different parametric distributions may be suitable for modeling a product's lifetime, as they effectively capture its behavior and natural degradation characteristics. Each distribution has unique properties and features, making it more or less appropriate depending on the specific product being analyzed. Assessing whether the assumed distribution is appropriate for the data is essential to ensure a fair and accurate analysis. This step ensures that the model generalizes effectively beyond the dataset used for its development, which is crucial for making reliable predictions in real-world scenarios. On the other hand, since different distribution families may fit the observed data well, selecting the most suitable model becomes equally important. A proper selection ensures that the analysis aligns with the underlying data structure and adequately addresses the problem at hand. Therefore, the process of model selection and validation involves conducting goodness-of-fit tests, evaluating model selection criteria, and addressing the effect of potential model misspecifications. These steps are fundamental in statistical modeling as they ensure the reliability, interpretability, and predictive accuracy of the chosen model. In this regard, few advances have been made in the literature, either from maximum likelihood or from robust inferential perspective. In the following, we highlight some key open research problems related to model selection and validation in the present context.

Open Problem 8 (Goodness-of-fit Test)
Balakrishnan and Chimitova (2017) detailed three goodness-of-fit test statistics, including the Cramer-von Mises-Smirnov-type statistic, Chi-square-type statistic, and White's statistic for one-shot device testing data collected from constant-stress ALTs,

and Balakrishnan et al. (2021) proposed a bootstrap procedure for the same form of extremely-censored data. Those tests were based on binomial data observed from the ALT experiments. Similar methods could be developed for step-stress ALT under interval-censoring, where the data are also discretized into counts of failures. Additionally, these tests were based on MLEs, but robust tests using the MDPDE could also be considered in this context.

Open Problem 9 (Model Selection) *Several potentially reasonable lifetime distributions can be fitted to lifetime data and a suitable model selection criterion must be used to select the best model for the observed data. In addition to leveraging expert knowledge, the utilization of model selection criteria becomes essential for accurate analysis.*

Several studies have focused on discriminating between Gamma and Weibull models when fitting lifetime data. Relevant discussions along these lines can be found in Bain and Engelhardt (1980), Fearn and Nebenzahl (1991), and Balasooriya and Abeysinghe (1994).

Gamma and Weibull distributions are flexible enough to model lifetimes with increasing, decreasing, and constant failure rates. Both of them include the exponential distribution as a special case. Therefore, choosing the best distribution family is not trivial in many practical situations.

Chapter 4 in Balakrishnan et al. (2021) reviews to model misspecification analysis and model selection in the context of one shot-devices, employing a likelihood-based approach. In particular, these authors proposed the use of AIC and distance-based criteria to select the best parametric family for a given data. Other model selection methods, such as BIC could also be used.

However, the AIC, BIC and distance-based model selection methods are based on likelihood inference, which lacks robustness. Felipe et al. (2023) proposed a model selection criteria based on divergence measures in general statistical models. One could therefore consider the MDPDE, a natural extension of MLE, for model selection between different distribution families, such as Weibull and Gamma, but also more general lifetime distributions as lognormal or Birnbaum-Saunders distributions. This will be an interesting research direction for the future.

Open Problem 10 (Model Misspecification)

Various lifetime distributions, such as exponential, Weibull, Gamma, and Lognormal, have been extensively studied and discussed in the literature. However, assuming a specific parametric family without thorough validation can lead to model misspecification.

This issue is far from trivial, as model misspecification can significantly distort inferential results regarding key lifetime characteristics. Such inaccuracies can, in turn, adversely affect the planning and outcomes of ALT strategies. Selecting a statistical model that does not adequately represent the underlying data generation process can result in biased estimates, incorrect inferences, and potentially misleading conclusions.

Given the widespread application of the exponential, Weibull, Gamma, and Lognormal distributions in reliability and survival analysis, addressing model misspecification within these prominent families is particularly important in practice. For instance, Balakrishnan et al. (2022a) explored model misspecification and the determination of optimal constant ALT strategies in the context of extremely censored one-shot device testing, assuming Gamma, Weibull, Lognormal, and Birnbaum-Saunders lifetime distributions.

A still open problem is examining the impact of model misspecification across these families of lifetime distributions in the context of step-stress ALT under interval-censoring frameworks. This too will be a very useful future research.

12 Analyses of real data

We present several reliability analyses using a real-data example provided in Section 3 to illustrate the application of robust methods across various lifetime distributions. We use the Electronic Components dataset described earlier in 3 As we have explicitly derived the MDPDE for four parametric families-namely, exponential, Weibull, gamma, and lognormal-we analyze this example assuming each parametric family and later address their estimated characteristics and model adequacy. It is important to note that many of the datasets include a limited number of devices under test, which is common in industrial experiments. Nevertheless, these examples provide valuable insights into the behavior of inferential methods, emphasizing the key advantages and limitations of the approaches employed. We present the estimates of the corresponding distribution parameters, along with the estimated mean time to failure (MTTF), median, and reliability at a fixed mission time. Additionally, we provide CIs and standard errors for these estimates. For the lifetime characteristics, we also include transformed confidence intervals to enhance interpretability and accuracy.

Besides fitting data under the assumption of a specific lifetime distribution, we conduct a chi-square goodness-of-fit test based on the MLE and the observed multinomial sample. The resulting p-values from the test provide insight into how well each parametric family fits the data. It is worth noting that the choice of the assumed family may be guided by the nature of the product (expert knowledge) or informed by observed behavioral patterns.

12.1 Reliability analysis

This dataset includes failure data from a total of $N = 100$ units tested under a simple step-stress ALT with stress levels $x_1 = 100°C$ and $x_2 = 150°C$, switched at $\tau_1 = 910$. To evaluate the performance of the estimators under interval monitoring, we transformed the exact failure times into counts of failures between the following inspection times:

$$IT = 227.5, 455, 682.5, 910, 936.57, 963.14,$$

Table 7 Point estimates, standard errors (se), and confidence intervals of the log-linear relationship parameter (θ_0, θ_1) for a grid of tuning parameter values. The MDPDE with $\beta = 0$ corresponds to the MLE. p-values of chi-square goodness of fit tests are also presented

β	θ_0 (se)	IC(θ_0)	θ_1 (se)	IC(θ_1)	p-value
Exponential					
0	10.86 (0.71)	[9.47, 12.24]	-3.02 (0.58)	[-4.16, -1.89]	0.67
0.2	10.85 (0.71)	[9.46, 12.23]	-3.01 (0.58)	[-4.15, -1.88]	
0.4	10.84 (0.71)	[9.45, 12.23]	-3.01 (0.58)	[-4.14, -1.87]	
0.6	10.83 (0.71)	[9.44, 12.22]	-3.00 (0.58)	[-4.14, -1.86]	
0.8	10.81 (0.71)	[9.42, 12.21]	-2.99 (0.58)	[-4.13, -1.85]	
1	10.80 (0.71)	[9.41, 12.20]	-2.98 (0.59)	[-4.13, -1.83]	
Weibull					
0	10.81 (1.07)	[8.72, 12.90]	-2.99 (0.82)	[-4.59, -1.39]	0.67
0.2	10.79 (1.07)	[8.70, 12.88]	-2.98 (0.82)	[-4.58, -1.37]	
0.4	10.77 (1.06)	[8.69, 12.86]	-2.96 (0.82)	[-4.56, -1.36]	
0.6	10.75 (1.06)	[8.67, 12.83]	-2.94 (0.82)	[-4.54, -1.34]	
0.8	10.72 (1.06)	[8.65, 12.80]	-2.92 (0.82)	[-4.52, -1.32]	
1	10.70 (1.06)	[8.62, 12.78]	-2.90 (0.82)	[-4.50, -1.30]	
Gamma					
0	10.73 (1.21)	[8.37, 13.10]	-2.96 (0.76)	[-4.45, -1.47]	0.68
0.2	10.71 (1.20)	[8.36, 13.07]	-2.95 (0.76)	[-4.44, -1.45]	
0.4	10.69 (1.20)	[8.33, 13.04]	-2.93 (0.76)	[-4.42, -1.44]	
0.6	10.66 (1.20)	[8.31, 13.01]	-2.91 (0.76)	[-4.40, -1.42]	
0.8	10.63 (1.20)	[8.29, 12.98]	-2.89 (0.76)	[-4.38, -1.40]	
1	10.60 (1.20)	[8.26, 12.95]	-2.87 (0.76)	[-4.36, -1.38]	
Weibull					
0	11.06 (1.05)	[9.00, 13.11]	-3.47 (0.83)	[-5.10, -1.85]	0.80
0.2	11.03 (1.05)	[8.98, 13.08]	-3.45 (0.83)	[-5.07, -1.83]	
0.4	10.99 (1.04)	[8.95, 13.04]	-3.42 (0.83)	[-5.05, -1.80]	
0.6	10.96 (1.04)	[8.91, 13.00]	-3.39 (0.83)	[-5.02, -1.77]	
0.8	10.91 (1.04)	[8.87, 12.96]	-3.36 (0.83)	[-4.99, -1.73]	
1	10.87 (1.05)	[8.82, 12.92]	-3.32 (0.83)	[-4.96, -1.69]	

$$989.71, 1016.3, 1042.88, 1069.43, 1096.$$

Table 7 presents the estimates of the log-linear relationship for the four different lifetime distributions. While each parameter represents distinct lifetime characteristics depending on the distribution, they share some similarities. Notably, if the third parameter of the Weibull or gamma distributions equals 1 (a plausible scenario as we will discuss later), both distributions reduce to the exponential, resulting in common parameter interpretations. On the other hand, the scale parameter defined by the log-linear relationship in the lognormal case corresponds to the median of the distribution, akin to the exponential distribution. Importantly, the estimates across the

Table 8 Point estimates, standard errors (se), and confidence intervals of the third parameter θ_2 of each distribution for a grid of tuning parameter values. The MDPDE with $\beta = 0$ corresponds to the MLE

β	θ_2 (se)	IC(θ_2)
Weibull		
0	1.01 (0.2064)	[0.61, 1.42]
0.2	1.01 (0.2067)	[0.61, 1.42]
0.4	1.02 (0.2071)	[0.61, 1.42]
0.6	1.02 (0.2076)	[0.611, 1.43]
0.8	1.02 (0.2082)	[0.61, 1.43]
1	1.02 (0.2087)	[0.61, 1.43]
Gamma		
0	1.03 (0.26)	[0.52, 1.55]
0.2	1.04 (0.26)	[0.52, 1.55]
0.4	1.04 (0.26)	[0.52, 1.55]
0.6	1.04 (0.27)	[0.52, 1.56]
0.8	1.04 (0.27)	[0.52, 1.56]
1	1.05 (0.27)	[0.53, 1.57]
Lognormal		
0	1.53 (0.28)	[0.98, 2.07]
0.2	1.52 (0.28)	[0.98, 2.07]
0.4	1.52 (0.28)	[0.98, 2.06]
0.6	1.52 (0.27)	[0.98, 2.06]
0.8	1.52 (0.27)	[0.98, 2.05]
1	1.51 (0.27)	[0.98, 2.05]

distributions are remarkably similar, both in terms of point estimates and estimated standard errors. Naturally, the inherent differences among the distributions lead to variations in asymptotic variances, as extensively discussed earlier. This consistency among estimators suggests two key points: first, the absence of significant outlying observations, as both robust and non-robust estimators perform similarly; and secondly, the appealing performance of the MDPDE even in uncontaminated datasets. Moreover, the high p-values observed across all distributions imply that any of the four could potentially fit the data. Thus, the selection of the appropriate model should rely on a model selection criterion, still an open problem as mentioned in Section 11. It is important to emphasize that this choice will significantly influence the estimates of lifetime characteristics of interest.

On the other hand, table 8 reports the estimated values of the shape parameter for the Weibull, gamma, and lognormal distributions, each with a distinct interpretation depending on the distribution. Notably, for the Weibull and gamma distributions, the point estimates of the shape parameter are approximately 1, indicating behavior closely aligned with the exponential distribution. This result suggests that the simpler exponential distribution may be sufficient to adequately fit the observed data.

Figure 3 displays pp-plots for the four parametric families, namely exponential, gamma, lognormal, and Weibull, using the fitted c.d.f.'s under step-stress conditions. A pp-plot compares the empirical cumulative probabilities of the observed data with

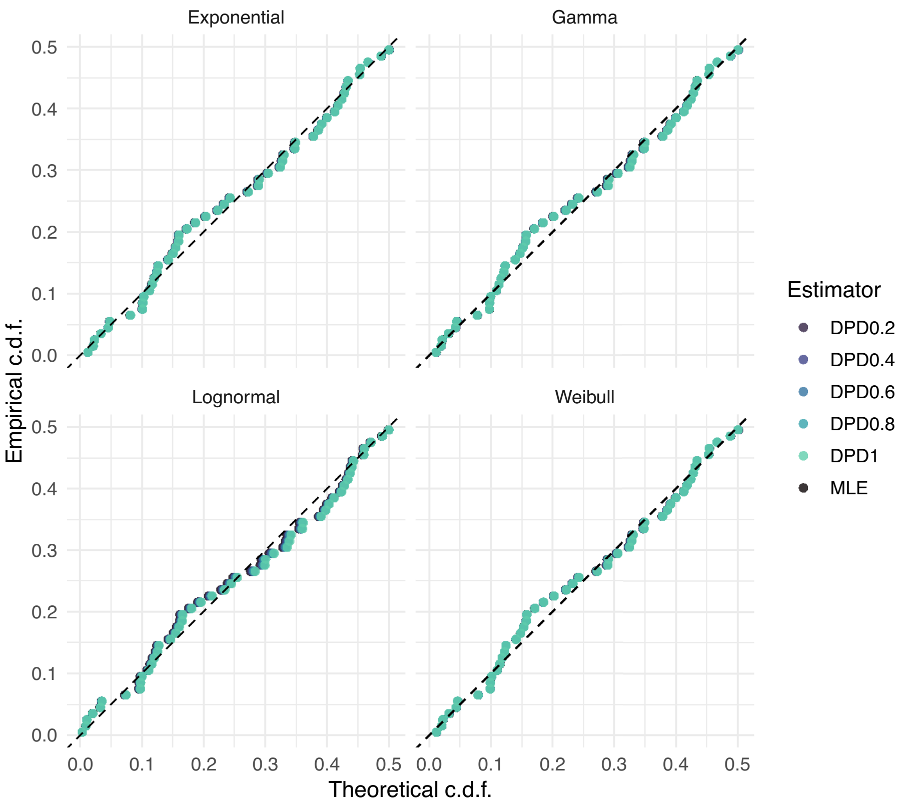


Fig. 3 PP-plots under the four parametric families

those implied by the fitted model at the same probability levels. Points lying near the diagonal reference line indicate good overall agreement. Results are provided for both MLE and DPD-based estimators; the close overlap of the points across estimators indicates that (possible) outliers have little impact, providing no evidence of influential outlying observations. In this setting, the exponential distribution provides the best overall fit.

As previously mentioned, the choice of distribution significantly impacts the point estimation of lifetime characteristics, especially when extrapolating results to normal operating conditions. In this experiment, where temperature is the stress factor, normal operating conditions correspond to a temperature of $x_0 = 25°$. Tables 9, 10 and 11 present the estimates, standard error and CI of the MTTF (in hours), median (in hours), and reliability at $t = 5.55$ hours, respectively, for devices operating at x_0 obtained based on different MDPDE. Graphical representation of the same quantities are presented on Figure 4.

Because the estimates of the shape parameter of the Weibull and gamma distributions is approximately equal to 1, the exponential, Weibull, and gamma models

Table 9 Estimated MTTF based on the MDPDE with different values of β under the four distributions

β	MTTF Estimate (se)	CI	Transformed CI
Exponential			
0.0	6.76 (3.83)	[0.00, 14.27]	[2.23, 20.50]
0.2	6.71 (3.80)	[0.00, 14.15]	[2.21, 20.34]
0.4	6.67 (3.78)	[0.00, 14.07]	[2.20, 20.24]
0.6	6.61 (3.75)	[0.00, 13.96]	[2.17, 20.10]
0.8	6.54 (3.72)	[0.00, 13.84]	[2.15, 19.96]
1.0	6.49 (3.71)	[0.00, 13.77]	[2.12, 19.90]
Weibull			
0.0	6.48 (6.03)	[0.00, 18.30]	[1.05, 40.16]
0.2	6.38 (5.92)	[0.00, 17.99]	[1.03, 39.37]
0.4	6.27 (5.81)	[0.00, 17.66]	[1.02, 38.55]
0.6	6.16 (5.69)	[0.00, 17.31]	[1.01, 37.70]
0.8	6.03 (5.56)	[0.00, 16.93]	[0.99, 36.78]
1.0	5.91 (5.44)	[0.00, 16.57]	[0.97, 35.94]
Gamma			
0.0	6.27 (5.15)	[0.00, 16.37]	[1.25, 31.37]
0.2	6.19 (5.08)	[0.00, 16.14]	[1.24, 30.88]
0.4	6.07 (4.97)	[0.00, 15.81]	[1.22, 30.17]
0.6	5.95 (4.86)	[0.00, 15.47]	[1.20, 29.46]
0.8	5.84 (4.76)	[0.00, 15.16]	[1.18, 28.83]
1.0	5.72 (4.65)	[0.00, 14.83]	[1.16, 28.14]
Lognormal			
0.0	23.65 (28.38)	[0.00, 79.26]	[2.25, 248.41]
0.2	23.09 (27.62)	[0.00, 77.22]	[2.21, 240.81]
0.4	22.38 (26.69)	[0.00, 74.70]	[2.16, 231.80]
0.6	21.61 (25.70)	[0.00, 71.99]	[2.10, 222.33]
0.8	20.77 (24.63)	[0.00, 69.05]	[2.03, 212.38]
1.0	19.89 (23.56)	[0.00, 66.07]	[1.95, 202.78]

perform similarly, yielding comparable estimates for the three characteristics. However, the additional complexity introduced by the Weibull and gamma assumptions, compared to the exponential distribution, is reflected in the larger standard errors of their estimates.

Now, let us examine the key differences when assuming a lognormal distribution. First, note that the estimated medians are quite similar because both the exponential and lognormal distributions use their scale parameter as the median. However, the estimation of the MTTF differs significantly, underscoring the critical impact of the chosen distribution on lifetime characteristics. Finally, it is worth noting that the transformed confidence intervals yield narrower intervals for all characteristics, highlighting the advantage of using this transformation based approach.

Table 10 Estimated median of the lifetime based on the MDPDE with different values of β under the four distributions

β	Median estimate (se)	CI	Transformed CI
Exponential			
0	6.76 (3.83)	[0.00, 14.27]	[2.23, 20.50]
0.2	6.71 (3.80)	[0.00, 14.15]	[2.21, 20.34]
0.4	6.67 (3.78)	[0.00, 14.07]	[2.20, 20.24]
0.6	6.61 (3.75)	[0.00, 13.96]	[2.17, 20.10]
0.8	6.54 (3.72)	[0.00, 13.84]	[2.15, 19.96]
1.0	6.49 (3.71)	[0.00, 13.77]	[2.12, 19.90]
Weibull			
0.0	4.53 (3.68)	[0.00, 11.74]	[0.93, 22.21]
0.2	4.47 (3.62)	[0.00, 11.56]	[0.92, 21.84]
0.4	4.40 (3.56)	[0.00, 11.37]	[0.90, 21.45]
0.6	4.33 (3.49)	[0.00, 11.17]	[0.89, 21.04]
0.8	4.25 (3.42)	[0.00, 10.95]	[0.88, 20.59]
1.0	4.17 (3.35)	[0.00, 10.74]	[0.86, 20.17]
Gamma			
0.0	4.40 (4.50)	[0.00, 13.22]	[0.59, 32.66]
0.2	4.35 (4.44)	[0.00, 13.05]	[0.59, 32.18]
0.4	4.27 (4.35)	[0.00, 12.79]	[0.58, 31.44]
0.6	4.19 (4.26)	[0.00, 12.54]	[0.57, 30.74]
0.8	4.12 (4.18)	[0.00, 12.31]	[0.56, 30.10]
1.0	4.03 (4.09)	[0.00, 12.05]	[0.55, 29.41]
Lognormal			
0	7.39 (6.25)	[0.00, 19.65]	[1.41, 38.80]
0.2	7.23 (6.10)	[0.00, 19.18]	[1.38, 37.78]
0.4	7.02 (5.92)	[0.00, 18.62]	[1.35, 36.59]
0.6	6.81 (5.73)	[0.00, 18.04]	[1.31, 35.40]
0.8	6.59 (5.54)	[0.00, 17.43]	[1.27, 34.20]
1.0	6.35 (5.34)	[0.00, 16.82]	[1.22, 33.04]

Notably, this large difference becomes apparent when extrapolating to normal operating conditions. Although all distributions fit the observed step-stress data well, their inherent characteristics emerge when the results are extrapolated.

12.2 Stability in the estimation

We finally examine the potential impact on estimation when a group of devices fails simultaneously due to an external cause, resulting in an outlying count of failures. To explore this effect, we designate the fourth inspection to be the interval with a higher-than-expected number of failed units. Note that we have selected an inspection interval

Table 11 Estimated reliability at $t = 5.55$hours of the lifetime based on the MDPDE with different values of β under the four distributions

β	Reliability estimate (se)	CI	Transformed CI
Exponential			
0.0	0.44 (0.20)	[0.04, 0.84]	[0.13, 0.80]
0.2	0.44 (0.20)	[0.04, 0.84]	[0.13, 0.80]
0.4	0.43 (0.21)	[0.03, 0.84]	[0.13, 0.80]
0.6	0.43 (0.21)	[0.03, 0.83]	[0.13, 0.80]
0.8	0.43 (0.21)	[0.02, 0.83]	[0.13, 0.80]
1.0	0.43 (0.21)	[0.02, 0.83]	[0.12, 0.80]
Weibull			
0.0	0.43 (0.55)	[0.00, 1.00]	[0.01, 0.98]
0.2	0.42 (0.55)	[0.00, 1.00]	[0.01, 0.98]
0.4	0.42 (0.56)	[0.00, 1.00]	[0.01, 0.98]
0.6	0.41 (0.56)	[0.00, 1.00]	[0.01, 0.98]
0.8	0.40 (0.56)	[0.00, 1.00]	[0.01, 0.98]
1.0	0.39 (0.56)	[0.00, 1.00]	[0.01, 0.98]
Gamma			
0.0	0.42 (0.38)	[0.00, 1.00]	[0.03, 0.94]
0.2	0.41 (0.38)	[0.00, 1.00]	[0.03, 0.94]
0.4	0.40 (0.38)	[0.00, 1.00]	[0.03, 0.94]
0.6	0.40 (0.38)	[0.00, 1.00]	[0.03, 0.94]
0.8	0.39 (0.38)	[0.00, 1.00]	[0.03, 0.94]
1.0	0.38 (0.38)	[0.00, 1.00]	[0.03, 0.94]
Lognormal			
0.0	0.57 (0.21)	[0.17, 0.98]	[0.20, 0.88]
0.2	0.57 (0.21)	[0.16, 0.98]	[0.20, 0.87]
0.4	0.56 (0.21)	[0.15, 0.97]	[0.19, 0.87]
0.6	0.55 (0.21)	[0.14, 0.97]	[0.19, 0.87]
0.8	0.54 (0.21)	[0.13, 0.96]	[0.18, 0.87]
1.0	0.54 (0.22)	[0.11, 0.96]	[0.17, 0.86]

with relatively few failures observed in the uncontaminated sample. As a result, we introduce an outlier in the fourth count and evaluate its impact on the estimation of the MTTF, a key measure of interest in many reliability analysis. Such scenarios are commonly encountered in real experiments, where external factors to the experiment may result in abnormal group failures.

To simulate this scenario, we artificially increase the number of failures in the fourth inspection by $\epsilon\%$ of the 50 surviving units. Two key points are worth noting: first, the proportions in the other inspection intervals remain unchanged, thereby preserving the original model. Second, the magnitude of the change in estimation should increase with the severity of the outlier. Figure 5 depicts the changes in the MTTF estimates

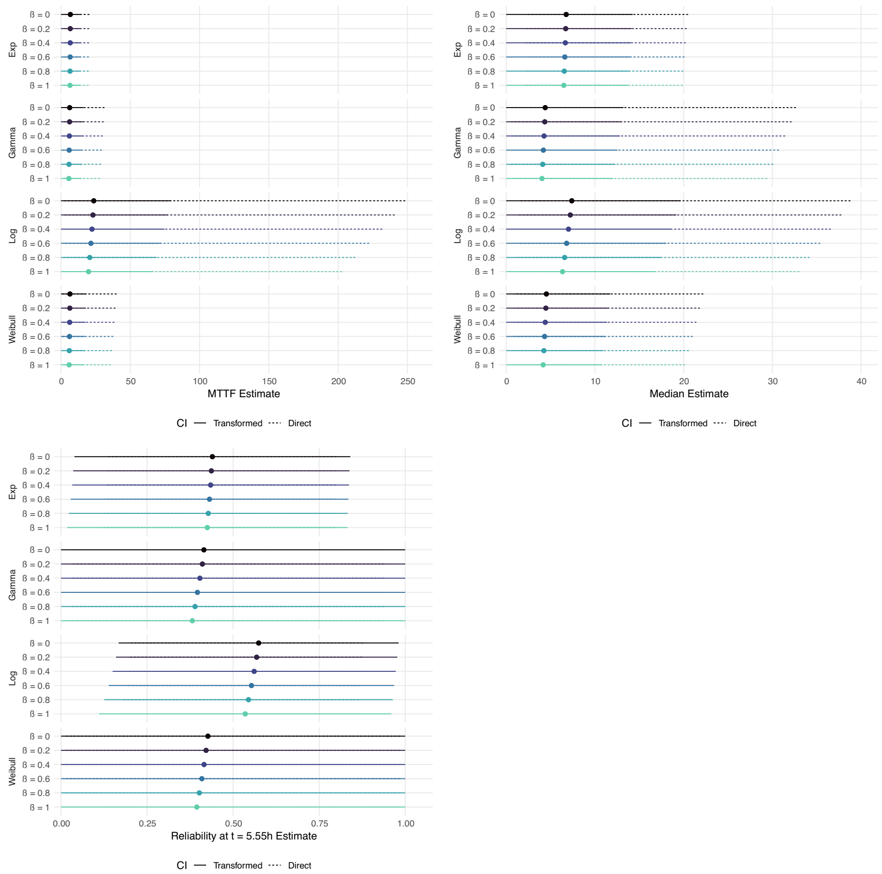
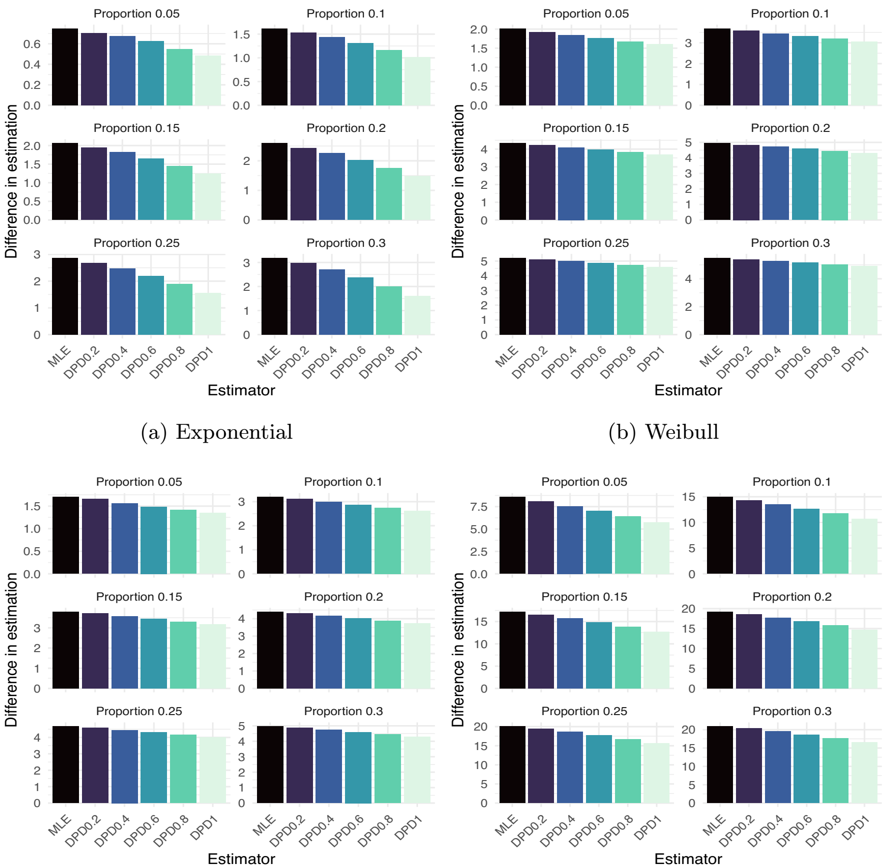


Fig. 4 Graphical comparison of lifetime characteristics estimates under the four parametric families

obtained using various MDPDEs, including the MLE, across the four parametric families (exponential, Weibull, gamma, and lognormal) as the proportion of outliers in the 4th inspection increases. It can be observed that, the MDPDE with larger values of the tuning parameter β demonstrate greater stability, being less affected by a single outlier. Furthermore, differences between the parametric families are evident: the changes are smaller for the exponential distribution, while they are more pronounced for the lognormal distribution.



(a) Exponential (b) Weibull (c) Gamma (d) Lognormal

Fig. 5 Absolute differences in the estimated MTTF across varying MDPDE values for the four parametric families of lifetime distributions

13 Concluding remarks

This paper elaborated in detail robust inferential methods for step-stress ALT under interval-censoring. In particular, explicit expressions of robust point estimates and approximate confidence intervals for four important distributions have been provided. Some important research gaps have been highlighted, including the derivation of robust estimates for more general families of distributions, the generalization of the competing risks model to other underlying distributions, as well as the derivation of model selection and validation criteria.

Our analysis demonstrates the importance of robust estimation techniques, particularly in handling outliers and contaminated datasets, which often arise in real-world reliability experiments. The MDPDE with moderate values of the tuning parameter shows enhanced stability compared to the classical MLE against outlying data, while keeping a similar performance in the absence of contamination, making it a valuable tool for reliability practitioners. Moreover, the analysis of confidence intervals further highlights the advantages of transformed intervals, which provide narrower bounds and greater interpretability, reinforcing their usefulness in reliability studies. The absence of notable differences between robust estimators and the MLE in the absence of contamination validates the good performance of the MDPDE while highlighting its resistance to outliers. Another key finding reveals that the choice of distribution has a significant impact on the estimation of lifetime characteristics, especially when extrapolating the results to normal operating conditions. This underscores the critical need for careful model selection, guided by both theoretical criteria and empirical goodness-of-fit tests.

Lastly, by listing some main open problems in the ALT literature, this paper aims to inspire further research in robust inference for step-stress ALT. Future studies should focus on extending these methods to more complex experimental designs and exploring their applications to real-world scenarios involving highly reliable devices. It is our sincere hope that review paper will encourage and facilitate much further research in this important and challenging area.

Appendix A Proof of Theorem 2

Proof Following Basu et al. (1997) and Balakrishnan et al. (2023a), the matrices $\boldsymbol{J}_\beta(\boldsymbol{\theta}_0)$ and $\boldsymbol{K}_\beta(\boldsymbol{\theta}_0)$ are given by

$$\boldsymbol{J}_\beta(\boldsymbol{\theta}_0) = \sum_{j=1}^{L+1} \boldsymbol{u}_j \boldsymbol{u}_j^T \pi_j(\boldsymbol{\theta})^{\beta+1},$$

$$\boldsymbol{K}_\beta(\boldsymbol{\theta}_0) = \sum_{j=1}^{L+1} \boldsymbol{u}_j \boldsymbol{u}_j^T \pi_j(\boldsymbol{\theta}_0)^{2\beta+1} - \left(\sum_{j=1}^{L+1} \boldsymbol{u}_j \pi_j(\boldsymbol{\theta}_0)^{\beta+1}\right)\left(\sum_{j=1}^{L+1} \boldsymbol{u}_j \pi_j(\boldsymbol{\theta}_0)^{\beta+1}\right)^T,$$

where

$$\boldsymbol{u}_j(\boldsymbol{\theta}) = \frac{\partial \log(\pi_j(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}} = \frac{1}{\pi_j(\boldsymbol{\theta})}\frac{\partial \pi_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\boldsymbol{w}_j}{\pi_j(\boldsymbol{\theta})}.$$

Hence, we can write

$$\boldsymbol{J}_\beta(\boldsymbol{\theta}_0) = \sum_{j=1}^{L+1} \boldsymbol{w}_j \boldsymbol{w}_j^T \pi_j(\boldsymbol{\theta}_0)^{\beta-1} = \boldsymbol{W}^T D_{\boldsymbol{\pi}(\boldsymbol{\theta}_0)}^{\beta-1} \boldsymbol{W},$$

$$
\begin{aligned}
\boldsymbol{K}_{\beta}(\boldsymbol{\theta}_0) &= \sum_{j=1}^{L+1} \boldsymbol{w}_j \boldsymbol{w}_j^T \pi_j(\boldsymbol{\theta}_0)^{2\beta-1} - \left(\sum_{j=1}^{L+1} \boldsymbol{w}_j \pi_j(\boldsymbol{\theta}_0)^{\beta} \right) \left(\sum_{j=1}^{L+1} \boldsymbol{w}_j \pi_j(\boldsymbol{\theta}_0)^{\beta} \right)^T \\
&= \boldsymbol{W}^T \left(D_{\boldsymbol{\pi}(\boldsymbol{\theta}_0)}^{2\beta-1} - \boldsymbol{\pi}(\boldsymbol{\theta}_0)^{\beta} \boldsymbol{\pi}(\boldsymbol{\theta}_0)^{\beta T} \right) \boldsymbol{W},
\end{aligned}
$$

as required. $\square$

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Code Availability Code is publicly available at Github.

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