Journal für Mathematik-Didaktik (2025)46:9
https://doi.org/10.1007/s13138-025-00256-w

ORIGINALARBEIT/ORIGINAL ARTICLE

Children's Explorative Mathematical Argumentation: the Role of Gesture

Empirical Insights in 5- to 6-year-olds Use of Language, Gesture and Material-based Action in a Guided Argumentation Process

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Received: 2 May 2023 / Accepted: 29 November 2024

Abstract This paper reports on how a group of German preschool children involving in mathematical argumentation use gestures in the course of a guided learning situation. Drawing on the concept of *explorative mathematical argumentation (EMA)* (Reuter 2023) and Walkington et al.'s categorization of collaborative gestures (Walkington et al. 2019a), a case study of young children's mathematical argumentation about geometrical objects is presented.

The focal point of this paper will be the collaborative exchange of gestures between a preschool child and an adult learning guide during a process of geometrical argumentation and conceptualization. The data shows that dynamic gestures, passed on as collaborative gestures between learners and learning guides, can support preschool children's early explorative mathematical argumentation processes in terms of argumentative as well as mathematical competence. The findings suggest that gestures should be applied and observed consciously in mathematical learning environments for young children, especially with respect to argumentation processes.

Keywords Early mathematics education · Argumentation · Use of gesture · Preschool children · Geometry

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Published online: 28 April 2025

Frühes exploratives mathematisches Argumentieren: Die Rolle von Gesten

Empirische Einblicke in den Gebrauch von Sprache, Gestik und materialbasiertem Handeln bei 5- bis 6-jährigen Kindern in einem angeleiteten Argumentationsprozess

Zusammenfassung In diesem Artikel wird gezeigt, wie Kinder im letzten Kita-Jahr im Verlauf einer Situation mathematischen Argumentierens Gesten einsetzen. Die vorgestellte Fallstudie basiert auf dem Konzept des explorativen mathematischen Argumentierens (EMA) (Reuter 2023) und der Kategorisierung kollaborativer Gesten nach Walkington et al. (2019a).

Im Fokus des Artikels steht der kollaborative Austausch von Gesten zwischen einem Kind im Vorschulalter und einer erwachsenen Lernbegleitung im Rahmen einer Situation der geometrischen Argumentation und Begriffsbildung. Die Ergebnisse zeigen, dass eine kollaborative Nutzung von Gesten in Prozessen explorativen mathematischen Argumentierens sowohl argumentative als auch mathematisch-inhaltliche Kompetenzen fördern kann. Die Ergebnisse unterstützen Forderungen nach einem bewussten Einsetzen und Beobachten von Gesten in elementardidaktischen mathematischen Lernsituationen, insbesondere in Bezug auf Argumentationsprozesse.

Schlüsselwörter Frühe mathematische Bildung · Argumentieren · Einsatz von Gesten · Kita Kinder · Geometrie

1 Introduction

Highlighting the fundamental role of argumentation in mathematics education, Krummheuer (2007) claims that “learning mathematics is *argumentative* learning” (Krummheuer 2007, p. 62). This holds true even for young learners. Mathematical argumentation can already be found in pre-school children’s mathematical activities (Böhringer 2021; Brunner 2019; Krummheuer 2018; Benz et al. 2015; Lindmeier et al. 2015).

A definition of argumentation often used is that by van Eemeren et al. (1996) as “a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a ‘rational judge’” (van Eemeren et al. 1996, p. 5). Thus, argumentation is seen as a socially embedded activity of dialectical nature guided by reason. This pragma-linguistic view on argumentation serves the purposes of educational studies well and includes the explorative, knowledge-based nature of the discourse that is suitable especially to science learning. In mathematics education, argumentation has been described by Schwarzkopf (2000) as an interpersonal process in which a need for justification is expressed and satisfied. Both definitions stress the procedural aspect of argumentation as opposed to an argument as a product of an argumentation process, the latter constituting the object of observation e.g. within the framework of the Toulmin model (see below, 3.1).

Building on these definitions, in an attempt to offer an approach to the identification and analysis of young learners' mathematical argumentation, the concept of explorative mathematical argumentation (EMA) has been previously introduced (Reuter 2023). In this study, the concept is applied with a focus on the use of gesture as one modality of communication in a situation of geometrical conceptualization.

The social nature of argumentation has an effect on the choice of methods for the analysis of argumentation processes in mathematics learning. Social communication is not limited to speech, but also implemented through gesture and material-based actions. Fetzer (2019) claims that objects can play important roles in collective mathematical argumentations (Fetzer 2019, p. 151), and Schwarz et al. (2010) emphasize the multimodality of learners' as well as professionals' mathematical argumentation, stating that "argumentation in mathematics is absolutely crucial and is multimodal" (p. 138).

In the case study portrayed this article, a group of five- to six-year-old preschool children engage in the task of allocating shapes to either triangles or quadrilaterals. A concave quadrilateral poses a challenge that calls for a deeper understanding of the concept of a vertex. This understanding is established in a multimodal process of explorative mathematical argumentation.

2 Theoretical Framework

In the following, the concept of explorative mathematical argumentation is introduced as a concept for describing and analysing mathematical argumentation processes of young learners. Preceding the presentation of the case study, the use of gesture in mathematics education will be discussed and the concept of collaborative gestures is introduced, followed by a presentation of selected studies on children's geometrical conceptualization.

2.1 Explorative Mathematical Argumentation

Knowledge construction is a significant objective of learners' mathematical argumentation. Alongside with the awareness that *learning to argue* is an important objective of school education (e.g. Mercer 2009), the concept of *arguing to learn* has gained increased interest in science and mathematics education (e.g. Andriessen 2006; Asterhan and Schwarz 2016), with argumentation being considered a collaborative, constructive activity amongst learners and learning guides with the aim of knowledge construction. Baker (1999) states that "an interaction is constructive if it literally leads to the (co-)construction or building of something—meaning, understanding, solutions to problems and sometimes knowledge" (Baker 1999, p. 180). It seems suitable for the field of early mathematics education to concentrate on argumentation processes that aim at collaborative knowledge construction, as opposed to mere rhetorical persuasion. A method of analysing learners' mathematical argumentation has to account for the specific characteristics of mathematical argumentation.

Based on Ehlich's (2014) distinction between persuasive and explorative argumentation, young learners' mathematical argumentation can be described as an explorative activity with the aim of collaborative knowledge construction. While Ehlich uses terms like the (boundary) extension of knowledge systems ("Weiterung", Ehlich 2014, p. 47; "Grenzerweiterung", p. 49), with regard to the constructivist foundation of the research project, I use the term "knowledge construction" which is widely prevalent in educational research on argumentation in mathematics as well as other fields. The dialectic nature of the argumentation is not initiated by knowledge systems in conflict, but rather by knowledge systems in contrast (Ehlich 2014). The goal of an EMA process does not lie in imposing one's own opinion on an interlocutor so that one party loses, but in exploring different aspects together in order to construct new mathematical knowledge, so that all parties experience knowledge construction. The concept of explorative argumentation offers an approach that meets the requirements of the explorative nature of children's mathematical learning, like the aim of collaborative knowledge construction. However, it has been specified according to the given conditions of mathematics learning, so it can be applied for research purposes in this field (Reuter 2023). To adapt Ehlich's concept of explorative argumentation to the subject-specific requirements of mathematical argumentation, it is important to specify *how* knowledge is being collaboratively constructed in this context. Tools that learners can use in such processes comprise

- making and communicating discoveries,
- challenging existing knowledge,
- finding and using (relational) patterns,
- developing and testing hypotheses,
- drawing on analogies, and
- drawing conclusions (Reuter 2023).

Conclusions can be drawn using different types of reasoning, such as inductive, abductive, and deductive reasoning.

For *inductive reasoning*, empirical observation is used to infer patterns or regularities and thus form conclusions concerning a given issue. Franklin (2013) offers an example in a geometry context: "All equilateral (plane) triangles so far measured have been found to be equiangular. This triangle is equilateral. Therefore, this triangle is equiangular." (Franklin 2013, p. 14).

Abductive reasoning is what Harman (1965) calls the "inference to the best explanation" (p. 88). This way of reasoning is often observed in mathematics learners' argumentation, when an unexpected observation calls for an explanatory hypothesis. Abduction and deduction can go hand in hand within an argumentation process: "Abductions allow reasoning backward from a desired conclusion to establish data that further deductions can be based on" (Knipping 2003).

Deductive reasoning implies that "if the premises are true, the conclusion is necessarily true as well" (Datsogianni et al. 2020, p. 2). It is strongly related to formal mathematical proof, which, as a specific kind of argumentation (Aberdein 2005), can be considered "outcomes of socially agreed upon sets of rules and mathematical objects (such as definitions, axioms, and theorems)" (Stylianides 2007, p. 12). Based on several empirical studies, Stylianides (2007) shows that elementary school children are already able to successfully perform deductive reasoning in the mathematics

Table 1 Analysing scheme for explorative mathematical argumentation (Reuter 2023, p. 429)

	MODALITIES		
TOOLS	Verbal statements	Gestures	Material-based actions
Making discoveries			
Communicating discoveries			
Challenging existing knowledge			
Finding (relational) patterns			
Using (relational) patterns			
Developing hypotheses			
Testing hypotheses			
Drawing on analogies			
Drawing conclusions			

classroom. The case study introduced in this article will also show an example of a pre-schooler's logical deduction from a definition during an argumentation process.

Argumentation is a multimodal activity that is not merely limited to verbal utterances. For example, visual materials can function as means of mathematical argumentation (Welsing 2020) and gesture can be an important aspect of mathematical reasoning processes (e.g. Billion and Huth 2023; Huth 2022; Walkington et al. 2019a; Elia 2018).

Summarizing the above, explorative mathematical argumentation (EMA) is defined as a mathematically anchored, socially embedded and multimodal process in which the participants make use of argumentative and mathematical abilities, like communicating discoveries, challenging existing knowledge, developing and testing hypothesis, finding and using relational patterns and drawing on analogies, with the collaborative aim of knowledge construction. Table 1 shows the corresponding analysing scheme first presented in Reuter (2023).

This study focusses on the use of gestures in an explorative mathematical argumentation process among pre-school children. So, before applying the above scheme to an exemplary situation of EMA, some existing studies on the use of gesture in mathematics education are presented with a focus on activities of mathematical argumentation.

2.2 Research On the Use of Gesture in Mathematics Education

As one modality of communication, gesture can "play a key role in mathematical reasoning" (Walkington et al. 2019a). It can serve social- as well as self-oriented purposes: "gesturing is relevant in communication and thinking processes" (Sabena 2018, p. 543). Sabena (2018) offers a broad definition of the term "gesture" that meets the requirements of mathematics educational research by describing it as "all those movements of hands and arms that subjects (students and teachers) perform during their mathematical activities and which are not a significative part of any other action (i.e. writing, using a tool, ...)" (p. 21).

Based on a psychological point of view, McNeill (1992) proposes different dimensions of imagistic gesture (gesture that represents meaningful aspects within a communicative situation, as opposed to movements that do not bear content meaning, like so-called beats). McNeill offers the following distinction between deictic, iconic and metaphorical gestures, which, according to Elia (2018) are "the ones of most interest in mathematics" (p. 160):

Deictic gestures are pointing gestures that can be directed towards concrete objects, but also towards a region within the gesture space that was assigned a certain meaning. A child pointing at a certain shape when asked to identify a triangle would be an example of a deictic gesture. Referring to the existence of abstract pointing gestures, i.e. those that indicate an imaginary object in empty space, Sabena (2018) states that although "[a]pparently simple, pointing is indeed a complex act" (p. 544). *Iconic gestures* display concrete objects or activities and usually accompany speech. An example would be forming a triangle with both hands while talking about triangles. This dimension is also sometimes called *depictive gestures*. Walkington et al. (2014) make a further distinction and talk of *static depictive gestures* when the represented object is not moved or changed, and *dynamic depictive gestures* when the presented object is transformed. Dynamic gestures are usually observed in experts, although learners have been shown to apply them with geometrical argumentation (Walkington et al. 2019a).

In contrast to iconic or depictive gestures, *metaphorical gestures* display abstract ideas, also usually accompanied by speech. Sabena (2018) provides an example of a fourth-grader using a metaphorical gesture in a mathematical context, who indicates a subtraction of three with a right-to-left movement with three extended fingers. Thus, the abstract ideas of subtracting and the number three are combined in one metaphorical gesture.

McNeill also introduces "cohesives" (McNeill 1992, p. 16), recurring gestures that structure the discourse by tying thematically related parts together and are therefore described as "discourse gesture" (p. 16). Deictic, iconic or metaphoric gestures can all be used as cohesives. Based on McNeill's ground-breaking work, some approaches to the use of gesture and it's role in mathematics learning can be found already in the nineties. Two of those early studies, both conducted with fourth graders, have shown that gesture can be "a vehicle through which children express their knowledge" (Gather et al. 1998), even if that knowledge is still emerging and implicit (Alibali et al. 1993). Newer research in this field emphasizes two different functions of gesture in learning processes: Apart from facilitating communication with others, gestures also serve self-oriented purposes such as supporting cognitive functions in mathematics learning (Krause and Salle 2019; Goldin-Meadow and Wagner 2005)[1]. Summarizing a number of studies on the function of gesture, Church and Goldin-Meadow (2017) claim that "[o]ne might say that gesture is in between worlds—the world of the mind and the world of concrete engagement. This in between place may serve a particularly important purpose for cognition." (p. 403).

[1] Krause and Salle (2019) note that "[d]espite the growing body of research on gestures in mathematics education, little is known about the self-oriented functions gestures can play in mathematical cognition" (p. 1).

Consequently, gestures can be seen as "symbols that serve both to communicate and to affect the gesturer's cognition" (Williams et al. 2012, p. 183).

Gestures facilitate communication, support learning processes and express emerging or implicit knowledge, which renders them very interesting for the analysis of preschool children's mathematical argumentation. However, in 2001, Roth claimed: "Whereas considerable research has focused on the way children express conceptual understandings in verbal form, very little is known about the role of gestures in expressing abstract mathematical and scientific knowledge" (p. 381). Now, in the past years, research on children's use of gesture in mathematics education has considerably increased, but studies on the use of gesture in mathematical argumentation processes still mainly focus on older learners.

For example, Williams et al. (2012) explore how university students with an average age of 20 years use gestures and actions in proof. They state that mathematical proof is a multimodal activity, claiming that gesture and action should be considered when analysing proof production. Furthermore, the authors suggest that gestures and actions may be especially helpful for students struggling with abstract notation—a precondition that holds true especially for younger learners. While Walkington et al. (2014) could show that university students' spontaneous production of dynamic depictive gestures was significantly linked to higher scores in the correctness of justifications, their study also showed that explicitly directing students to perform dynamic gestures did not have a significant effect on their performance in justification and proof.

As for adult learning guides, first grade teachers have been shown to use up to seven non-spoken representations (gestures, pictures, objects, and writing) per minute in place-value lessons in the mathematics classroom (Flevares and Perry 2001). The authors of this study claim that "[t]eachers' gestures have the potential to serve a crucial role in building bridges between words and external representations" (Flevares and Perry 2001, p. 332). When Walkington et al. (2019b) conducted a study with pre- and in-service teachers, they found that gestures can be collaboratively passed on between learners in mathematical reasoning.

In recent years, younger children have occasionally come into focus. Elia (2018) presents a case of a five-year-old interacting with her kindergarten teacher while solving a shape configuration problem, and emphasizes the role of deictic gestures as a "major component of the child's spatial communication and thinking" (p. 168), conveying information that was not expressed verbally. Iconic gestures also played an important role in the analysed situation with regard to the transformation of shapes.

The case study presented in this study sheds light on the role of collaborative gestures in a process of geometrical conceptualization among preschool children with an adult learning guide.

2.3 Collaborative Gestures

Walkington et al. (2019a) define "gestural exchanges that take place as learners discuss and explore mathematical ideas, using their bodies in concert to accomplish a shared goal" (p. 1) as *collaborative gestures*.

With EMA pursuing the shared goal of collaborative knowledge construction, collaborative gestures are of high interest for the analysis of EMA processes: “Learners use collaborative gestures to extend mathematical ideas over multiple bodies as they explore, refine, and extend each other’s mathematical reasoning” (Walkington et al. 2019a, p. 1).

The authors propose four distinct categories of collaborative gestures partly divided into subcategories: Echo (*Simple Echo* and *Echo and Build*), Mirror (*Simple Mirror* and *Anticipation*), Alternate (*Alternate and Build* and *Alternate and Redirect*) and Joint. The subcategory *Echo and Build* is defined as follows:

> “One learner makes a gesture, and then a second learner makes part or all of the same gesture (or a very similar gesture) afterwards. However, the second learner also changes or adds to the gesture in some way, in addition to echoing the original gesture.” (Walkington et al. 2019a, p. 8)

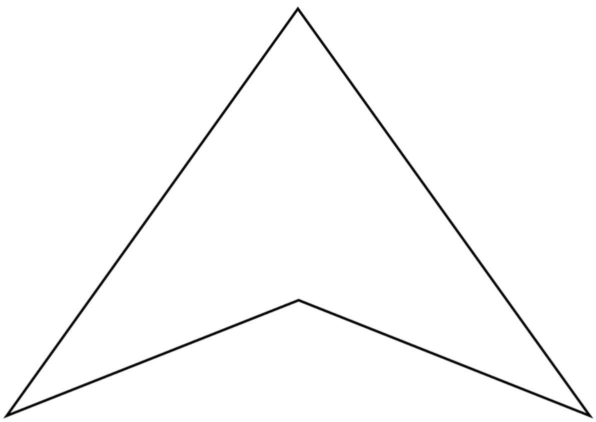
While Walkington et al. (2019a) focus on the use of gestures between adult learners, the study presented below demonstrates the use of Echo gestures of the type *Echo and Build* and confirms their important role for the outcome of the argumentation process, but in this case between a child and an adult learning guide.

2.4 Young Children’s Geometrical Conceptualization of Triangles and Quadrilaterals

Before taking a look at studies on young children’s competencies in geometrical conceptualization, it should be noted that in German, there is no verbal distinction between the mathematical term vertex and the word used for a corner in everyday language. Both can be translated as “Ecke”. In the German term for triangle (Dreieck), part of the word “Ecke” is used: Drei|eck = three|vertex or three|corner. The word for quadrilateral (Viereck) follows the same pattern: Vier|eck = four|vertex or four|corner.

Now, Unterhäuser (2019) shows in a study with 120 German children from the age of four to six years, that the term “Ecke” (vertex/corner) was used more often

Fig. 1 Concave quadrilateral



by the children than the term "Seite" (side), and that a vertex was associated with something pointed. When asking 4- to 6-year-old children in England and Germany to decide whether a given shape was a triangle or not, Maier (2019) found that 47% of the English children and 65% of the German children falsely identified a concave quadrilateral (Fig. 1) as a triangle. The author concludes that the number of recognized vertices was often more relevant to the children than the properties of a side, like its straightness (p. 254).

Elia (2018) points out that there is still a great need for gaining insight into young learners' development in geometry, especially with regard to gestures. The situation analysed in the following was chosen because it poses an example for a need for argumentation that leads to an explorative argumentation process with collaborative gestures playing an important role for the conceptualization of the vertex as a geometric property.

3 Case Study: the Concave Quadrilateral

Drawing on the concept of EMA and the insights of the studies presented above, the question is pursued how gestures are used in a guided situation of early mathematical argumentation with pre-school children. The analysed situation promotes geometrical conceptualization and offers a rich potential for argumentation. It was chosen for thorough analysis because the use of collaborative gestures in the argumentation process leads to a remarkable process of geometrical conceptualization. Before the sequence is described and analysed in detail, the process of data collecting and sampling is presented, displaying benefits and limitations of the Toulmin model.

3.1 Data Collection and Sampling

The sampling process focussed on situations of mathematical argumentation. Out of a broad sampling of video material conducted in the course of several semesters in a mathematical join-in studio at the University of Education Karlsruhe, Germany[2], sequences of mathematical argumentation could be identified applying the Toulmin model (Toulmin 2003). The Toulmin model provides a structured presentation of the constituting elements of an argument (data, claim, and a sometimes implicit warrant) as well as additional components like backing, rebuttal and qualifier. Homer-Dixon and Karapin (1989) have introduced the "attack" as an additional component of an argument scheme. It can be defined as a challenge to any other component of the argument.[3] Apposite to the dialectic nature of explorative argumentation with knowledge systems in contrast, attacks create a need for argumentation. They can appear as inter- or intrapersonal challenges, like irritations, disbelieve etc. and seem to relate to what Gellert and Krummheuer (2019) describe as a "condensed course of interaction" (p. 210) that often requires crisis management and provides good condi-

2 For further information, see Benz (2016).

3 The distinction of attacks from rebuttals was discussed in Reuter (2023).

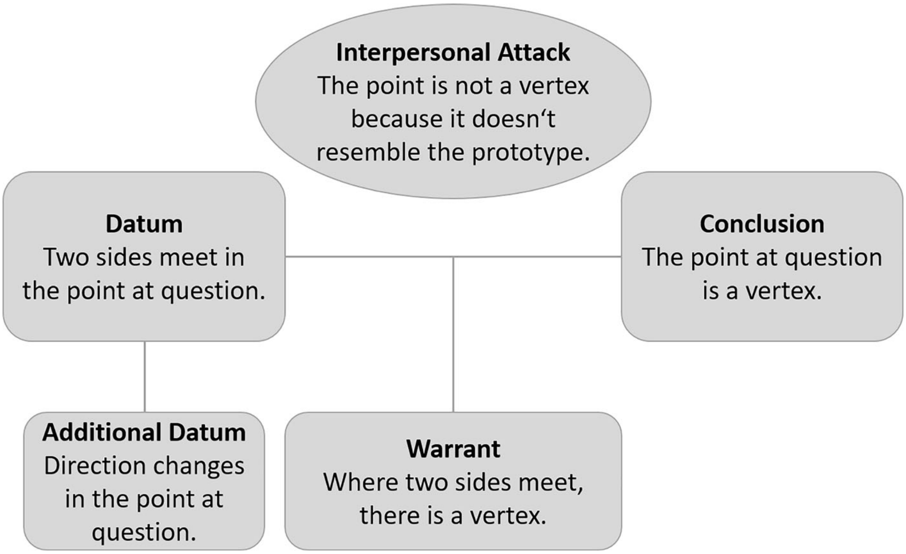


Fig. 2 Toulmin model 1

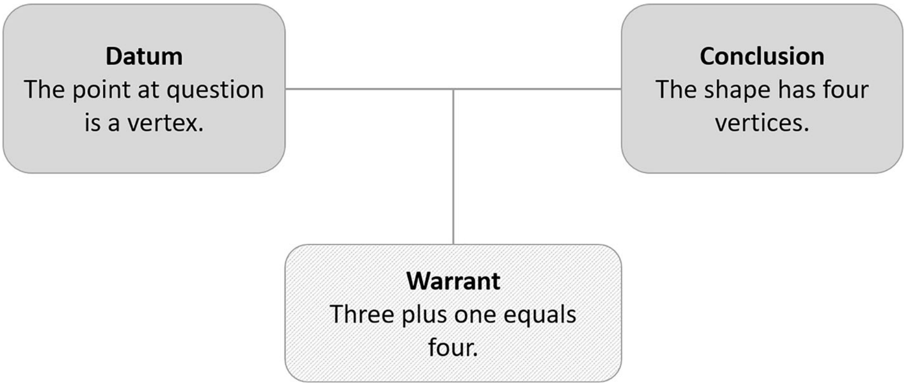


Fig. 3 Toulmin model 2

tions for learning, which makes argumentation processes so fruitful for mathematical learning (Gellert and Krummheuer 2019).

In order to be considered mathematical argumentation, a sequence had to contain at least a claim, a datum and an implicit or explicit warrant, all three of which had to be mathematically anchored. An example would be this fictive argument:

Claim: That ladybird has six dots.
Data: Because there are three dots on each of the two wings.
Warrant: Two times three equals six.

Whereas the following argument would not be categorized as mathematical argumentation:

Claim: That ladybird has six dots.
Data: Because the teacher said so.
Warrant: The teacher is always right.

However, the Toulmin model has been criticized as being too rigid to adequately portray the complexity of a dynamic argumentation process in a social dialogical setting (e.g. Gronostay 2017). For example, assertions may change from being a claim, to serving as data for another claim, as can be seen in the following examples that visualize individual arguments in the course of the argumentation process presented below, the first one covering the question if a certain point in a convex quadrilateral (Fig. 2) can be considered a vertex, the second one dealing with the allocation of the shape to the group of quadrilaterals. The lighter shaded component in Fig. 3 remains implicit in the process, but can be inferred from the situation.

	Verbal utterance	**gesture**	**Material based action**	**Implicite content**
Challenging existing knowledge	Where are the vertices?			
Using (relational) patterns	vertex – side	point (3x), slide, point	pointing, running along	
Using (relational) patterns	vertex – side	triangle, static representation		
	side – vertex	slide/point	running along, pointing	
Drawing on analogies	„That side and that side (...) in THIS corner"	adopting slide/point gesture to the concave angle	running along, pointing	The concave angle meets the same criteria as the convex angles.
Making discoveries/ communicating discoveries	„And THERE (...) that side and that side meet in THIS corner."	slide, point	pointing material	The point in qurestion is a vertex, too.
Drawing conclusions	„Then it belongs to the quadrilaterals."		placing material	If the point with the concave angle is a vertex, the shape has four vertices and is a quadrilateral.

Fig. 4 Analyzing scheme "Concave quadrilateral"

In the first scheme, the assertion that the point in question is indeed a vertex forms the conclusion, while in the second scheme, it fulfils the function of a datum. Also, the chronological order of the process within each argument remains unclear. However, in order to address the role of collaboratively shared gestures, it is of great importance to depict the chronological order of the argumentation process. In this study, the Toulmin model has thus only been used to identify mathematical arguments in young children's activities, postulating that at least a datum, a claim and an (implicit) warrant have to be inferable and mathematically anchored (Reuter 2023) in a situation of mathematical argumentation.

3.2 Description and Analysis of the Situation

The presented video sequence was analysed in the style of video interaction analysis (Knoblauch and Tuma 2020) combined with the categories from the EMA analysing scheme introduced in Table 1. Verbal expressions, gestures and material-based actions performed during the argumentation process by the children as well as the adult guiding the situation were allocated to the respective tools of reasoning. Thus, corresponding to the multimodality of the mathematical argumentation, the role of different modalities within the argumentation process can be uncovered (see Fig. 4).

The videotaped situation takes place in a join-in studio at university. A kindergarten teacher training student has prepared the learning situation focussing on geometrical conceptualization and is now accompanying the process, asking questions and giving impulses if needed. The use of gestures was not part of the preparation and planning, thus we can assume gestures emerged rather spontaneously.

A group of 12 pre-school children (5 to 6 years old) are presented several triangles and quadrilaterals from thin foam rubber, spread out randomly in front of them on the floor, and the assignment is to sort them. Beforehand, the student teacher discussed the terms corner and side with the children and also addressed how many vertices and sides quadrilaterals and triangles have. In accordance with the above-mentioned specifics of the German terminology, in the following the word "Ecke" will be translated as "corner" when it is verbally used in the learning situation, and as "vertex" when the situation is being described and analysed.

The task of sorting the shapes has been made more difficult by including a concave quadrilateral looking quite similar to a prototypical triangle except for the concave vertex in one side (Fig. 1), in order to provoke argumentation processes.

The transcript sets in when the children have already sorted the shapes into two groups of triangles and quadrilaterals. As expected, the concave quadrilateral was allocated to the triangles. The children are asked by the student teacher to confirm that every triangle has three vertices and three sides. When this has been done for some of the triangles, the following situation occurs (Tab. 2):

Luca is making and communicating a discovery here: The shape has four sides and three vertices, which does neither correspond to the previously acquired description of a triangle, nor to that of a quadrilateral. Thus, it is considered "strange".

An intrapersonal conflict seems to arise for several children, leading to two children labelling the concave quadrilateral as "strange", with Paul referring to the non-prototypical appearance caused by the concave vertex ("because here it goes inside

Table 2 Transcript 1

Marie	(Points to every side of the concave quadrilateral one by one)	
	One, two, three, four.	Eins, zwei, drei, vier.
	(shrugs one shoulder)	
Tim	(Points to one of the convex vertices of the concave quadrilateral, then lifts it slightly up from the floor)	
	This one here, this is somehow strange.	Das hier, das hier ist irgendwie komisch.
Paul	(Points to the concave vertex of the concave quadrilateral)	
	Yes, because here it goes inside like this.	Ja, weil es hier so rein geht.
Luca	(Lifts the concave quadrilateral from off the floor and turns it in his hands)	
	(wrinkles brows)	
	Um, well, this one here, this is somehow a bit strange after all.	Ähm, also, das hier, das ist irgendwie doch schon ein bisschen komisch.
	(Puts the shape back down with the triangles)	
	(Points at the sides of the concave quadrilateral one by one)	
	Because that has one, two, three, four sides	Weil, das hat eins, zwei, drei, vier Seiten
	(Points at the convex vertices of the concave quadrilateral one by one)	
	And one, two, three corners.	Und eins, zwei, drei Ecken.
	(Looks at student teacher)	

like this") and Luca providing another, mathematically anchored datum: "Because that has one, two, three, four sides and one, two, three vertices." Note that although Luca's datum is not mathematically correct, it is clearly mathematically anchored. Up to this point, we can see that many deictic pointing gestures accompany the discussion, mostly in connection with counting processes, presumably facilitating the one-to-one correspondence. When Paul points to the concave vertex and says: "(...) here it goes inside like this", he may be trying to specify Tim's claim that the shape is "somehow strange" by generating attention for a special property of the shape.

By looking directly at the student teacher, Luca now actively includes her in the interaction. The student teacher returns the eye contact and joins the discussion (Tab. 3).

In terms of geometrical conceptualization, Tim is showing a remarkable alteration of the other children's gestures here. By using the side of his hand when pointing to the sides of the shape, he implies a difference between the mathematical objects of a vertex being a point and a side being a line segment. The function of this gesture could go beyond mere pointing. However, maybe because he doesn't produce any speech accompanying his gestures, Tim's idea seems to go unnoticed. Instead, the student teacher focusses on the vertices and asks the children (Tab. 4):

By placing the concave quadrilateral between the two groups of shapes, the student teacher opens up the possibility of not classifying it as a triangle. Luca verbally offers a definition of a vertex: It is located "where the sides meet". This last bit of the statement is accompanied by a more dynamic gesture than the previous pointing gestures, possibly displaying the idea of the sides "meeting" in the vertex. This

Table 3 Transcript 2

S	(smiles, looks at Luca)	
	That has (.) four sides and three corners.	Das hat (.) vier Seiten und drei Ecken.
Children	(Nod their heads)	
	Yes.	Ja.
Tim	(Points at the four sides one by one with the side of his hand)	
	(points on the three convex vertices one by one with his finger)	

Table 4 Transcript 3

S	So what is special about the corners?	Was ist denn das Besondere an den Ecken?
	(takes up the concave quadrilateral, holding it on two convex vertices and puts it between the group of triangles and the group of quadrilaterals, then taps on the shape with her hand)	
	Where are the corners always located?	Wo lie(.) Wo sind denn die Ecken immer?
Luca	(Points to the three convex vertices one by one, then makes a quick sliding movement along one side and towards one of the vertices with his right index finger while holding his left index finger on that vertex.)	
	Always at those, where then... Well, always where the sides meet then.	Immer an denen, wo sich dann... Also, immer wo dann die Seiten sich treffen.
	(Looks at student teacher)	
S	(Looks at Luca, nods)	
	Where the sides meet, exactly.	Wo die Seiten sich treffen, genau.

sliding gesture will play an important role in the ongoing argumentation. However, first the student teacher points to the four sides again and then forms a triangle with her hands in a static iconic gesture (Tab. 5):

Now the student teacher adopts Luca's sliding gesture, although she carries it out with both index fingers meeting in the vertices (Echo and Build) (Tab. 6).

An interpersonal conflict appears to arise between Karim and Luca at this point. However, before taking a closer look at it, I will pre-empt to the end of this learning situation (Tab. 7):

3.3 Application of the Analysing Scheme

With the help of the analysing scheme, we can have a close look at what is happening in the EMA process between the student teacher and Luca in the course of the past few sequences concerning the different tools and modalities of argumentation (Fig. 4):

The grayed-out table cells show actions of the student teacher, the cross-hatched cells show actions carried out by Luca.

The primary impulse comes from the student teacher asking where the vertices are located. In his definition, Luca uses the relation between vertices and sides, saying that the vertices are "always where the sides meet". After pointing to the three convex vertices, he uses a dynamic sliding gesture towards one of them to accompany this

Table 5 Transcript 4

S	(Points to the four sides of the concave quadrilateral one by one)	
	And... What did you say? We have four sides: this side, this side, this side and this	Und... Was hast du gesagt? Wir haben vier Seiten: die Seite, die Seite, die Seite, und die.
(...)		
S	(Sits up straight, looks at Luca)	
	(Forms a triangle with both hands, thumbs overlapping and forming the base, other fingertips touching)	
	But you just said the corners are always where the sides meet.	Aber, du hast ja gerade eben gesagt, die Ecken sind immer da, wo die Seiten sich treffen.
Luca	Exactly.	Genau.

Table 6 Transcript 5

S	So let's see where the sides meet.	Dann lass uns mal gucken, wo sich die Seiten treffen.
	(slides along two sides of the shape simultaneously with both index fingers meeting in one of the convex vertices)	
	So, HERE the sides meet, therefore we have ONE corner here.	Also, HIER treffen sich die Seiten, deswegen haben wir hier EINe Ecke.
Luca	Yes.	Ja.
S	(slides along two other sides of the shape simultaneously with both index fingers meeting in another one of the convex vertices)	
	And HERE the sides meet, then we have a corner here, TOO.	Und HIER treffen sich die Seiten, dann haben wir hier AUCH eine Ecke.
Luca	(Moves closer to the shape)	
	(points to the side on the right of the concave vertex, then to the side on the left of the concave vertex, then makes a quick sliding motion with both index fingers towards the concave vertex and finally taps on the concave vertex with his right finger several times)	
	And THERE, this cor... this side and this side meet in THIS corner.	Und DA treffen sich die Eck... die Seite und die Seite an DIEser Ecke.
	(Looks at student teacher)	
S	(Raises eyebrows, sits up straight, smiles at Luca)	
	So that is a corner, TOO?	Also ist das AUCH eine Ecke?
Karim	No.	Nein.
Luca	Yes, it is.	Doch.

statement. Now the student teacher, after making a static iconic gesture with both hands depicting a triangle, which does not allow any dynamic action, adopts Luca's sliding gesture on the material. However, she slightly alters Luca's approach, which Walkington et al. (2019a) call "Echo and Build" (see Table 1). Instead of first pointing to a vertex and then sliding towards it with the other hand, the student teacher uses both index fingers to slide along two sides of the shape until they meet in the respective vertex. She also performs this change in her verbal utterances: After

Table 7 Transcript 6

S	And what kind of shape is it now, if we have four corners and four sides?	Und was ist das jetzt für eine Form dann, wenn wir jetzt vier Ecken haben und vier Seiten?
Luca	(takes the shape up from the floor and places it with the quadrilaterals)	
	Then it belongs to the quadrilaterals.	Dann gehört es zu den Vierecken.

repeating Luca's statement: "[Y]ou just said the corners are always where the sides meet", she accompanies her gestures on the material saying "So, HERE the sides meet, therefore we have ONE corner here". This shift of direction from *vertex* → *sides* to *sides* → *vertex* seems to allow a shift in Luca's reasoning from an inductive procedure of identifying prototypical vertices and exemplarily applying his idea to them, to a deductive application of a definition: Wherever two sides of a geometric shape meet, there is a vertex—no matter what it looks like.

We can see in the analysing scheme how, after the student teacher introduces this shift by echoing and altering Luca's gesture as well as the verbal utterance it accompanied, Luca is able to carry out the rest of the argumentation by himself. Gestures are collaboratively passed on between Luca and the student teacher in the course of this explorative argumentation.

Lastly, I want to take a closer look at the previously mentioned interpersonal conflict arising in the course of the argumentation process.

3.4 Gestures and the Development of Hypotheses

After it seemed clear to Luca that the concave vertex is a vertex, too, another child was not convinced (see above) (Tab. 8).

When analysing this sequence focussing on Luca and what he does to support his argument, it can be concluded that Luca formulates a hypotheses ("[I]f there were no corners, then it would just always be a long stick") and accompanies it with a *metaphorical gesture*, representing the idea of infinity. Immediately after that, Marie also formulates a hypothesis and accompanies it with a *dynamic depictive gesture*, as folding the shape is a concrete idea. However, by stating that "[i]f you fold that over like this, those two, then it would be like a triangle" and imitating the respective folding movement, Marie seems to show considerable competence in mental folding (Harris et al. 2013) and finding relations between shapes. Luca's use of a metaphorical gesture starting on the material, but ultimately going beyond, may have prompted Marie's idea to mentally manipulate the shape by folding it, which is another example of the co-constructive nature of the argumentation and the role that gesture plays in the process. Looking at Luca's and Marie's verbal utterances, it is interesting to see that both children use a subjunctive at this point. Thus, both children formulate hypotheses, marked on the verbal level by the use of a subjunctive, which is an essential skill for abductive reasoning and thus shows a development in argumentation skills (learning to argue).

Table 8 Transcript 7

S	(Raises eyebrows, sits up straight, smiles at Luca)	
	So that is a corner, TOO?	Also ist das AUCH eine Ecke?
Karim	No.	Nein.
Luca	Yes, it is.	Doch.
S	(nods)	
	Yes, it is.	Doch.
Tim	But there is nothing sharp.	Aber da ist nichts Spitzes.
S	There is nothing sharp, but it's a corner anyway.	Da ist nichts Spitzes, aber es ist trotzdem eine Ecke.
Luca	Yes. The c... sides meet there anyway.	Ja. Da treffen sich trotzdem die E... Seiten.
S	Exactly.	Genau.
Luca	AND by that the direction is changed.	UND dadurch wird die Richtung geändert.
	(slides along a side leading to a convex vertex and continues the movement)	
	Because if there were no corners, then it would just always be a long stick.	Weil, wenn es keine Ecken gäbe, dann würde es ja immer nur ein langer Stab werden.
S	(nods)	
Marie	(performs an arc movement with both hands over the shape, indicating folding it at the axis of symmetry)	
	If you fold that over like this, those two, then it would be like a triangle.	Wenn man das da so hin klappt, die zwei, dann würde es wie ein Dreieck sein.
	(looks at the student teacher)	
S	(nods)	
	That's right.	Stimmt.

4 Discussion

The growing interest in young learners' mathematical argumentation calls for empirical evidence of its impact on learning processes in different mathematical content areas, different age groups and different educative settings. Especially concerning pre-school children, empirical research is only just emerging. Case studies covering specific mathematical content and educative settings allow a qualitative approach to the subject.

The presented data show how young children construct mathematical knowledge making use of gestures and material-based actions in a complex mathematical argumentation process. Materials used to foster mathematical learning processes should be chosen with regard to its argumentative potential. Here, the presentation of the concave quadrilateral creates a need for justification and leads to a process of knowledge construction in the form of geometrical conceptualization. Also, gestures are conducted in a collaborative manner that promotes the argumentation process as well as the mathematical learning process.

Considering mathematical thinking of 4- to 6-year-old children, Krummheuer (2018) claims that successful learning processes are constituted by "the increased autonomous participation in such collective argumentation in the process of a current interaction and/or in the following interaction that is thematically imbedded in the actual situation" (Krummheuer 2018, p. 113). Explorative mathematical argumentation processes, consciously prepared and accompanied by a learning guide as presented here hold great potential for procedural as well as content-specific learning. Luca shows a considerable development in his argumentation skills (learning to argue), but we also find him arguing to learn, as geometrical conceptualization was fostered and actively enhanced in the course of the argumentation process. Especially the adoption and alteration of gestures (Echo and Build) between children and the learning guide seem fundamentally important for the children's progress. The dynamic sliding gesture representing the idea of sides "meeting" in a vertex plays a key role in the argumentation and conceptualization process. As a reoccurring cohesive gesture, it ensures the participants' mutual focus on the definition of a vertex as a prerequisite for an extended understanding of the term *quadrilateral*. It was introduced by Luca, echoed and altered by the learning guide, and again adopted by Luca in its new form which allows a deductive approach to the presented problem and facilitates a considerable progress of argumentation skills finally leading to an enhanced geometrical conceptualization.

We find both above-mentioned functions of gestures in this situation: facilitated interpersonal communication and intrapersonal cognitive support. It can be summarized that gestures that are collaboratively passed on and altered between learners and learning guides can support early explorative mathematical argumentation processes in a situation of geometric conceptualization with pre-school children. The argumentation itself was enhanced, with children engaging in forming hypotheses and one child (Luca) even conducting deductive reasoning, when he draws on the definition of a vertex for correctly classifying the shape that he previously described as "somehow a bit strange after all" (transcript 1).

The findings also suggest a need for a conscious, well-considered use of gesture when guiding and supporting mathematical learning processes, and show that it is of great importance to reflect on the role of gesture in fostering children's mathematical argumentation.

Funding The researchis financed via a qualification position by the programme "Lehrerbildung in Baden-Württemberg", provided by the Ministerium für Wissenschaft, Forschung und Kunst. The specific project is "Lehr-Lern-Labore in den MINT-Fächern als Innovations- und Vernetzungsfeld in der Lehrerbildung am KIT und an der PH Karlsruhe" (MINT2KA). Reference Number: 43-6700-2/18/1.

Funding Open Access funding enabled and organized by Projekt DEAL.

Conflict of interest F. Reuter declares that she has no competing interests.

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