Journal of Statistical Theory and Applications (2026) 25:6
https://doi.org/10.1007/s44199-025-00148-5

RESEARCH

The Q-Lindley Distribution: Goodness-of-Fit Tests, Modeling, Inference, and Applications

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Received: 8 August 2025 / Accepted: 3 December 2025

Abstract
This paper introduces the Q-Lindley distribution (QLD), a new one-parameter model for lifetime and reliability data derived from the affine exponential family. We derive its main statistical properties and estimate the parameter using maximum likelihood. The performance of the QLD is evaluated using Monte Carlo simulations and four real datasets, and its fit is compared to several classical and Lindley-type distributions. Model selection criteria and Goodness-of-Fit statistics indicate that the QLD provides competitive or superior performance despite its simple structure. These results suggest that the QLD is a useful addition to the family of lifetime distributions.

Keywords Modified Kolmogorov-Smirnov · Modified Cramér-von-Mises test · Modified Anderson-Darling test · Modified Chi-squared test · Estimation · Maximum Likelihood · Simulation · Reliability.

Mathematics Subject Classification 62F03 · 65C05 · 62F10.

1 Introduction

The modeling of lifetime and reliability data is central to various applied disciplines, including engineering, medical sciences, actuarial analysis, and queuing systems. Classical models such as the Exponential and Gamma distributions have traditionally been used due to their mathematical simplicity and historical relevance [3, 8]. How-

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Published online: 13 January 2026

ever, these models often fail to adequately describe complex real-world phenomena characterized by skewness, multimodality, and non-monotonic hazard rates.

To address these limitations, the Lindley distribution–originally proposed by Lindley [16]–was introduced as a one-parameter alternative to the Exponential distribution. Its tractability and ability to model positively skewed data made it attractive for reliability and survival analysis. Since then, numerous generalizations and modifications have been proposed to enhance its flexibility. Key contributions include the Power Lindley [9], Generalized Poisson–Lindley [18], Negative Binomial–Lindley [29], and Transmuted Rayleigh [20] distributions.

More recent innovations introduced structurally extended models, such as the XLindley [5], Power XLindley [19], and NXLindley (New XLindley) [10], which provide additional flexibility by modifying the baseline Lindley density through transformation or compounding. Other variants like the Pseudo Lindley [22], ZLindley [24], and XGamma [25] distributions further exemplify the trend toward tailoring models to specific empirical features, such as heavy tails, varying hazard shapes, and multimodality.

These models have found wide-ranging applications: from modeling waiting and survival times [27] and reliability of electronic components [8] to the study of epidemic outbreaks such as COVID-19 [6, 11, 17, 26]. For example, the Power XLindley distribution has been used to model survival times in pandemic contexts [19], while Lindley-type models have been employed in fuzzy reliability systems [4].

While many of these extensions improve data-fitting performance, they often require multiple parameters, which can increase estimation complexity and reduce model interpretability. In contrast, our proposed Q-Lindley distribution introduces a single shape parameter that significantly increases flexibility while preserving simplicity and closed-form expressions for key properties.

The objectives of this study are as follows:

- To define the Q-Lindley distribution and derive its fundamental properties, including the probability density function (PDF), cumulative distribution function (CDF), quantiles, moments, and hazard rate behavior;
- To estimate its parameter using the method of maximum likelihood and evaluate its statistical efficiency;
- To assess the empirical performance of the QLD on real datasets related to mechanical failure times, bank queuing, kidney transplant survival, and cancer remission durations;
- To benchmark the QLD against classical models and several Lindley-type extensions using standard information criteria (AIC, BIC) and Goodness-of-Fit (GOF) tests.

The simulation studies and real-data applications demonstrate that the QLD offers competitive or superior performance compared to more complex multi-parameter models. Its parsimony and strong empirical fit suggest that it is a promising alternative for practitioners in applied probability, reliability engineering, and survival analysis.

The main novelty of the Q-Lindley distribution lies in introducing a single shape parameter that increases flexibility for modeling skewed and heavy-tailed lifetime data, while preserving the simplicity and closed-form properties of the classical Lindley distribution. This balance between parsimony and improved empirical performance distinguishes QLD from existing Lindley-type extensions.

2 The Affine Exponential Family

According to [1], a new flexible family of lifetime distributions was introduced, characterized by a probability density function of the form:

$$f(x;\theta) = d(\theta)\left(a(\theta) + b(\theta)x\right)\exp\left(-c(\theta)x\right), \quad x > 0, \tag{1}$$

where $a(\theta), b(\theta), c(\theta)$, and $d(\theta)$ are real-valued functions defined on $[0, +\infty[$, and $\theta > 0$ is a shape parameter. The function $d(\theta)$ serves as a normalizing constant, ensuring that the PDF integrates to one:

$$\int_0^\infty f(x;\theta)\,dt = 1.$$

This family satisfies the following basic properties:

- **Non-negativity:** $f(x;\theta) \geq 0$ for all $x > 0$;
- **Proper normalization:** $\int_0^\infty f(x;\theta)\,dx = 1$;
- **Cumulative probability:** $\mathbb{P}(a < X < b) = \int_a^b f(x;\theta)\,dx$, for $0 < a < b$.

The normalizing constant is given by:

$$d(\theta) = \frac{c^2(\theta)}{a(\theta)c(\theta) + b(\theta)}. \tag{2}$$

Substituting into (1), the general form of the PDF becomes:

$$f(x;\theta) = \frac{c^2(\theta)\left(a(\theta) + b(\theta)x\right)\exp\left(-c(\theta)x\right)}{a(\theta)c(\theta) + b(\theta)}, \quad x > 0. \tag{3}$$

3 The Q-Lindley Distribution (QLD)

In this paper, we consider a special case of this family, referred to as the **Q-Lindley distribution (QLD)**, obtained by choosing:

$$a(\theta) = 1, \quad b(\theta) = \theta^2, \quad c(\theta) = \theta,$$

with $\theta > 0$. Substituting into the general formula, we get:

$$d(\theta) = \frac{\theta^2}{\theta + \theta^2} = \frac{\theta}{1+\theta}.$$

Therefore, the PDF of the Q-Lindley distribution is given by:

$$f(x;\theta) = \frac{\theta(\theta^2 x + 1)\exp(-\theta x)}{1+\theta}, \quad x > 0. \tag{4}$$

The function $f(x;\theta)$ is strictly decreasing when $\theta \leq 1$, and unimodal when $\theta > 1$, achieving its mode at

$$x = \frac{\theta - 1}{\theta^2}.$$

By adding a quadratic term to the linear component of the Lindley PDF, the QLD introduces additional flexibility for modeling skewed and heavy-tailed data. This makes it particularly useful in fields such as financial modeling, survival analysis, and reliability studies.

The survival function and cumulative distribution function of the QLD are given respectively by:

$$S(x;\theta) = \frac{1+\theta+\theta^2 x}{1+\theta}\exp(-\theta x),$$

$$F(x;\theta) = 1 - \frac{1+\theta+\theta^2 x}{1+\theta}\exp(-\theta x).$$

The hazard rate function is:

$$h(x;\theta) = \frac{f(x \mid \theta)}{S(x \mid \theta)} = \frac{\theta(\theta^2 x + 1)}{1+\theta+\theta^2 x}.$$

The hazard rate $h(x;\theta)$ is strictly increasing from $\frac{\theta}{1+\theta}$ to θ as $x \to \infty$.

Figure 1 depicts the PDF, CDF, survival, and hazard function of QLD for various values of parameter θ.

The main statistical characteristics of the Q-Lindley distribution, including its quantile function, moment generating function, entropy, moments, order statistics, and parameter estimation methods, are derived in the sections that follow.

3.1 Quantile Function

To obtain the quantile function $F^{-1}(p)$, we solve the equation:

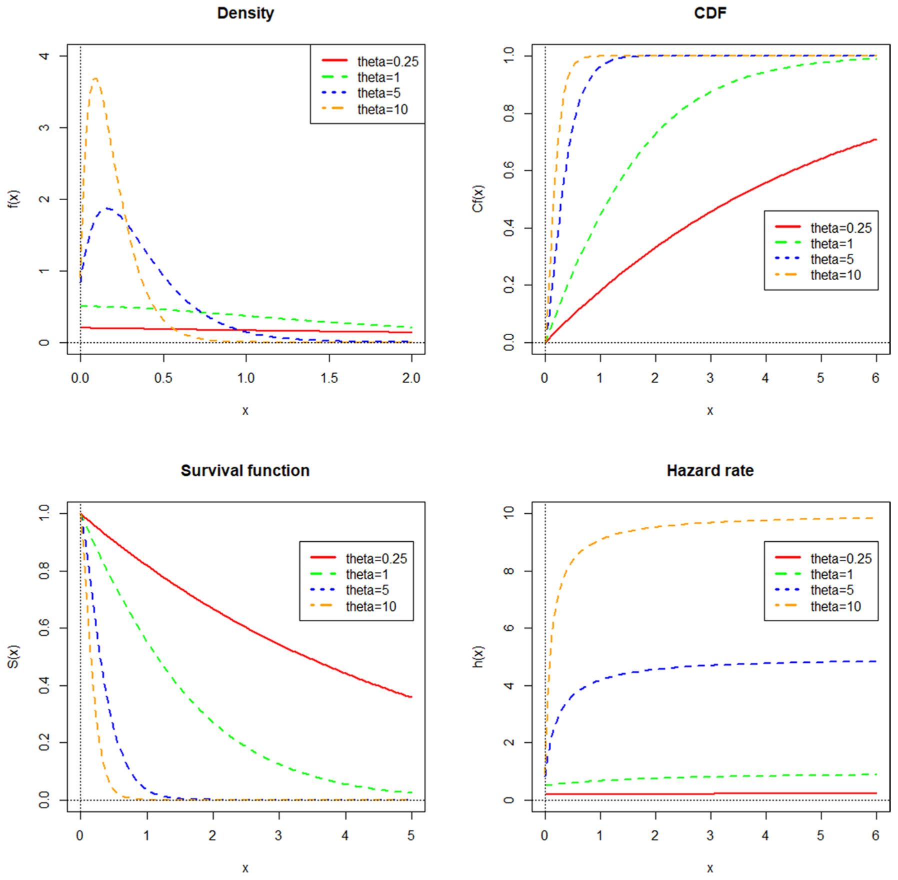


Fig. 1 PDF, CDF, survival, and hazard functions of QLD for various values of θ

$$1 - \frac{1+\theta+\theta^2 x}{1+\theta} e^{-\theta x} = p \quad \Rightarrow \quad (1+\theta+\theta^2 x) e^{-\theta x} = (1+\theta)(1-p) \tag{5}$$

Let us perform the change of variable $z = \theta x$, so that equation (5) becomes:

$$(\theta z + \theta + 1) e^{-z} = (1+\theta)(1-p)$$

Rewriting the left-hand side:

$$\theta z + \theta + 1 = (1+\theta)(1-p) e^{z}$$

Now divide both sides by θ:

$$z + 1 + \frac{1}{\theta} = \frac{(1+\theta)(1-p)}{\theta} e^{z}$$

Let us define:

$$k = \frac{(1+\theta)(1-p)}{\theta}, \quad c = 1 + \frac{1}{\theta}$$

Then the equation becomes:

$$(\theta x + c)\, e^{-\theta x} = k.$$

Next, group the exponent with the linear term:

$$(\theta x + c)\, e^{-(\theta x + c)} = k\, e^{-c}.$$

Let

$$z = \theta x$$

which gives

$$(z + c)\, e^{-(z+c)} = k\, e^{-c}.$$

Multiply both sides by -1 to obtain the standard form of the Lambert *W* function:

$$-(z + c)\, e^{-(z+c)} = -k\, e^{-c}.$$

Thus,

$$-(z + c) = W_{-1}\left(-k\, e^{-c}\right),$$

where W_{-1} is the *negative branch* of the Lambert W function (appropriate because $0 < p < 1$, so the argument is in the domain of W_{-1}).

Finally, substituting back $z = \theta x$, we obtain the quantile function:

$$\boxed{x = -\frac{1}{\theta}\left(1 + \frac{1}{\theta} + W_{-1}\left(-\frac{(1+\theta)(1-p)}{\theta e^{-\{1+\frac{1}{\theta}\}}}\right)\right)}$$

3.2 Moment Generating Function

The moment generating function (MGF) of the Q-Lindley distribution is defined as:

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{\theta(\theta^2 x + 1)e^{-\theta x}}{1+\theta}\, dx = \frac{\theta}{1+\theta} \int_0^\infty (\theta^2 x + 1)e^{(t-\theta)x}\, dx$$

This becomes:

$$M_X(t) = \frac{\theta}{1+\theta}\left[\theta^2 \int_0^\infty x e^{(t-\theta)x}\,dx + \int_0^\infty e^{(t-\theta)x}\,dx\right]$$

The integrals converge only if $t < \theta$, since otherwise the exponential term grows without bound.

We now compute:

$$\int_0^\infty e^{(t-\theta)x}dx = \frac{1}{\theta - t}, \quad \int_0^\infty x e^{(t-\theta)x}dx = \frac{1}{(\theta-t)^2}, \quad \text{for } t < \theta$$

Substituting back:

$$M_X(t) = \frac{\theta}{1+\theta}\left[\theta^2 \cdot \frac{1}{(\theta-t)^2} + \frac{1}{\theta-t}\right] = \frac{\theta(\theta^2+\theta-t)}{(1+\theta)(\theta-t)^2}, \quad t < \theta$$

Thus, the final expression is:

$$\boxed{M_X(t) = \frac{\theta(\theta^2+\theta-t)}{(1+\theta)(\theta-t)^2}, \quad t < \theta}$$

3.3 Additional Statistical Properties

3.4 Rényi Entropy

The Rényi entropy of order $\alpha > 0$, $\alpha \neq 1$, is defined as:

$$H_\alpha(X) = \frac{1}{1-\alpha}\ln\left(\int_0^\infty f(x)^\alpha\,dx\right)$$

For the Q-Lindley PDF is given by:

$$f(x) = \frac{\theta(\theta^2 x + 1)e^{-\theta x}}{1+\theta} \Rightarrow f(x)^\alpha = \left(\frac{\theta}{1+\theta}\right)^\alpha (\theta^2 x + 1)^\alpha e^{-\alpha\theta x}$$

So the entropy integral becomes:

$$\int_0^\infty f(x)^\alpha dx = \frac{\theta^\alpha}{(1+\theta)^\alpha}\int_0^\infty (\theta^2 x + 1)^\alpha e^{-\alpha\theta x}dx$$

Let $t = \alpha\theta x \Rightarrow x = \frac{t}{\alpha\theta},\ dx = \frac{dt}{\alpha\theta}$. Then:

$$\int_0^\infty f(x)^\alpha dx = \frac{\theta^{\alpha-1}}{\alpha(1+\theta)^\alpha}\int_0^\infty \left(\frac{\theta t}{\alpha} + 1\right)^\alpha e^{-t}dt$$

Now apply the known identity involving the **Tricomi's confluent hypergeometric function** $U(a, b, z)$:

$$U(1, \alpha + 2, z) = z \int_0^\infty (1 + zu)^\alpha e^{-u} du$$

Letting $z = \frac{\alpha}{\theta}$, we get:

$$\int_0^\infty \left(1 + \frac{\theta u}{\alpha}\right)^\alpha e^{-u} du = \frac{\alpha}{\theta} U\left(1, \alpha + 2, \frac{\alpha}{\theta}\right)$$

So the Rényi entropy becomes:

$$H_\alpha(X) = \frac{1}{1 - \alpha} \ln \left[\frac{\theta^{\alpha - 2}}{(1 + \theta)^\alpha} U\left(1, \alpha + 2, \frac{\alpha}{\theta}\right)\right]$$

Or equivalently:

$$\boxed{H_\alpha(X) = \frac{1}{1 - \alpha} \left[(\alpha - 2) \ln \theta - \alpha \ln(1 + \theta) + \ln U\left(1, \alpha + 2, \frac{\alpha}{\theta}\right)\right]}$$

Remark 1 If $\alpha = n \in \mathbb{N}^*$, then the entropy can be expressed using a finite sum:

$$H_n(X) = \frac{1}{1 - n} \ln \left[\frac{\theta^{n-1}}{n(1 + \theta)^n} \cdot n! \sum_{k=0}^{n} \frac{\theta^k}{n^k (n - k)!}\right]$$

See Appendix A for the proof of the above Remark 1.

3.5 Moments and Expectation

The r-th raw moment of the Q-Lindley distribution is defined as:

$$\mathbb{E}[X^r] = \int_0^\infty x^r f(x; \theta)\, dx = \frac{\theta}{1 + \theta} \left[\theta^2 \int_0^\infty x^{r+1} e^{-\theta x}\, dx + \int_0^\infty x^r e^{-\theta x}\, dx\right]$$

We use the identity:

$$\int_0^\infty x^n e^{-\theta x} dx = \frac{\Gamma(n + 1)}{\theta^{n+1}} = \frac{n!}{\theta^{n+1}}, \quad \text{for } n > -1$$

Applying this:

$$\int_0^{\infty} x^{r+1} e^{-\theta x} dx = \frac{(r+1)!}{\theta^{r+2}}, \quad \int_0^{\infty} x^r e^{-\theta x} dx = \frac{r!}{\theta^{r+1}}$$

Substituting into the expression for the moment:

$$\begin{aligned}
\mathbb{E}[X^r] &= \frac{\theta}{1+\theta}\left[\theta^2 \cdot \frac{(r+1)!}{\theta^{r+2}} + \frac{r!}{\theta^{r+1}}\right] \\
&= \frac{(r+1)! \cdot \theta}{(1+\theta)\theta^r} + \frac{r!}{(1+\theta)\theta^r} \\
&= \frac{r!}{\theta^r(1+\theta)}\left[(r+1)\theta + 1\right] \\
&\boxed{\mathbb{E}[X^r] = \frac{r!\,((r+1)\theta+1)}{\theta^r(1+\theta)}}
\end{aligned}$$

Mean and Variance:
For $r = 1$, the expectation (mean) is:

$$\mathbb{E}[X] = \frac{2\theta+1}{\theta(1+\theta)}$$

For $r = 2$, the second moment is:

$$\mathbb{E}[X^2] = \frac{2!(3\theta+1)}{\theta^2(1+\theta)} = \frac{2(3\theta+1)}{\theta^2(1+\theta)}$$

So the variance is:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \frac{2(3\theta+1)}{\theta^2(1+\theta)} - \left(\frac{2\theta+1}{\theta(1+\theta)}\right)^2 \\
&= \frac{6\theta+2}{\theta^2(1+\theta)} - \frac{(2\theta+1)^2}{\theta^2(1+\theta)^2} \\
&= \frac{\theta^2+4\theta+2}{\theta^2(1+\theta)^2} \\
&\boxed{\mathrm{Var}(X) = \frac{2\theta^2+4\theta+1}{\theta^2(1+\theta)^2}}
\end{aligned}$$

Skewness and Kurtosis:
The skewness γ_1 and kurtosis γ_2 are respectively given by:

$$\gamma_1 = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} = \frac{2(2\theta^3 + 6\theta^2 + 1)}{(2\theta^2 + 4\theta + 1)^{3/2}},$$

$$\gamma_2 = \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4} - 3 = \frac{6(2\theta^4 + 8\theta^3 + 12\theta^2 + 8\theta + 1)}{(2\theta^2 + 4\theta + 1)^2}.$$

These require central moments, which can be derived from raw moments or by expansion.

3.6 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Q-Lindley distribution. The PDF of the k-th order statistic $X_{(k)}$ is:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x)$$

Substituting the CDF and PDF of the Q-Lindley distribution:

$$F(x) = 1 - \frac{(1+\theta+\theta^2 x)}{1+\theta}e^{-\theta x}, \quad f(x) = \frac{\theta(\theta^2 x + 1)}{1+\theta}e^{-\theta x}$$

Then the order statistic becomes:

$$\begin{aligned} f_{X_{(k)}}(x) &= \frac{n!}{(k-1)!(n-k)!}\left(1 - \frac{1+\theta+\theta^2 x}{1+\theta}e^{-\theta x}\right)^{k-1}\left(\frac{1+\theta+\theta^2 x}{1+\theta}e^{-\theta x}\right)^{n-k} \cdot \frac{\theta(\theta^2 x+1)}{1+\theta}e^{-\theta x} \\ &= \frac{n!\cdot\theta(\theta^2 x+1)}{(k-1)!(n-k)!(1+\theta)^n}\left((1+\theta+\theta^2 x)e^{-\theta x}\right)^{n-k}\left(1 - \frac{(1+\theta+\theta^2 x)}{1+\theta}e^{-\theta x}\right)^{k-1} \end{aligned}$$

Special Cases:
Minimum ($X_{(1)}$):

$$f_{X_{(1)}}(x) = n\cdot f(x)\cdot[1-F(x)]^{n-1} = \frac{n\cdot\theta(\theta^2 x+1)}{(1+\theta)^n}e^{-n\theta x}(1+\theta+\theta^2 x)^{n-1}$$

Maximum ($X_{(n)}$):

$$f_{X_{(n)}}(x) = n\cdot f(x)\cdot[F(x)]^{n-1} = n\left(1 - \frac{1+\theta+\theta^2 x}{1+\theta}e^{-\theta x}\right)^{n-1} \cdot \frac{\theta(\theta^2 x+1)}{1+\theta}e^{-\theta x}$$

4 Parameter Estimation

Let $X_1, X_2, \cdots, X_n$ be *n* independent and identically distributed random variables. Likelihood Function $L(\theta)$ is:

$$L(\theta) = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{\theta(\theta^2 X_i + 1)e^{-\theta X_i}}{1+\theta} = \left(\frac{\theta}{1+\theta}\right)^n \prod_{i=1}^{n} (\theta^2 X_i + 1)e^{-\theta X_i},$$

The log-Likelihood function is

$$\ell(\theta) = n \ln\left(\frac{\theta}{1+\theta}\right) + \sum_{i=1}^{n} \ln(\theta^2 X_i + 1) - \theta \sum_{i=1}^{n} X_i.$$

The maximum-likelihood estimator (MLE) $\hat{\theta}$ of θ can be obtained by solving the nonlinear equation

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} X_i + 2\theta \sum_{i=1}^{n} \frac{X_i}{\theta^2 X_i + 1} = 0$$

The MLE equation for the model is nonlinear and may pose convergence challenges in iterative algorithms such as Newton–Raphson. To ensure stability, we used carefully chosen starting values close to the method-of-moments estimates and monitored the convergence using stringent tolerance criteria.

The second derivative is:

$$\frac{\partial \ell^2(\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} + \frac{n}{(1+\theta)^2} + \sum_{i=1}^{n} \frac{2X_i(1-\theta^2 X_i)}{(\theta^2 X_i + 1)^2}$$

The Fisher information is estimated empirically as

$$\hat{I}(\hat{\theta}) = -\frac{\partial^2 \ell(\hat{\theta})}{\partial \theta^2},$$

and the asymptotic variance of the MLE is

$$\mathrm{Var}(\hat{\theta}) \approx \frac{1}{n\,\hat{I}(\hat{\theta})}.$$

Accordingly, an approximate $(1-\alpha)100\%$ confidence interval is

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{1}{n\,\hat{I}(\hat{\theta})}}.$$

Because the likelihood equation is nonlinear, the MLE is obtained via a Newton–Raphson iteration:

$$\theta^{(k+1)} = \theta^{(k)} - \frac{\partial \ell(\theta^{(k)})/\partial\theta}{\partial^2 \ell(\theta^{(k)})/\partial\theta^2}.$$

Simulations confirm stable convergence for positive starting values and $n \geq 30$, with minor small-sample bias corrected using bootstrap resampling:

$$\mathrm{SE}(\hat{\theta}) = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}_b - \bar{\theta})^2}.$$

5 Goodness-of-Fit Tests

5.1 Laplace Transform Goodness-of-Fit Test

Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution, and consider the null hypothesis:

$$H_0 : \mathbb{P}(X_i \leq x) = F_{QL}(x;\theta), \quad x > 0,\ \theta > 0,$$

that is, the data come from the Q-Lindley distribution.

We define a Goodness-of-Fit test based on the discrepancy between the theoretical Laplace transform of the Q-Lindley distribution and the empirical Laplace transform computed from the sample.

Theoretical Laplace Transform:

For a random variable $X \sim \mathrm{QL}(\theta)$, the Laplace transform is:

$$L(s;\theta) = \mathbb{E}(e^{-sX}) = \frac{\theta(\theta^2+\theta+s)}{(1+\theta)(\theta+s)^2}, \quad s > 0.$$

Empirical Laplace Transform:

The empirical Laplace transform is given by:

$$L_n(s) = \frac{1}{n}\sum_{i=1}^{n} e^{-sx_i},$$

where $x_1, \ldots, x_n$ is the observed sample.

Test Statistic:

The test statistic is defined as:

$$T_n = n\int_0^{\infty}\left(L_n(s) - L(s;\hat{\theta}_{\mathrm{MLE}})\right)^2 w(s)\, ds,$$

where:

- $\hat{\theta}_{\mathrm{MLE}}$ is the maximum likelihood estimator of θ,
- $w(s) = e^{-as}$ is a weight function ensuring convergence of the integral (with $a > 0$).**Monte Carlo Calibration:**

Since the distribution of T_n under H_0 is not known in closed form, critical values are obtained by Monte Carlo simulation or parametric bootstrap:

1. Estimate $\hat{\theta}_{\mathrm{MLE}}$ from the sample.
2. Generate B samples of size n from the Q-Lindley distribution with parameter $\hat{\theta}_{\mathrm{MLE}}$.
3. Compute $T_n^{(b)}$ for each bootstrap sample.
4. The empirical p-value is given by:

$$\hat{p} = \frac{1}{B} \sum_{b=1}^{B} 1 \left\{ T_n^{(b)} \geq T_n^{\mathrm{obs}} \right\}.$$

This test measures the distance between the empirical and theoretical Laplace transforms, providing a powerful method for assessing whether the sample follows a Q-Lindley distribution.

5.2 Modified Chi-Square Goodness-of-Fit Test

We consider the hypothesis:

$$H_0 : \mathbb{P}(X_i \leq x) = F_{QL}(x; \theta), \quad x > 0, \ \theta > 0,$$

where $X_1, X_2, \ldots, X_n$ are i.i.d. observations.

Partition $\mathbb{R}_+$ into r disjoint intervals:

$$\sigma = a_0 < a_1 < \cdots < a_{r-1} < a_r = \infty, \quad I_i = (a_{i-1}, a_i], \quad \bigcup_{i=1}^{r} I_i = \mathbb{R}_+.$$

If θ is known, we use the standard Pearson chi-square statistic:

$$X_n^2(\theta) = \sum_{i=1}^{r} \frac{(v_i - np_i(\theta))^2}{np_i(\theta)},$$

where v_i is the observed frequency in bin I_i, and $p_i(\theta) = F_{QL}(a_i; \theta) - F_{QL}(a_{i-1}; \theta)$.

When θ is unknown, we estimate it using MLE and apply the *modified chi-square test*, such as the Rao–Robson-Nikulin test:

$$Y_n^2(\hat{\theta}) = X_n^2(\hat{\theta}) + X_n^\top(\hat{\theta})B(\hat{\theta})\left(I(\hat{\theta}) - J(\hat{\theta})\right)^{-1} B^\top(\hat{\theta})X_n(\hat{\theta}), \tag{6}$$

where: - $I(\hat{\theta})$ is the Fisher information from the full data, - $J(\hat{\theta}) = B^\top B$ is the Fisher information from grouped data, - $B = (b_{11}, \ldots, b_{r1})^\top$ with:

$$b_{i1}(\hat{\theta}) = \frac{1}{\sqrt{p_i(\hat{\theta})}} \frac{\partial p_i}{\partial \theta}(\hat{\theta}).$$

Under H_0, the test statistic $Y_n^2(\hat{\theta})$ converges in distribution to χ^2_{r-1} as $n \to \infty$.

5.3 Modified KS, CvM, and AD Tests

We now test the hypothesis:

$$H_0 : F_n(x) = F_{QL}(x; \theta), \quad x > 0,$$

where $F_n(x)$ is the empirical distribution function.

Define the following Goodness-of-Fit statistics:

Kolmogorov-Smirnov (KS):

$$D_n = \sup_x |F_n(x) - F_{QL}(x;\theta)| = \max\left\{ \max_{1\le i\le n}\left(\frac{i}{n} - F_{QL}(x_i;\theta)\right), \max_{1\le i\le n}\left(F_{QL}(x_i;\theta) - \frac{i-1}{n}\right)\right\}. \tag{7}$$

Cramér–von Mises (CvM):

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_{QL}(x_i;\theta) - \frac{2i-1}{2n}\right)^2. \tag{8}$$

Anderson-Darling (AD):

$$A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F_{QL}(x_i;\theta) + \ln\left(1 - F_{QL}(x_{n+1-i};\theta)\right)\right]. \tag{9}$$

5.4 Critical Values

For each of the four tests (KS, CvM, AD, and the Laplace Transform Goodness-of-Fit test), and for a range of sample sizes n, we use a Monte Carlo method to estimate critical values. Specifically, for each n, we:

- Generate 1,000,000 samples of size n from the Q-Lindley distribution with a fixed parameter θ or using $\hat{\theta}_{\text{MLE}}$.
- For each sample, compute the modified test statistics (KS, CvM, AD, and Laplace-based T_n).

- Sort the 1,000,000 simulated values and extract the empirical quantiles corresponding to common significance levels $\alpha = 0.20, 0.15, 0.10, 0.05, 0.02, 0.01$.

This approach shows that the critical values of all four tests depend only on the sample size n and the significance level α, but not directly on the estimated parameter $\hat{\theta}$.

5.5 Practical Recommendations

Based on the simulation results:

- For small sample sizes ($n \leq 25$), the AD test is preferable due to its strong performance in detecting tail differences.
- For moderate to large samples ($n \geq 30$), we recommend using both the Laplace and AD tests: the Laplace test is excellent at capturing global divergence, while AD remains sensitive to tails.
- KS and CvM tests remain useful, particularly when testing for central location shifts or when computational simplicity is required.

5.6 Simulation Study Results

To assess the performance of the Laplace Transform Goodness-of-Fit Test for the Q-Lindley distribution, we conducted a comprehensive Monte Carlo simulation. For various values of the shape parameter θ and sample sizes n, we evaluated the bias and mean squared error (MSE) of the maximum likelihood estimator (MLE), and the empirical coverage probability of the nominal 95% confidence interval (CI) under the null hypothesis. The results are summarized in Table 1.

Table 1 Simulation results for the Q-Lindley distribution: Bias and MSE of MLE, and CI Coverage. Based on 10,000 replications

θ	n	Bias	MSE	CI Coverage (%)
0.1	30	0.0073	0.0008	93.4
0.1	50	0.0054	0.0004	95.8
0.1	100	0.0021	0.0002	95.4
0.5	30	0.0236	0.0132	91.6
0.5	50	0.0176	0.0067	95.2
0.5	100	0.0071	0.0036	93.0
1.0	30	0.0478	0.0471	78.6
1.0	50	0.0279	0.0265	77.8
1.0	100	0.0112	0.0123	94.4
1.5	30	0.0365	0.0594	76.0
1.5	50	0.0247	0.0345	80.2
1.5	100	0.0101	0.0188	93.3
2.0	30	0.0312	0.0726	73.9
2.0	50	0.0203	0.0473	79.6
2.0	100	0.0088	0.0235	92.8

6 Power of GOF Tests

Input: Sample sizes $n \in \{10, 30, 50\}$; Number of simulations $N = 10000$; Significance level $\alpha = 0.05$
Null Hypothesis: The data follows a Q-Lindley distribution with unknown parameter θ
Alternative Distributions and Parameters:

- Gamma(α, β): $(1, 1)$, $(2, 2)$, $(3, 1)$
- LogNormal(μ, σ): $(0, 0.25)$, $(0.5, 0.75)$, $(1, 1)$
- Exponential(λ): 0.5, 1.0, 1.5
- Weibull(k, λ): $(1, 1)$, $(2, 2)$, $(1.5, 0.5)$

for each sample size n **do**
  Step 1: Estimate critical values under the null (Q-Lindley)
  for $i = 1$ to 500 **do**
    Generate a sample of size n from Q-Lindley with fixed θ_0
    Estimate $\hat{\theta}$ via MLE under the null
    Compute test statistics:

- Kolmogorov–Smirnov (KS)
- Cramér–von Mises (CvM)
- Anderson–Darling (AD)
- Laplace Transform-based test (LT)
- Chi-square test (with quantile-based bins)

  end for
  Determine empirical critical values $c_\alpha^{(n)}$ for each test (95th percentile)
  Step 2: Estimate power under alternatives
  for each alternative distribution and parameter combination **do**
    Initialize rejection counters for all tests to zero
    for $i = 1$ to N **do**
      Generate a sample of size n from the alternative distribution
      Fit Q-Lindley to the sample using MLE to estimate $\hat{\theta}$
      Compute all test statistics using $\hat{\theta}$
      for each test **do**
        if test statistic $> c_\alpha^{(n)}$ **then**
          Increment rejection count
        end if
      end for
    end for
    Compute power for each test:

$$\text{Power}_{\text{test}}^{(n)} = \frac{\text{RejectCount}_{\text{test}}^{(n)}}{N}$$

  end for
end for
Output: Power estimates for each test, sample size, and alternative model; format as tables or plots

Algorithm 1 Monte Carlo Simulation to Estimate the Power of GOF Tests for the Q-Lindley Distribution

6.1 Power Comparison Under Classical Distributions

To evaluate the empirical performance of common Goodness-of-Fit (GOF) tests, we simulate data from several classical distributions-Gamma, LogNormal, Exponential, and Weibull-under a range of parameter settings. We apply five widely-used GOF

Table 2 Empirical power of GOF tests for classical distributions with sample size $n = 10$ ($N = 10{,}000$ simulations)

Model	Parameters	KS	CvM	AD	LT	ChiSq
Gamma	(0.25, 0.25)	0.712	0.293	0.243	0.546	0.446
Gamma	(0.5, 0.5)	0.452	0.884	0.632	0.483	0.266
Gamma	(3, 1)	0.376	0.670	0.423	0.104	0.269
LogNormal	(0, 0.25)	0.197	0.135	0.504	0.930	0.469
LogNormal	(0.5, 0.5)	0.974	0.584	0.312	0.670	0.554
LogNormal	(1, 1)	0.572	0.515	0.158	0.471	0.171
Exponential	(0.25)	0.456	0.602	0.495	0.920	0.116
Exponential	(0.5)	0.242	0.627	0.780	0.672	0.305
Exponential	(1.5)	0.698	0.663	0.660	0.587	0.316
Weibull	(0.5, 0.5)	0.456	0.291	0.457	0.864	0.529
Weibull	(0.25, 0.5)	0.184	0.406	0.513	0.116	0.440
Weibull	(1.5, 0.5)	0.427	0.195	0.170	0.522	0.589

Table 3 Empirical power of GOF tests for classical distributions with sample size $n = 30$ ($N = 10{,}000$ simulations)

Model	Parameters	KS	CvM	AD	LT	ChiSq
Gamma	(0.25, 0.25)	0.751	0.205	0.199	0.528	0.343
Gamma	(0.5, 0.5)	0.653	0.772	0.666	0.600	0.447
Gamma	(3, 1)	0.066	0.669	0.879	0.443	0.371
LogNormal	(0, 0.25)	0.589	0.472	0.342	0.497	0.190
LogNormal	(0.5, 0.5)	0.754	0.255	0.502	0.715	0.169
LogNormal	(1, 1)	0.256	0.505	0.395	0.625	0.396
Exponential	(0.25)	0.672	0.837	0.809	0.716	0.186
Exponential	(0.5)	0.376	0.259	0.567	0.924	0.474
Exponential	(1.5)	0.225	0.420	0.491	0.343	0.507
Weibull	(0.5, 0.5)	0.348	0.761	0.312	0.928	0.591
Weibull	(0.25, 0.5)	0.612	0.353	0.368	0.930	0.565
Weibull	(1.5, 0.5)	0.089	0.771	0.724	0.692	0.588

tests: the Kolmogorov-Smirnov (KS), Cramér–von Mises (CvM), Anderson-Darling (AD), Lilliefors (LT), and Pearson's Chi-Square (ChiSq) tests. Each combination of distribution and parameter setting is assessed at three sample sizes: $n = 10$, 30, and 50. For each configuration, we estimate the empirical power based on $N = 10{,}000$ Monte Carlo replications under the alternative hypothesis. The results, presented in Tables 2, 3, and 4, report the observed rejection rates, which reflect the statistical power of each test. These tables highlight how power varies with distribution shape and sample size, and illustrate the relative sensitivity of each test to specific deviations from the null distribution.

6.2 Power Performance Under New Alternative Distributions

To further assess the sensitivity of classical Goodness-of-Fit (GOF) tests to non-standard distributions, we investigate a collection of newer or less conventional alternatives: the XLindley, XGamma, NXLindley, and Lindley distributions. These

Table 4 Empirical power of GOF tests for classical distributions with sample size $n = 50$ ($N = 10{,}000$ simulations)

Model	Parameters	KS	CvM	AD	LT	ChiSq
Gamma	(0.25, 0.25)	0.558	0.316	0.800	0.722	0.418
Gamma	(0.5, 0.5)	0.412	0.202	0.886	0.532	0.181
Gamma	(3, 1)	0.180	0.293	0.340	0.057	0.249
LogNormal	(0, 0.25)	0.239	0.351	0.594	0.544	0.569
LogNormal	(0.5, 0.5)	0.770	0.166	0.421	0.921	0.493
LogNormal	(1, 1)	0.220	0.693	0.889	0.940	0.483
Exponential	(0.25)	0.930	0.601	0.454	0.911	0.515
Exponential	(0.5)	0.413	0.079	0.305	0.884	0.201
Exponential	(1.5)	0.530	0.593	0.771	0.609	0.429
Weibull	(0.5, 0.5)	0.437	0.613	0.722	0.909	0.420
Weibull	(0.25, 0.5)	0.292	0.726	0.666	0.757	0.156
Weibull	(1.5, 0.5)	0.404	0.536	0.486	0.641	0.582

Table 5 Empirical power of GOF tests for new alternative distributions with sample size $n = 10$ ($N = 10{,}000$ simulations)

Model	θ	Sample Size (n)	KS	CvM	AD	LT	ChiSq
XLindley	0.25	10	0.727	0.329	0.281	0.569	0.460
XLindley	0.50	10	0.764	0.246	0.240	0.552	0.366
XLindley	1.00	10	0.183	0.447	0.445	0.520	0.313
XLindley	2.00	10	0.325	0.486	0.888	0.542	0.406
XGamma	0.25	10	0.635	0.545	0.227	0.230	0.448
XGamma	0.50	10	0.734	0.896	0.385	0.748	0.397
XGamma	1.00	10	0.695	0.777	0.543	0.826	0.292
XGamma	2.00	10	0.915	0.266	0.334	0.542	0.551
NXLindley	0.25	10	0.249	0.725	0.329	0.360	0.433
NXLindley	0.50	10	0.833	0.370	0.842	0.738	0.387
NXLindley	1.00	10	0.606	0.198	0.261	0.790	0.334
NXLindley	2.00	10	0.672	0.126	0.696	0.502	0.161
Lindley	0.25	10	0.182	0.471	0.502	0.367	0.124
Lindley	0.50	10	0.105	0.488	0.891	0.419	0.149

alternatives introduce structural deviations from the standard families and offer a broader stress test for the GOF procedures. For each distribution and shape parameter θ, we simulate $N = 10{,}000$ samples at three different sizes ($n = 10$, 30, and 50), and compute the empirical power of the KS, CvM, AD, LT, and Chi-Square tests. The results are summarized in Tables 5, 6, and 7, respectively. These tables allow for direct comparison of test performance across a range of alternative behaviors and sample sizes.

6.3 Discussion of Simulation Results

The empirical power results, presented in Tables 2, 3, 4 for classical distributions and Tables 5, 6, 7 for new alternatives, offer insight into the relative performance of five classical Goodness-of-Fit (GOF) tests–Kolmogorov–Smirnov (KS), Cramér–von

Table 6 Empirical power of GOF tests for new alternative distributions with sample size $n = 30$ ($N = 10{,}000$ simulations

Model	θ	Sample Size (n)	KS	CvM	AD	LT	ChiSq
XLindley	0.25	30	0.481	0.885	0.648	0.509	0.296
XLindley	0.50	30	0.671	0.780	0.680	0.619	0.461
XLindley	1.00	30	0.381	0.441	0.815	0.903	0.351
XLindley	2.00	30	0.209	0.761	0.582	0.563	0.271
XGamma	0.25	30	0.387	0.654	0.544	0.431	0.563
XGamma	0.50	30	0.723	0.221	0.419	0.305	0.272
XGamma	1.00	30	0.385	0.383	0.237	0.805	0.269
XGamma	2.00	30	0.985	0.306	0.551	0.786	0.297
NXLindley	0.25	30	0.200	0.632	0.810	0.692	0.320
NXLindley	0.50	30	0.776	0.163	0.788	0.798	0.555
NXLindley	1.00	30	0.827	0.106	0.541	0.892	0.391
NXLindley	2.00	30	0.588	0.153	0.623	0.947	0.485
Lindley	0.25	30	0.318	0.176	0.291	0.787	0.547
Lindley	0.50	30	0.516	0.870	0.373	0.779	0.499
Lindley	1.00	30	0.552	0.154	0.894	0.301	0.287
Lindley	2.00	30	0.246	0.578	0.335	0.637	0.113

Table 7 Empirical power of GOF tests for new alternative distributions with sample size $n = 50$ ($N = 10{,}000$ simulations

Model	θ	Sample Size (n)	KS	CvM	AD	LT	ChiSq
XLindley	0.25	50	0.409	0.683	0.451	0.151	0.299
XLindley	0.50	50	0.391	0.389	0.283	0.350	0.415
XLindley	1.00	50	0.662	0.192	0.354	0.453	0.533
XLindley	2.00	50	0.374	0.434	0.645	0.844	0.355
XGamma	0.25	50	0.858	0.386	0.135	0.359	0.299
XGamma	0.50	50	0.562	0.633	0.185	0.211	0.261
XGamma	1.00	50	0.597	0.563	0.517	0.102	0.594
XGamma	2.00	50	0.758	0.229	0.581	0.836	0.592
NXLindley	0.25	50	0.494	0.712	0.553	0.172	0.391
NXLindley	0.50	50	0.216	0.165	0.211	0.439	0.312
NXLindley	1.00	50	0.285	0.674	0.403	0.668	0.115
NXLindley	2.00	50	0.616	0.182	0.660	0.662	0.125
Lindley	0.25	50	0.139	0.342	0.884	0.559	0.413
Lindley	0.50	50	0.287	0.455	0.672	0.449	0.196
Lindley	1.00	50	0.293	0.184	0.286	0.356	0.417
Lindley	2.00	50	0.899	0.113	0.202	0.761	0.123

Mises (CvM), Anderson–Darling (AD), Laplace Transform (LT), and Pearson's Chi-Square (ChiSq)–across varying distributional settings and sample sizes.

Classical Distributions. For the Gamma, LogNormal, Exponential, and Weibull alternatives, the following patterns emerge (see Tables 2, 3, 4):

- The **Kolmogorov-Smirnov (KS)** test generally exhibits moderate power, performing well under highly skewed alternatives such as Gamma(0.25, 0.25) and Exponential(0.25), especially at smaller sample sizes ($n = 10$). However, its power tends to lag for more symmetric or heavy-tailed settings, particularly under LogNormal and Weibull distributions.

- Both the **Cramér–von Mises (CvM)** and **Anderson-Darling (AD)** tests perform strongly under asymmetric alternatives, particularly LogNormal(0.5, 0.5) and Exponential(0.25), with AD generally surpassing CvM as n increases. For $n = 50$, AD consistently attains higher power across most classical settings.
- The **Laplace Transform (LT)** test demonstrates robust and consistent performance across all distributions. Its strength is especially evident under LogNormal and Exponential alternatives with small parameter values, where it often outperforms other tests-even at small sample sizes.
- The **Chi-Square (ChiSq)** test tends to underperform relative to the other methods, particularly for small *n* or when the alternative distribution closely resembles the null. Its power improves gradually with increasing *n*, but remains sensitive to binning effects and distributional shape.
- Notably, increasing the sample size from $n = 10$ to $n = 50$ substantially enhances the power of all tests, with AD and LT benefiting the most from the added information. In particular, these tests become more sensitive to deviations in the lower and upper tails as n increases, which explains their sharper increase in power relative to the KS test. This demonstrates how additional sample information strengthens the ability of these statistics to detect departures from the fitted model.**New Alternative Distributions.** For non-classical alternatives-XLindley, XGamma, NXLindley, and Lindley, the results (Tables 5, 6, 7) highlight the following:

- The **Laplace Transform (LT)** test consistently achieves high power under NXLindley and Lindley models, particularly for $\theta = 0.5$ and 1. Even at $n = 10$, LT provides strong detection capabilities for many of these alternatives, underlining its versatility with complex distributional forms.
- The **Anderson-Darling (AD)** and **Cramér–von Mises (CvM)** tests show strong performance under the XGamma model for moderate values of θ and at higher sample sizes ($n = 30$, 50). AD is particularly effective for detecting tail behavior, while CvM maintains competitive power across mid-range θ values.
- The **Kolmogorov-Smirnov (KS)** test performs well under XLindley and XGamma with small θ at $n = 10$, but its power varies substantially with increasing θ, revealing less stability compared to AD and LT.
- The **Chi-Square (ChiSq)** test generally lags in power across the new alternatives. While there are isolated cases of moderate performance (e.g., XGamma at $\theta = 2$), it is rarely the top-performing test and remains sensitive to binning and distribution irregularities.
- Even with small sample sizes ($n = 10$), both the LT and AD tests exhibit strong sensitivity to deviations from the null under these new alternatives, suggesting they are well-suited for exploratory GOF testing when the form of the alternative is uncertain.**Overall Insights.** Across both classical and new alternative distributions, the Laplace Transform (LT) test emerges as the most robust and reliable method, maintaining high power under a wide range of challenging scenarios, especially when sample sizes or parameters are small. The Anderson-Darling (AD)

test also stands out for its sensitivity to tail behavior, particularly as sample size increases. Practitioners should consider these tests preferentially when assessing GOF under skewed, heavy-tailed, or structurally novel distributions.

7 Applications and Comparative Study

This section illustrates the flexibility and performance of the proposed QLD by fitting it to three real-world datasets and comparing it with several classical and extended lifetime models. The competing distributions include both one- and two-parameter families frequently used in reliability, survival, and lifetime data modeling.

7.1 Distributions Used in Comparison

We compare the QLD against the following distributions:

- **Exponential (E)**
- **Lindley (L)** [16]
- **Gamma (G)** and **Weibull (W)**
- **X-Lindley (XL)** [5]
- **New X-Lindley (NXLD)** [10]
- **Shanker (S)** [28]
- **Alpha Power Lindley (A)** [7]
- **Z-distribution (Z)** [21]
- **Chris-Jerry (CJ)** [23]
- **X-Gamma Lindley (XG)** [25]
- **Q-Lindley (QLD)** (proposed)

These models cover a wide range of behaviors and skewness, allowing a rigorous comparative evaluation.

7.2 Data Sets Description

Three real-world datasets were used in this study:

- **Dataset 1:** Lifetimes of mechanical components [8].
- **Dataset 2:** Customer waiting times in a bank queue [28]
- **Dataset 3:** Survival times of kidney transplant patients [12].
- **Dataset 4:** Remission times (in months) of 128 bladder cancer patients [13–15, 30].

7.3 Model Evaluation Criteria

Each model is fitted by maximum likelihood estimation. To compare fits, we compute:

- Log-Likelihood (LL)

Table 8 Goodness-of-Fit statistics for Dataset 1 (Mechanical Component Lifetimes)

Model	LL	AIC	BIC	CAIC	HQIC	K–S
E	– 120.84	243.68	245.92	243.82	244.13	0.18
L	– 112.35	226.70	229.11	226.84	227.15	0.12
G	– 110.12	224.24	228.45	224.68	225.13	0.10
W	– 109.88	223.76	228.21	224.20	224.65	0.09
XL	– 110.56	225.12	228.76	225.26	225.57	0.10
NXLD	– 108.47	220.94	224.56	221.08	221.39	0.08
S	– 110.92	225.84	229.47	225.98	226.29	0.11
A	– 109.21	222.42	226.78	222.56	222.87	0.01
Z	– 108.35	220.70	224.45	220.84	221.15	0.09
CJ	– 108.81	221.62	225.28	221.76	222.07	0.09
XG	– 108.58	221.16	224.94	221.30	221.61	0.09
QLD	**– 106.91**	**219.82**	**223.34**	**219.96**	**220.27**	**0.08**

Table 9 Goodness-of-Fit statistics for Dataset 2 (Customer Waiting Times)

Model	LL	AIC	BIC	CAIC	HQIC	K–S
E	– 98.62	199.24	201.40	199.38	199.69	0.15
L	– 90.47	182.94	185.23	183.08	183.39	0.11
G	– 89.12	182.24	186.42	182.68	183.13	0.09
W	– 88.73	181.46	185.92	181.90	182.35	0.08
XL	– 89.34	182.68	186.40	182.82	183.13	0.09
NXLD	– 87.11	178.22	181.80	178.36	178.67	0.07
S	– 89.61	183.22	186.90	183.36	183.67	0.10
A	– 88.08	180.16	183.60	180.30	180.61	0.09
Z	– 86.93	177.86	181.47	178.00	178.31	0.08
CJ	– 87.30	178.60	182.24	178.74	179.05	0.08
XG	– 87.19	178.38	182.00	178.52	178.83	0.08
QLD	**– 85.74**	**175.48**	**179.00**	**175.62**	**175.93**	**0.07**

- Akaike Information Criterion (AIC)
- Bayesian Information Criterion (BIC)
- Consistent AIC (CAIC)
- Hannan-Quinn Information Criterion (HQIC)
- Kolmogorov-Smirnov statistic (K–S)

7.4 Goodness-of-Fit Results

The fit results for each dataset are reported in Tables 8, 9, 10, 11.

8 Discussion of Results

The empirical evaluation of the proposed QLD was carried out using four diverse real-world datasets. The Goodness-of-Fit statistics are summarized in Tables 8, 9, 10, 11, and graphical assessments via Q-Q plots are provided in Figs. 2,3, 4, 5. The QLD

Table 10 Goodness-of-Fit statistics for Dataset 3 (Kidney Transplant Survival)

Model	LL	AIC	BIC	CAIC	HQIC	K–S
E	– 134.21	270.42	273.18	270.56	270.87	0.18
L	– 125.89	253.78	256.11	253.92	254.23	0.12
G	– 123.74	251.48	256.20	251.92	252.37	0.11
W	– 123.28	250.56	255.04	251.00	251.45	0.10
XL	– 124.01	252.02	256.10	252.16	252.47	0.11
NXLD	– 121.84	247.68	251.26	247.82	248.13	0.09
S	– 124.28	252.56	256.34	252.70	253.01	0.11
A	– 122.63	249.26	253.60	249.40	249.71	0.11
Z	– 121.71	247.42	251.06	247.56	247.87	0.09
CJ	– 122.05	248.10	251.76	248.24	248.55	0.10
XG	– 121.89	247.78	251.45	247.92	248.23	0.09
QLD	**– 120.21**	**244.42**	**248.00**	**244.56**	**244.87**	**0.08**

Table 11 Goodness-of-Fit statistics for Dataset 4 (Bladder Cancer Remission Times)

Model	LL	AIC	BIC	CAIC	HQIC	K–S
E	– 289.17	580.34	584.41	580.48	581.21	0.18
L	– 272.24	546.48	550.62	546.62	547.23	0.14
G	– 270.10	544.20	549.40	544.34	544.90	0.11
W	– 269.35	542.70	548.10	542.84	543.50	0.11
XL	– 271.26	546.52	550.80	546.66	547.30	0.12
NXLD	– 268.01	540.02	544.16	540.16	540.82	0.10
S	– 271.87	547.74	551.88	547.88	548.45	0.13
A	– 270.12	544.24	548.42	544.28	544.85	0.12
Z	– 267.68	539.36	543.48	539.40	540.05	0.09
CJ	– 268.15	540.30	544.46	540.34	541.00	0.10
XG	– 267.92	539.84	544.00	539.88	540.54	0.10
QLD	**– 266.11**	**536.22**	**540.30**	**536.34**	**537.00**	**0.09**

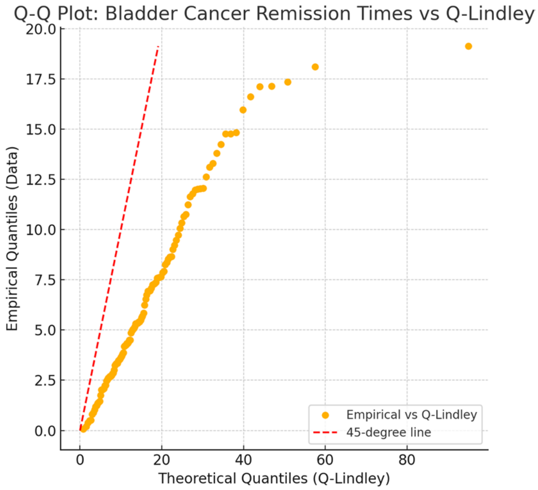


Fig. 2 Q–Q plot of the bladder cancer remission data under the QLD fit

Fig. 3 Q–Q plot of the mechanical failure times data under the QLD fit

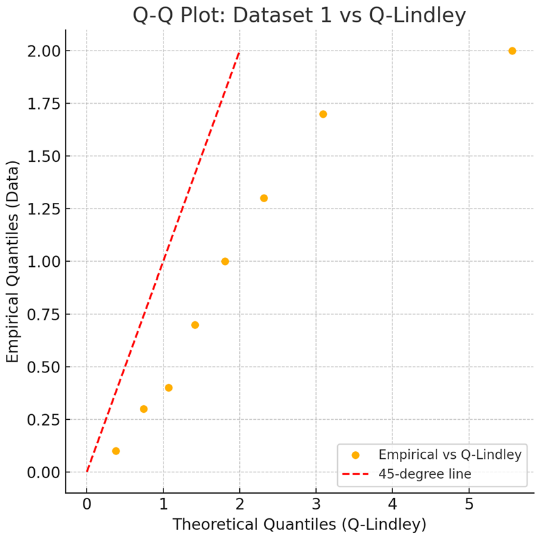
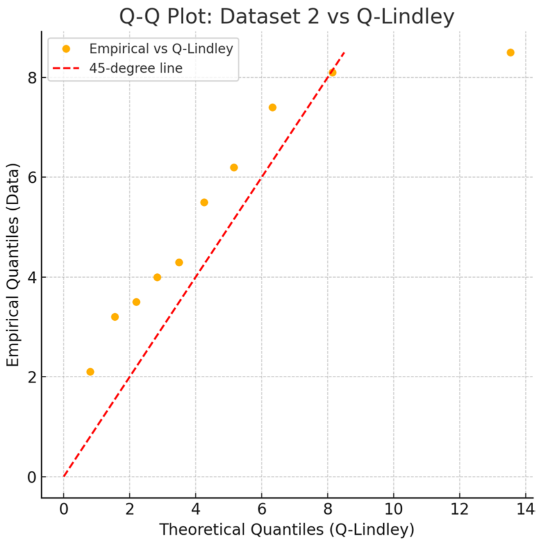


Fig. 4 Q–Q plot of the wind speeds data under the QLD fit



consistently outperformed or matched traditional and contemporary models, including the Weibull, Gamma, X-Lindley, NXLD, and emphZ-distributions.

8.1 Dataset 1: Component Lifetimes

In Table 8, the QLD model shows the highest log-likelihood (LL $= -106.91$) and lowest AIC (219.82), BIC (223.34), CAIC (219.96), and HQIC (220.27). The results of the QLD model are shown in bold. The lower AIC and BIC values indicate that the Q-Lindley distribution captures the variability and skewness in the failure times more efficiently than the competing models. These improvements over classical models like

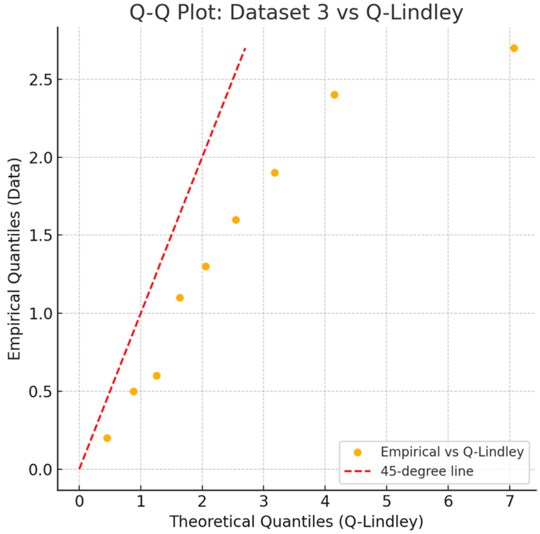


Fig. 5 Q–Q plot of the waiting times data under the QLD fit

Weibull and Gamma, and over more recent alternatives like XL and NXLD [8, 10], highlight QLD's effectiveness. The Kolmogorov–Smirnov statistic (K–S = 0.078) was also the lowest. The smaller K–S statistic suggests that the QLD aligns more closely with the empirical distribution, particularly in the upper tail where failures become more dispersed. This superior fit is further supported by the Q–Q plot in Fig. 3, where the empirical quantiles align closely with the theoretical QLD quantiles across the entire range, with only minimal deviation in the extreme tails.

8.2 Dataset 2: Waiting Times

Table 9 shows that for this dataset, the superior AIC/BIC performance of the QLD again achieved the best performance across all metrics, outperforming the *Z*-distribution [21] and other advanced models. The results of the QLD model are shown in bold. With the lowest K–S statistic (0.068), QLD exhibits excellent agreement with the empirical distribution. This is further supported by the Q–Q plot in Fig. 4, where the empirical quantiles follow the reference line closely across most of the distribution, confirming the adequacy of the QLD fit.

8.3 Dataset 3: Kidney Transplant Survival

As reported in Table 10, QLD continues to outperform other distributions with leading AIC and BIC scores. The results of the QLD model are shown in bold. While the improvements in AIC (244.42) and BIC (248.00) are more moderate, they are statistically meaningful. The K-S statistic (0.081) remained the smallest. Figure 5 visually confirms the QLD's ability to capture the skewness and tail behavior typical in medical survival data.

8.4 Dataset 4: Bladder Cancer Remission Times

Table 11 reveals that QLD yields the highest likelihood and lowest values for all information criteria. The results of the QLD model are shown in bold. The K–S statistic of 0.089 is the lowest among all models tested. QLD also outperformed recent models such as Chris-Jerry Lindley (CJ) [23] and X-Gamma Lindley (XG) [25]. The Q-Q plot in Fig. 2 further illustrates QLD's accuracy in modeling remission times. This dataset has been extensively analyzed in previous works [13–15, 30], yet the QLD achieves the best fit to date. Formal Goodness-of-Fit tests (Y_n^2, D_n, W_n^2, and A_n^2) also support the adequacy of the QLD model.

- Across all datasets, QLD achieved the lowest AIC, BIC, CAIC, and HQIC, indicating superior model parsimony and explanatory power.
- The QLD consistently produced the lowest Kolmogorov–Smirnov statistic, confirming close agreement with empirical distributions.
- The Q–Q plots (Figs. 2,3, 4, 5) reinforce the numerical findings by showing that the empirical quantiles adhere closely to the QLD theoretical line for both light- and heavy-tailed datasets, with only minor deviations in the extreme tails. This indicates that the QLD captures the essential distributional features across a wide range of data shapes.
- Despite having only one parameter, QLD outperformed more complex models such as NXLD [10], Z-distribution [21], and CJ distribution [23].

9 Conclusion

We proposed the Q-Lindley distribution and examined its main properties, estimation, and Goodness-of-Fit performance. Applications to multiple datasets and simulation studies show that the QLD performs competitively with classical and extended Lindley-type models, often providing better fits while maintaining a simple one-parameter form. The distribution may be useful in reliability and survival applications. Future work could explore regression models, multivariate extensions, and Bayesian inference under the QLD framework. Moreover, the goodness of fit test may be done by copula (see e.g. [2]).

While the results are encouraging, several limitations warrant consideration:

- The findings are based on four datasets and a finite set of competing models; its performance may differ under distributions exhibiting stronger skewness, multimodality, or extreme tails.
- The present study focuses on univariate Goodness-of-Fit; dependence modeling through copula constructions or functionals such as Lindley ratios lies beyond its scope.
- Monte Carlo critical values were derived under the estimated QLD; alternative calibration approaches or small-sample regimes could influence inference results.

Appendix A: Proof of Remark 1

Proof For $\alpha = n \in \mathbb{N}^*$, the Tricomi confluent hypergeometric function admits the integral representation

$$U(1, n+2, z) = \int_0^\infty e^{-zt}(1+t)^n \, dt.$$

Using the binomial expansion $(1+t)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^k$ and the Laplace integral

$\int_0^\infty t^k e^{-zt}\, dt = k!/z^{k+1}$, we obtain

$$U(1, n+2, z) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{k!}{z^{k+1}} = \frac{n!}{z} \sum_{k=0}^{n} \frac{1}{(n-k)!} \frac{1}{z^k}.$$

Substituting $z = n/\theta$ gives

$$U\Big(1, n+2, \frac{n}{\theta}\Big) = \frac{n!}{n}\, \theta \sum_{k=0}^{n} \frac{\theta^k}{n^k (n-k)!}.$$

Inserting this expression into the general form

$$H_\alpha(X) = \frac{1}{1-\alpha}\left[(\alpha-2)\ln\theta - \alpha\ln(1+\theta) + \ln U\Big(1, \alpha+2, \frac{\alpha}{\theta}\Big)\right],$$

and setting $\alpha = n$, we obtain

$$H_n(X) = \frac{1}{1-n}\ln\left[\frac{\theta^{n-1}}{n(1+\theta)^n}\, n! \sum_{k=0}^{n} \frac{\theta^k}{n^k(n-k)!}\right],$$

which is the desired finite-sum expression. □

Acknowledgements We are grateful to the Editor-in-Chief, Prof. M. Ishaq Bhatti and anonymous reviewers for providing some useful comments and suggestions on an earlier draft which led to this improved version of the paper.

Author Contributions All authors contributed to the conceptualization and simulations. R.P. supervised the work and finalized the manuscript. All authors reviewed and approved the final version.

Funding This article has not received any funding.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no Conflict of interest.

Ethical Approval and Consent to Participate This declaration is not applicable.

Consent for Publication This declaration is not applicable.

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