



From pre-ritual to exploration: young learner's gestural routine development in manipulative-based number discourse

Ying Zhang² · Oi-Lam Ng¹  · Biyao Liang²

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Abstract

This study characterises a number discourse developed between a 6-year-old child and an expert interlocutor (the researcher) in manipulative-based contexts from a commognitive perspective. Specifically, we focus on the child's development of counting and addition discourses as she engaged in a series of task situations that used the manipulatives of M&M candies and play coins. Adopting a multimodal perspective of discourse, we demonstrate the routine development of the child that changed from ritualistic to explorative routines. Furthermore, this study extends current research on routine development by proposing a pre-ritual phase, which is characterised by commognitive conflicts the learner encountered before the emergence of a ritualistic routine. Although the pre-ritual phase brings about challenges in communication, their resolution plays an essential role in engendering the child's learning. Implications for cultivating explorative participation in early arithmetic teaching and learning with manipulatives are also discussed.

Keywords Commognition · Manipulative · Early number learning · Primary mathematics education · Arithmetic · Counting · Addition · Commognitive conflicts

1 Introduction

Children develop numerical competencies in the early years of life, often starting before and outside formal schooling through play and other informal activities—a phenomenon that was not widely recognised by early mathematics educators until the late half of the twentieth century (Balfanz, 1999; Gelman & Gallistel, 1986; Ginsburg, 2006). Researchers in cognitive science have suggested that children as early as infants may possess innate number sense, such as subitising small numbers of items before explicit instruction (Starkey & Cooper, 1980). As children grow and interact with the

✉ Oi-Lam Ng
oilamn@cuhk.edu.hk

¹ Faculty of Education, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong SAR, China

² Faculty of Education, The University of Hong Kong, Pok Fu Lam, Hong Kong Island, Hong Kong SAR, China

environment, they become increasingly fluent working with numbers, from the sequence of numerals, cardinality with one-to-one correspondence, to arithmetic operations and problem-solving (Fuson, 1988). Along the trajectory of early number learning, arithmetic has received particular attention for its transitional role in prompting children to take up strategies ranging from basic counting procedures to the use of more structured number combinations (Clement & Sarama, 2014; Kullberg et al., 2024).

When initially encountered with simple addition problems, children have been found to primarily rely on two strategies, *counting-all* and *counting-on* (Carpenter et al., 1981; Secada et al., 1983). For the former, children determine the sum by denoting each addend with corresponding numbers of physical objects or fingers through counting, combining the two sets, and then counting the total number of the union set starting from one. This strategy is foundational as it establishes a one-to-one correspondence between each object and a number word. By contrast, children who employ a counting-on strategy start by counting from one of the addends and continue until they have enumerated the quantity of the second addend. This strategy is considered more advanced than the counting-all strategy, since the first addend is perceived by children as not only a cardinal amount but also part of a larger whole simultaneously (Clement & Sarama, 2014; Fuson & Secada, 1986). Although the counting-based strategies in early addition have been widely studied from a cognitivist or developmental standpoint (e.g., Baroody, 1987; Carpenter & Moser, 1984; Fuson, 1982; Secada et al., 1983), few studies have drawn upon a discursive approach to examine the evolving nature of arithmetic learning as it unfolds *in and through* communication (Sfard, 2008), which we believe to be crucial for further understanding early arithmetic thinking.

Sfard's (2008) discursive framework challenged the duality of thinking and communicating in mathematical activities, emphasising the inseparable nature of cognition and communication within mathematical discourse. Since this seminal work, researchers have taken up Sfard's discursive approach to examine mathematics learning as changes in one's mathematical discourse, characterised by specific *word use*, *visual mediators*, *routines*, and *narratives* used (see, for example, a recent special issue, "Advances in Commognitive Research", edited by Cooper & Kontorovich, 2022). From this viewpoint, physical manipulatives can be considered a form of (dynamic) *visual mediators* (Ng, 2016), realising mathematical meanings as they are moved from one location to another. In addition, as the child repeatedly acts upon objects, the actions become increasingly mathematical and can be taken as (gestural) *routines* associated with certain mathematical procedures. The significance of routine development in tool-based learning was highlighted by Sinclair and Moss (2012), who exemplified how dynamic geometry environments support the gradual development of children's geometric thinking as their routines for identifying shapes shift from simple visual recognition to discursively mediated identification. Other scholars have also delved into the genesis of routines from ritualistic to explorative, as well as teachers' roles in supporting children's development thereof (e.g., Christiansen et al., 2023; Lavie et al., 2019; Nachlieli & Tabach, 2019).

We consider that a discursive approach can offer valuable insights into the role played by manipulative-based contexts, where both children's verbal words and non-verbal actions (e.g., gestures) are mobilised for reasoning with mathematics through real-world

perceptual experiences (Goldin-Meadow et al., 2009; Gunderson et al., 2015). While children's manipulative learning has been regarded as central to studying early numeracy development (e.g., Boggan et al., 2010; Laski et al., 2015; McGuire et al., 2012), extant literature has yet to fully address the potentials of the context itself sensitive to different types of manipulatives, and the mediations afforded by the act of manipulation elicited within such contexts. Therefore, this study aims to fill the gaps mentioned above, by examining how early arithmetic discourses develop, in two distinct manipulative-based contexts, with M&M candies and play coins.

Taking a discursive lens, our investigation is twofold: First, we draw chronological attention to how learning occurs and discourse develops by observing the change in routines. Second, multimodal mediators such as the interlocutors' utterances and gestures are analysed to gain insights into how different manipulatives facilitate the mathematical discourse.

2 Theoretical framework

Sfard (2008) considered thinking a form of communication with others or oneself. Her commognitive framework sets mathematical discourse apart from other human communication by specifying its unique features, which we will elaborate on in the following subsections.

2.1 Thinking as communicating

Sfard (2008) challenged mind–behaviour dualism and proposed a non-dualist view by unifying thinking and communicating under the term *commognition*. In her book, Sfard (2008) defined thinking as the “individualised version of interpersonal communication” (p. 81), and as such, communicating is regarded as the act of thinking per se rather than a separate entity externalising internal thinking. Sfard (2008) pointed out four features of mathematical discourse: *word use* (keywords used with discipline-specific content, e.g., square, calculus); *visual mediators* (visible objects that can be seen or acted upon for facilitating communication, e.g., graphs, concrete objects); *routines* (repetitive patterns that characterise the form of discursive discourse); and *narratives* (axioms, definitions, and theorems that can be endorsed as true according to well-defined rules, which are the goals of mathematical discourse, e.g., $5 + 2 = 7$). Moreover, mathematical discourse entails signifier-realisation pairs, each of which can be translated into the other. While *signifiers* refer to symbols or words mentioned in discourse to denote a mathematical object, *realisations* of a signifier are perceptually accessible entities that can be operated upon for interlocutors constructing narratives about the signifier.

With the commognitive underpinning, our investigation of mathematical thinking becomes a study of development in discourse (Sfard, 2008). In essence, there is a contradiction between one's desires for self-expression and for mutual understanding with others, which challenges the effectiveness of communication. Specifically, the mismatch between individualisation and negotiation-based communication drives interlocutors to constantly

modify their behaviours in a shared effort to establish a discourse community, during which *commognitive conflicts* are likely to be produced. According to Sfard (2008), commognitive conflict refers to “the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules” (p. 161). Despite the incommensurable nature, Sfard (2007, 2008) argued that commognitive conflicts are a crucial impetus for triggering discourse change. Specifically, being exposed to commognitive conflicts may engender students to adjust previous discourse and establish new ones by individualising discursive rules of others. For this study, we pay particular attention to whether and how the learner may undergo commognitive conflicts in arithmetic learning, and seek to map the conflict, if any, onto the continuum of the learner’s routine development that we introduce in the following subsection.

2.2 De-ritualised routines: the gist of learning

A routine is a set of patterned actions that occur repeatedly in certain situations, which forms the discourse configuration and the basic unit of mathematics learning (Sfard, 2008). Research on routinised mathematical discourse has characterised two types of routines, *ritual* and *exploration* (e.g., Christiansen et al., 2023; Heyd-Metzuyanim et al., 2016; Viirman & Nardi, 2019). *Rituals* are routines generated by complying with the rules of others (e.g., mimicking teacher actions) without recognising the intention of those behaviours. *Explorations* are routines aimed at constructing or substantiating an endorsed mathematical narrative independently. The former is motivated by complying with social norms, while the latter is driven by the desire to know about the (mathematical) world for the sake of oneself.

The study of *ritualisation* allows us to investigate children’s mathematical thinking and learning by examining shifts in the types of routines performed. This progression is often characterised by a shift from ritualistic to explorative routines, a process referred to as *de-ritualisation* of routine development (Lavie et al., 2019). An example of de-ritualisation in the context of early number learning can be seen in a study by Sinclair and Heyd-Metzuyanim (2014), who investigated two 5-year-old children’s embodied interaction with TouchCounts, a touchscreen-based application designed for supporting early number sense. In one case, a child initially formed her *ritualistic* routine of skip-counting by five through imitating the researcher’s pinching gestures, and over time, it progressed into the *explorative routine* for successive addition of five in a self-initiated manner for reasoning with numbers.

Lavie et al. (2019) operationalised routines as task–procedure pairs; that is, each routine constitutes a *task* situation interpreted by the discussant—regardless of how the task is articulated by others—and the corresponding *procedure* in dealing with the interpreted *task*. Identifying the task–procedure pair underlying a routine is necessary for scrutinising *when* a routine is likely to be evoked (under which interpreted task) and *how* the routine is enacted (with what procedure performed). With this task–procedure construct, we are able to observe the existence of routines through repeated actions across similar task situations, and in turn to trace the trajectory of one’s routine development by detecting any changes in their recurrent actions. Meanwhile, the construct can be used to analyse the existence of

commognitive conflicts or *contextual miscommunication* that arise from interlocutors interpreting the task differently. This exemplifies the value of a discursive lens instead of a deficit lens, and the need to look through children's eyes by speculating on how children participate in mathematical discourse with their interpretation of the *task* underlying their *procedures* performed (Lavie et al., 2019; Sfard, 2008).

Furthermore, Lavie et al. (2019) characterised routines along six dimensions, among which the performer's *agentivity* is selected as the focus of this study when examining de-ritualised routines. In particular, agentivity refers to the extent to which learners independently decide what actions to take and when, without external assistance from others (Christiansen et al., 2023; Lavie et al., 2019; Nachlieli & Tabach, 2022), which offers a lens for us to foreground students' autonomy in mathematical discourse. In summary, the task-procedure construct and agentivity of routines allow us to adopt a child-centred perspective and avoid the pitfall of discussing what students are *not able to do* by bringing attention to *what they actually do and why* when they engage in the given task (Baccaglioni-Frank, 2021; Sfard & Lavie, 2005). Guided by this perspective, we attend to *what* discursive patterns develop—and *how* they develop—throughout the discourse between a researcher-child dyad, which we investigate through the following research questions (RQs):

1. How are ritualistic routines developed between a researcher-child dyad, and how do they evolve into explorative routines individualised by the child?
2. What is the role of manipulatives (M&M candies and play coins) in the development of the young learner's counting-addition discourses?

3 Methodology

3.1 Context and participants

The data source comes from a larger research project which examined children's mathematical communication during teaching interventions in various learning environments (e.g., physical manipulatives and screen-based applications such as TouchCounts, TouchTimes, and GeoGebra). Taking place in Hong Kong, the researchers conducted nine 1.25-h sessions of mathematics lessons with eight children at a local community centre over nine weekends. Each week, the sessions were allocated at three separate time slots for two to four children each. At each session, three researchers worked with one or two children at different tables in the same room. In this study, we report on one of the eight participants, a first-grader, Susan (pseudonym), who was aged six at the time of the study and is ethnic Pakistani. We chose to analyse Susan's discourse because her attendance at the sessions was the highest among participants, and she showed significant learning progress over the sessions. All children provided parental consent to participate in the study.

Given the nature of the informal learning setting and the qualitative nature of the study, the teaching intervention took the form of semi-structured, task-based teaching sessions (Goldin, 2000; Hunting, 1997). Our task design was informed by the following considerations. First, they generally began with elementary and open-ended tasks

and progressed to more challenging and task-specific ones, for we intended to assess children's baseline knowledge and the evolution of their mathematical thinking. Second, tasks were designed to foster *explorative participation* (Baccaglioni-Frank, 2021), thereby constructing a meaningful narrative about numbers; this was achieved in multimodal environments by inviting children to tinker with physical manipulatives or digital tools. Third, tasks were designed to focus on competence beyond arithmetic operations. For example, we designed tasks with physical manipulatives to inquire into children's developing discourse as a *counter* (able to count and read the last number), *producer* (able to produce a set with a given), and *counter from any number* (able to count starting from any number other than one; see Clements and Samara, 2014). The sessions were videotaped to capture participants' and researchers' verbal expressions and bodily movements, and were transcribed for data analysis.

3.2 The task

In response to the research questions, we selected and analysed the session with Susan around counting and arithmetic tasks. The task involved two kinds of concrete objects, M&M candies and play coins, and we considered them to be conceptually different, with the former mediating one-to-one correspondence and the latter representing multiples (a \$5 coin = 5*\$1 coin) in arithmetic thinking. In the first part, Susan was engaged in the following tasks:

- (1) counting and redividing 10 and 18 M&Ms for two dolls in different ways;
- (2) figuring out how many M&Ms each person gets if sharing 20 M&Ms to 4 and 21 M&Ms to 3.

During the second part, we involved Susan in the following tasks:

- (1) comparing the value of coins (10c, 20c, 50c, \$1, \$2, \$5, \$10);
- (2) using a given set of coins to make \$10 in different ways;
- (3) figuring out how much she should pay for 5 packets of M&Ms, each of which costs \$3, and how much change she would get from \$20.

The semi-structured interview protocol prompted Susan to explain her thinking by asking, "What did you do?" "How do you know that?" and "Why is that?" and posed further questions to advance her thinking.

3.3 Method of data analyses

The research team first transcribed the session using MAXQDA, a qualitative analysis software. The transcription convention followed that of Ng (2016), which captured and organised simultaneous spoken words and gestures within the dyad's communication in a tool-based environment. Next, taking Sfard's commognitive framework, we iteratively watched and discussed the video as a team and generated several rounds of field notes. In

the initial round, the team coded any communicative acts via speech and gestures in the researcher–child dyad. In subsequent rounds of discussions, the goal was to unpack the mathematical meanings associated with communicational acts, as well as the mathematical significance of procedures or routines enacted, in the multimodal forms of speech, gestures, and actions with manipulatives in the mathematical discourse.

Following Arzarello (2006), we performed a *synchronic* multimodal analysis of relationships among participants' in-the-moment gestures and utterances simultaneously activated during the task; additionally, we conducted a *diachronic* analysis to reveal the evolution of researcher–child communication over time, taking the routine—along with its level of agentivity—as the basic unit of analyses. Based on the aforementioned definition of agentivity, we labelled routines as exhibiting *low agentivity* when the child followed instructions or relied on assistance from others, and as entailing *high agentivity* when the child independently made decisions and initiated actions without external guidance during the task. Together, the analyses allowed us to foreground how the child's routines developed with various visual mediators over time.

4 Findings








In this section, we provide fine-grained accounts of Susan's mathematical discourse from four interview episodes, selected for their analytical relevance, in the sense that they offer particularly rich instances of discourse where shifts in Susan's routines were observed. In doing so, we illustrate the process of de-ritualisation, comprising three routines presented through four chronological episodes. Episode 1 depicts Susan's routine 1 (ritual) for *counting-from-any-number-with-M&M*. Episode 3 demonstrates the emergence of routine 2 (ritual) for *adding-coin-values-by-counting* upon resolving commognitive conflicts that Susan encountered in episode 2 (pre-ritual). Episode 4 exhibits Susan's routine 3 (exploration) that coordinates key features from routines 1 and 2. For each routine, we determine whether it is ritualistic or explorative according to the underlying *task-procedure* pair and characterise it by the degree of the learner's *agentivity*.

4.1 Routine 1: counting-from-any-number-with-M&M

During the baseline assessment, we noted that Susan could proficiently count the cardinality of small sets but started losing track of M&Ms when dealing with more than 30 of them. Every time she lost track, she tended to recount from 1.

4.1.1 Episode 1

Episode 1 captures how Susan produced a ritualistic routine, namely R1 (*counting-from-any-number*), upon resolving a contextual miscommunication with the researcher. The dyad was engaged in a counting task with two sets of M&Ms, each containing 7 and 11 candies.

Turn	Speaker	Gesturer	What was said <what was done>	Figure 1
1	R	R	This is what? <R took out a yellow M&M from the pile of 11 M&Ms, and moved it back and forth between the two sets>	 (a)
2	S	R and S	One? < S pointed to the candy; R continued with the <i>moving-gesture</i> to incorporate the M&M into the pile of 7>	 (b)
3	R	R	It's one? Okay. But this is seven, <R used one hand to cover the collection of 7 M&Ms> in here already. <R traced circles in the air around the set>	 (c)  (d)
4	R	R	So, this is —? <R used <i>moving-gestures</i> with the same candy>	 (e)
5	S		Eight!	
6	R	R	Yes! Do it. Go ahead. <R made <i>moving-gestures</i> back and forth in the air>	 (f)
7	S	S	Nine, <S moved a candy> ten, <S moved a candy> eleven, <S moved a candy> [so on and so forth] eighteen. <S moved a candy>	 (g)

S Susan, R Researcher, “?” rising intonation, “/” resolute intonation

Turns 1–5 Procedure: In turn 1, the researcher enacted *moving-gestures* upon the M&M candy and asked Susan what number it denoted. In response, Susan pointed to the M&M with her finger and uttered “one” with a rising intonation, showing a degree of uncertainty. The researcher then clarified her intention by realising the set of seven M&Ms as a unit using the number word “seven” and the spatial–temporal expression “in here already,” accompanied by deictic gestures (Figure 1c and 1d shown in the table above). Following that, in response to the researcher’s repeated question, Susan immediately answered “eight” with a resolute tone—even before the researcher had finished asking the question in turn 4.

Task: The above procedures in turns 1, 3, and 4 indicate that the researcher’s intended task was to prompt Susan to *count from the set of seven M&Ms*. However, Susan initially appeared to interpret the task as *counting the number of M&Ms being moved*, which can be inferred from her response “one” in synchrony with her pointing to the M&M that had been moved (turn 2). At that point, we suggest that a *contextual miscommunication* occurred due to the interlocutors’ different interpretations of the task situation. This was subsequently resolved through the researcher’s procedures in turn 3, where she used spoken words and gestural mediators, including palm-covering (Figure 1c) and circle-making gestures (Figure 1d) to clarify the idea of grouping and realise the signifier “seven” as the set of M&Ms to also be counted.

The resolution of the contextual miscommunication indicates that Susan had aligned her task interpretation with that of the researcher, as evidenced by the change in her response from “one” (turn 2) to “eight” (turn 5). Moreover, this change suggests that she had recognised the group of seven M&Ms as a unit and that the mathematical meaning realised through the movement of the same M&M had shifted from *counting from one* to *counting from any number*. Hence, it is fair to say that, with the use of gestural mediators, the mathematical object being communicated was eventually agreed upon within the discourse community.

Turns 6–7 Procedure: Upon resolving the contextual miscommunication, the researcher encouraged Susan to count on by demonstrating a kind of *moving-gesture* back and forth in the air alongside verbal prompts (turn 6). Subsequently, Susan recalled and mimicked the researcher’s *moving-gestures* ten times in about ten seconds, each time moving one M&M while uttering the corresponding number from nine to eighteen (turn 7).

Task: According to the procedures described above, we suggest that the task situation was mutually unified between the interlocutors as *counting from the set of seven M&Ms*.


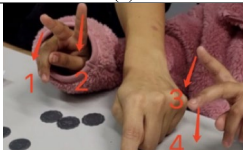


Routine: Our analysis evidenced the emergence of Susan’s ritualistic R1 for *counting-from-any-number-with-M&M*, which comprised the task to count from any number and the procedure of performing a *moving-gesture* with each M&M. Specifically, Susan adopted a discursive pattern with each of Susan’s ten repeated *moving-gestures*, driven by her interpreted task of counting from the last number she had just uttered (another recurrence of this task–procedure pair can be found in episode 4). This routine could be interpreted as a ritual entailing a low level of agentivity, as its occurrence heavily relied on Susan replicating the researcher’s modelled actions upon following the researcher’s command.

4.2 Routine 2: adding-coin-values-by-counting

In this section, we detail two episodes that demonstrate how Susan's second ritualistic routine, R2, used for adding values of multiple coins, took place with the help of the researcher. In episode 2, Susan went through a *pre-ritual* phase, characterised by a com-cognitive conflict when she encountered a new discourse presented by the researcher. Then, episode 3 shows how the interlocutors resolved this conflict, marking the end of the pre-ritual phase and the emergence of R2.

4.2.1 Episode 2

Before this episode, we observed that Susan had been able to effortlessly make 10 with M&Ms, such as $2 + 8$ or $3 + 7$, within seconds. However, when changed to coin-based environments, the "making 10" task proved challenging for her to complete. In the task of determining the total value of four \$2 coins, Susan initially answered "six," which prompted the researcher's intention to revisit this task with her.

Turn	Speaker	Gesturer	What was said <what was done>	Figure 2
8	R	<u>R</u> and S	Do it again. I will show you. [...] <u>Two</u> . <R moved over one \$2 coin; S lifted four fingers>	 (a)
9	S	R and <u>S</u>	<u>One, two, three, four</u> . <S tapped the four fingers in the air one by one (b)> <u>Five, six</u> . <S then lifted two more fingers (c); R moved over \$2 coin>	 (b)  (c)
10	R	R and S	<u>Two</u> . <S continued to hold the six fingers up; R moved over \$2 coin>	
11	R	<u>R</u>	Six? <u>Another two</u> ? <R moved over the fourth \$2 coin>	 (d)
12	S		Six? Four?	

Transcripts underlined Utterances synchronising with gestures of the same person, "... " time lapse, *italicised text* overlapping communication with the other interlocutor

Procedure: In this episode, the researcher was moving the \$2 coins while uttering “two” to indicate the value of each coin, with her speech overlapping with Susan’s (see italicised texts in turns 8 to 10). For Susan, it is observed that she counted six fingers in synchrony with the researcher’s moving of \$2 coins (turns 8 to 10) and lost track of the fourth coin (turns 11 and 12). Specifically, we note that Susan first signified the value of two \$2 coins by raising four fingers and counting them from one to four by tapping her fingertips successively in the air (Figure 2a–b shown in the table above). Then, Susan counted on from four—“five, six”—while synchronising this with raising two more fingers to denote the third \$2 coin, arriving at the answer “six” (Figure 2c).

Task: The above procedures indicate that the researcher attempted to probe how Susan might respond to the task of calculating the total value of four \$2 coins. For Susan, we infer from the last number she uttered—“six” in turn 9—produced upon her raising four and then two fingers, that her interpretation of the task was aligned with the researcher’s, that is, to determine the total value (how much) rather than the number of coins (how many).





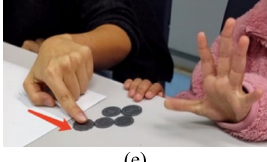

However, although Susan’s procedures seemed to work to identify the total values of the first three coins, they were enacted asynchronously with the researcher. The researcher’s rising intonations with confusion and Susan’s uncertain guesses of “six? four?” in turns 11 and 12 provided further evidence of their incommensurable discourse. The commognitive conflict was also evident in the different mathematical meanings associated with the interlocutors’ use of the same number word, “two.” That is, the researcher uttered “two” to denote the value of each coin being added, which was visually mediated by coin movements. In contrast, Susan’s “two” (turn 9) referred to the second step in her one-to-one counting sequence (“1, 2, 3, 4, 5, 6”), which was visually mediated by her finger gestures.

Routine: From the observation that Susan counted by ones (the value of an individual unit) instead of by twos (the value of each coin), we conjecture that Susan spontaneously enacted a mixed strategy of *counting all* (to get “four”) and *counting on* (to get “six”) as her routines. Susan relied on her familiar counting routine based on one-to-one correspondence that treated each \$2 coin as comprising two separate units, evidenced by her raising of two fingers at a time. Yet, it is evident that Susan kept track of only three \$2 coins but not the fourth (turns 11 and 12), suggesting her unfamiliarity with the intermediate discourse between adding and counting in the coin-based context. Although the coin environment provides a natural setting for early addition, it also poses challenges for Susan in taking care of multiple quantities simultaneously. As shown in Figure 2a–c, Susan repeated the procedure of displaying the value of each \$2 coin with two fingers three times, after which she seemed to lose track of the number of twos and became confused towards the end of the episode.

To conceptualise the connection between commognitive conflict and routine development, we refer to this transitional phase as the *pre-ritual*: a fuzzy, germinating stage where the learner’s existing discourse (counting) is confronted by a new discourse (adding). During the pre-ritual phase, the learner’s familiar routine, while functional in prior contexts (e.g., M&M), becomes increasingly insufficient for the discursive demands of the new task, thereby giving rise to commognitive conflicts. In this sense, the pre-ritual stage is foundational, as it marks the learner’s initial exposure to a new discursive rule and precedes the emergence of the relevant ritual (R2, as detailed below), provided that the discursive tension is successfully addressed.

4.2.2 Episode 3

This episode illustrates how Susan's routine (R2) for *adding-coin-values-by-counting* comes into being upon passing through the pre-ritual stage, marked by the resolution of the commognitive conflict. In this instance, the dyad was discussing the total value for one \$5 coin, three \$1 coins, and one \$2 coin.

Turn	Speaker	Gestureur	What was said <what was done>	Figure 3
13	R	<u>R</u> and S	<u>Try with the five first. Show me five.</u> <R touched a \$5 coin and instructed S to raise five fingers of the right hand>	 (a)
14	R	<u>R</u> and S	<u>Show me one.</u> <R performed a <i>moving-gesture</i> with a \$1 coin; S retracted her right hand and placed it on the edge of the table, and <i>raised a finger of another hand</i> >	 (b)
			<u>One more.</u> <R performed a <i>moving-gesture</i> with another \$1 coin; S <i>raised another finger</i> >	 (c)
			<u>One more.</u> <R performed a <i>moving-gesture</i> with another \$1 coin; S <i>raised another finger</i> >	 (d)
15	R	<u>R</u> and S	<u>Two more.</u> <R performed a <i>moving-gesture</i> with a \$2 coin; S <i>lifted two more fingers</i> >	 (e)
			<u>So how much is these?</u> <R drew circles in the air around all five coins; S kept lifting the fingers>	 (f)
16	S		Ten.	

Procedure: The researcher first instructed Susan to show five fingers to signify the value of the \$5 coin (turn 13). Susan then retracted her right hand and placed it on the edge of the table, which we interpret as signifying the addend of 5 that had already been included (Figure 3b–f shown in the table above); this interpretation is based on her subsequent answer of “ten” in turn 16. Following that, each time the researcher performed the procedure of moving each of the remaining coins while uttering the corresponding number word, Susan would subsequently take her procedure of displaying the coin value with fingers. As a result, when the researcher asked, “How much is these?”, Susan looked at her raised fingers and confidently uttered “ten” without hesitation (turns 15–16), marking the successful completion of the addition task.

Task: From the procedures presented above, we infer that the researcher helped Susan decompose the task into two subtasks aligned with Susan’s familiar routine of counting, which included subtask (1) to signify the value of each coin to be added (turns 13 and 14), and subtask (2) to determine the total value as denoted by Susan’s fingers (turns 15 and 16).

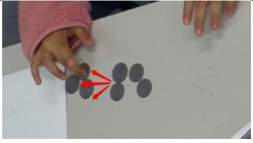

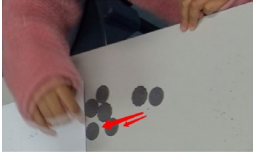
Routine: Based on the analysis, we suggest that Susan constructed a routine (R2) for *adding-coin-values-by-counting* with the support of the researcher, in which she repeated the same finger-lifting procedure four times in response to the task of signifying the value of each coin to obtain the total. We argue that this routine can be considered a ritual characterised by a low level of agentivity because Susan’s actions consisted entirely of executing the researcher’s instructions that helped keep track of all the addends with fingers. As a result, we infer that Susan may not have been fully aware of the overarching mathematical goal behind her actions, as her procedures appeared primarily motivated by the desire to comply with the researcher’s guidance.

Furthermore, it is worth noting that during this process, simultaneous communication between the interlocutors was observed, which played a key role in making R2 take shape. Specifically, in synchrony with the researcher’s procedure of moving each coin (a visual mediator of adding) while verbally signifying the value of each coin (the addend), Susan kept her already raised fingers static in the air to track the accumulating coin values. Following this, Susan steadily lifted the corresponding number of fingers on her left hand (Figure 3b–e), signifying the newly added coin values as uttered by the researcher. The simultaneous communication played an essential role in resolving the previous commognitive conflict. Now, we argue that the multiple variables that had earlier confused Susan were systematically signified through a range of mediators. This resolution indicates a discourse change, where Susan progressed through the pre-ritual stage (episode 2) towards the generation of a ritualistic routine, R2 (episode 3), as supported by the researcher.

4.3 Routine 3: A compound routine combining R1 and R2

4.3.1 Episode 4

This episode presents how Susan developed an explorative routine (R3) independently, combining key features of R1 and R2 with flexible modifications. In what follows, Susan was tasked with making \$10 using multiple coins.

Turn	Speaker	Gesturer	What was said <what was done>	Figure 4
17	S	<u>S</u>	Five. <S picked up a \$5 coin and placed it on the other side> Two. <S used a <i>moving-gesture</i> with a \$2 coin> One. <S used a <i>moving-gesture</i> with a \$1 coin>	 (a)
18	S	<u>S</u>	<u>Eight</u> . <S lifted five, two, and one fingers successively>	 (b)
19	S	S	Ten. <S used <i>moving-gestures</i> to add two more \$1 coins>	 (c)

Procedure: As demonstrated above, Susan first took the procedure of moving the three coins one by one to the other side (Figure 4a) and then raising five, two, and one finger(s) successively (Figure 4b) to calculate the total value of the coins, “eight.” Following that, we note that Susan performed the procedure of counting on from 8, the accumulative value, to successively join two more \$1 coins into the existing group (Figure 4c), after which she verbally uttered “ten” to indicate the total of \$10.

Task: According to the procedures carried out, we conjecture that Susan decomposed the task into two subtasks: subtask (1) to calculate the total value of three coins ($\$5 + \$2 + \$1$) (turns 17 and 18), and subtask (2) to count on from the previous sum until making \$10 (turn 19).

Routine: Taking a closer look at the task–procedure pairs, it is possible to argue that Susan applied a modified version of R2 to solve subtask (1), followed by the implementation of R1 for subtask (2), suggesting that she produced a compound routine combining key features of R1 and R2 to address the new task. Specifically, in turns 17 and 18, Susan appeared not only to lift the corresponding number of fingers and take over the researcher’s role in moving individual coins, but also to slightly re-sequence the steps. That is, she first moved the three coins (turn 17) and then raised her fingers three times to successively display the value of each coin to find the sum (turn 18). In contrast, recall that in turn 14 of episode 3, when R2 was first produced, each *moving-gesture* with one coin was immediately followed by Susan lifting the corresponding number of fingers, and this procedure was repeated for each coin to realise its value. We infer that Susan re-sequenced the steps here because, as the sole actor in this episode, it may not have been feasible for her to fully replicate the collaborative procedure enacted in episode 3.

Moreover, as shown in turn 19 for solving the subtask (2), Susan counted on from the previously obtained total of \$8, using *moving-gestures* to incorporate two more \$1 coins into the existing group until reaching \$10. This reflects her use of the *counting-from-any-number* routine that initially formed in episode 1. In this case, it is suggested that Susan independently activated the procedure associated with the earlier R1 and now applied it in the coin context without external assistance. This demonstrated her competence to detach the procedure from its original task situation (with M&Ms where R1 first emerged) and to adapt it in a new circumstance for solving an addition problem. Therefore, in this episode, since Susan's actions were both motivated by mathematical outcomes and guided by a series of independent decisions made, we contend that R3 could be described as a full-fledged exploration, entailing a high level of agentivity. Through this, Susan appears to have evolved into an explorer who can recognise, modify, and integrate multiple routines of counting and adding in response to new task situations.

5 Discussion

The study examines how the focal child, Susan, developed number discourse with an expert interlocutor in manipulative-based contexts. By focusing on Susan's routine development, we characterise how the explorative routine R3 gradually took shape by conjoining the ritualistic routines R1, which arose from the resolution of contextual miscommunication between interlocutors, and R2, which emerged from a pre-ritual stage involving a commognitive conflict. In this section, we discuss the findings in response to the RQs.

5.1 Evolution from ritual to exploration

With regard to RQ1, we observed Susan's routines evolving from rituals (R1 and R2) to a full-fledged exploration (R3), during which her focus shifted from complying with social expectations and mimicking the researcher's actions to linking her procedures with mathematical outcomes, characterised by increasing levels of agentivity. This finding corroborates Lavie et al.'s (2019) statement that children act "on behalf of other people" (p. 170) when engaging in rituals; moreover, as the discourse becomes de-ritualised, learners are able to make increasingly independent decisions with less assistance from others. We also shed light on the intersection between routines and visual mediators. That is, during the de-ritualised process, we highlight how Susan gradually individualised the gestural mediators, such as *moving-gestures* initiated by the researcher, for generating her own explorative routines.

Inspired by Lavie and Sfard's (2019) work on how cardinality emerges through the merging of two distinct discourses, in our case, the discourses under examination are counting and adding. We characterise Susan's gradual transformation and consolidation of counting and adding routines to shed light on the evolving nature of arithmetic thinking as emerging in communication with both the expert interlocutor and herself. In particular, as Susan participated in the de-ritualised discourse, her discursive rule developed from *(re)counting-from-one* (a familiar routine observed in the baseline assessment), to *counting-from-any-number* (R1), to *adding-by-counting* (R2) as prompted in coin-based contexts, and eventually to a compound routine for addition that integrates both

counting-all and counting-on routines (R3), with the latter building upon her prior routine R1, *counting-from-any-number*.

Unlike previous studies that conceptualise children's counting-based strategies for addition problems as the acquisition of procedural subskills (e.g., Martins-Mourão & Cowan, 1998; Secada et al., 1983), this study contributes to the body of literature by offering a participationist account of this learning as a discursive transformation. Specifically, we adopt a commognitive perspective to elucidate how counting-on and counting-all emerge as the child internalises discursive rules modelled and guided by an expert interlocutor, where she was prompted to mobilise various communicative mediators (e.g., gestures, number words) as interaction with manipulatives. During this process, the child gradually objectified number (the first addend) as an operable entity stably retaining its value that simultaneously functions as part of a larger total (counting-on), thereby reducing the need to always (re) count every concrete object or finger that signifies both addends starting from 1 (counting-all); this was evidenced by Susan's de-ritualised learning process throughout the episodes.

5.2 Beyond rituals: tracing the path from pre-ritual to de-ritualisation

This study extends the de-ritualisation of learning by proposing a new stage, *pre-ritual*, to capture the foundational phase that precedes ritual, de-ritualisation, and exploration. The pre-ritual phase is characterised by commognitive conflicts arising from interlocutors' application of different discursive rules to the same mathematical task (Sfard, 2007). We argue that the miscommunication during the pre-ritual stage lays crucial foundations for Susan to participate in subsequent ritualisation upon its resolution with the help of the expert interlocutor. While previous research has examined commognitive conflicts in terms of their productive role in triggering discursive shifts (e.g., Cooper & Lavie, 2021; Nachlieli & Tabach, 2019), as well as their nuanced classification depending on the degree to which the conflict is explicit to the discussants and whether it is reflected upon (Nachlieli & Heyd-Metzuyanim, 2022), our conceptualisation of *pre-ritual* seeks to map the learner's experience of commognitive conflict onto the continuum of routine development from a child-centred lens. It is noteworthy that not all occasions with commognitive conflicts qualify as pre-rituals; rather, only those that precede and give rise to the formation of a relevant new ritual (in this case, R2) fall within the scope of our definition of pre-ritual.

As shown in episode 2, during the pre-ritual, Susan encountered a commognitive conflict when her familiar rule of counting was newly exposed to the researcher's routine of adding, which challenged the adequacy of Susan's existing rule to meet the discursive demands of the coin-based addition task. Although ultimately unsuccessful, Susan's attempt in the pre-ritual suggested that she was prompted to begin to recognise the need to transition from her familiar routine of counting-as-cardinality to a more advanced one that treats counting as a tool for solving addition problems. In response to this need, we speculate that she spontaneously invented a mixed strategy of counting all four fingers raised and then counting on from 4 to 6. Although she turned out to lose track of the total value of the coins due to unfamiliarity with the new discourse, her actions marked an early form of arithmetic thinking, expressed in and through her finger gestures.

According to Sfard (2007), although the commognitive conflict can serve as a catalyst for discourse change, it can also be pedagogically risky, as children often struggle to smoothly adopt unfamiliar discursive rules to replace their existing ones. To that end, learners' early participation in a new discourse may need to be ritualised, socially driven by teachers' guidance (Cooper & Lavie, 2021). As shown in episode 3, with the

researcher's support in decomposing the task into two subtasks that were more friendly to Susan's familiar counting routine, Susan was then able to move beyond her pre-ritual stage in episode 2 and therefore begin to participate in the new discourse of adding-coin-values-by-counting, as marked by the emergence of her ritual R2. Hence, the findings of this study complement the literature on the importance of teachers, in supporting children's development from ritual to exploration (e.g., Christiansen et al., 2023; Lavie et al., 2019), by also highlighting their crucial role in facilitating children's initial entry into a new discourse through recognising and guiding the resolution of commognitive conflicts.

5.3 The role of manipulatives in routine development

Regarding RQ2, M&M candies and play coins provided perceptually enriched environments where Susan was supported to communicate mathematical ideas upon physical manipulation with concrete objects. Although physical manipulatives are widely employed in early mathematics education for their affordances to help children connect numbers to real-world experiences (Cheung et al., 2023; Clement & Samara, 2014), one should not assume that merely giving manipulatives to children will automatically enhance mathematical learning (Brown et al., 2009; Dowker, 2005). Rather, it is crucial to ensure that manipulatives are purposefully selected and employed in a way that aligns with the instructional needs (Björklund, 2014). Or, in a Sfardian sense (2008), the learning environment should feature targeted rules that are intended to be adopted by children, such that they can internalise these new rules through participation in the discourse.

The use of manipulatives presented in this study contributes to the literature by providing an instance of *interdiscursivity* in children's counting–addition discourses. As introduced by Cooper and Lavie (2021), *interdiscursivity* refers to a teaching principle for scaffolding children as they navigate a “midway discourse” (p. 8), an in-between space where learners switch from precedent discursive rule to a new one, by drawing on elements (e.g., keywords, routines, visual mediators) from both the learner's current and the target discourses. Rather than forcing a shift, our findings suggest that coins could serve as interdiscursive visual mediators, bridging the learner's counting and emerging addition discourses. This manipulative enables the learner's discourse to transition from communicating about concrete objects using existing counting-as-cardinality routines (such as counting-from-one or counting-from-any-number, similar to routines enacted with M&Ms) to entering the discourse of value-based addition (e.g., counting-all and counting-on). This also responds to previous research (e.g., Carpenter & Moser, 1984; Martins-Mourão & Cowan, 1998; Siegler & Jenkins, 2014) that children may possess counting-on skills but fail to demonstrate them, highlighting the need for task design that reveals rather than underestimates children's capabilities.

Furthermore, the use of coins alongside gestures enables simultaneous communication between interlocutors, which broadens communicational possibilities (Ng, 2016) for resolving commognitive conflicts in the pre-ritual stage. Specifically, the simultaneous communication, as demonstrated in episode 3, allowed the discourse community to enact the complex realising procedure of adding coin values using multimodal mediators, where the researcher's verbal number words (addend) and moving-gesture with each coin (act of adding) were synchronised with Susan's already raised fingers (cumulative value). Through this mode of simultaneous communication, all Susan subsequently needed to do was raise the corresponding number of fingers, as requested by the researcher, to signify the value of each newly added coin. This differs from conventional

classroom conversations—where one speaker typically follows another without overlap, driven by politeness or cultural norms—thereby reducing the commognitive demands required for Susan in the coin-based addition task.

6 Conclusion

Taking a commognitive perspective, this study characterises a young learner's counting-addition discourses within manipulative-based environments, evolving from pre-ritual to ritual, and eventually into full-fledged exploration with an expert interlocutor. Our proposal of pre-ritual highlights those conflicts in discourse change as not necessarily negative deficits but rather learning opportunities, which necessitate educators to be sensitive to those moments and to support children in transforming pre-rituals into rituals, such that the newcomers can participate in subsequent de-ritualised discourse. More studies on professional development are needed to equip school mathematics teachers with the necessary knowledge and skills to support routines occurring and evolving (Sfard, 2023a, 2023b), with attention to learners' agentivity during the process. Future research is also expected to enrich the conceptualisation of pre-ritual in other mathematical discourses.

Moreover, this study sheds light on the role of manipulatives in communicating ideas of counting and adding, which enables the learner to make sense of abstract mathematical objects through embodied gesture use when physically interacting with M&M candies and play coins. Future endeavours shall delve into how other types of manipulatives may afford or hinder the development of early arithmetic discourse, as well as what instructional goals and strategies are compatible with the nature of their manipulation.

Author contribution The second and third authors collected the data and then guided the first author in data analysis. The first and second authors wrote the initial draft of the manuscript, with the third author providing critical insights throughout. All authors read and approved the final manuscript.

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Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author upon reasonable request.

Code availability Not applicable.

Declarations

Ethics approval All procedures performed with human subjects followed the ethical standards of The Chinese University of Hong Kong, Survey and Behavioural Research Ethics Committee.

Informed consent Informed consent was obtained from all human participants in the study.

Consent for publication Not applicable.

Conflict of interest The authors declare no competing interests.

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