Differential Equations and Dynamical Systems (2026) 34:499–524
https://doi.org/10.1007/s12591-025-00751-z

ORIGINAL RESEARCH

Positive Solutions for a Class of Two-term Fractional *q*-derivative Boundary Value Problems with an Integral Condition

L. P. Castro[1]

Received: 4 November 2023 / Accepted: 22 December 2025 / Published online: 9 March 2026

Abstract
We deduce different conditions to ensure positive solutions for a class of nonlinear two-term Riemann-Liouville fractional q-derivative boundary value problems which depends on a parameter and an integral condition. As a first step, we start by bringing this class of problems through a q-integral equation where its kernel is explicitly identified and its most characteristic properties are analysed. We then use an upper and lower solutions method, an appropriate cone and other techniques to obtain different types of conditions that ensure the existence of a positive solution to the class problems under study. Specific Lipschitz conditions are also identified to ensure the uniqueness of a continuous and positive solution. Finally, using a non-negative continuous concave functional on a cone and comparing the images it produces with the norm of potential solutions to the class of problems, other conditions are obtained that guarantee the existence of three (different) positive solutions to the class of problems under analysis.

Keywords Positive solution · Riemann-Liouville fractional q-derivative · q-integral · Boundary value problem · Upper and lower solutions

Mathematics Subject Classification 34B18 · 05A30 · 26A33 · 34A08 · 34B09 · 34B10 · 34L05 · 39A13 · 45J05 · 47H10

Dedicated to Professor Roland Duduchava on the occasion of his 80th birthday.

✉ L. P. Castro
castro@ua.pt

[1] CIDMA - Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Campus Universitário, 3810-193 Aveiro, Portugal

Introduction

The q-calculus is commonly known as "calculus without limits", and has the potential to –instead of using the classical derivative– use the q-derivative operator which even allows us to deal with non-differentiable functions (cf. the relevant and systematic works of Jackson and Carmichael, where several essential notions and some of their consequences were formally introduced for the first time [11, 24, 25]). In fact, from an even more global perspective, it can be also said that in q-calculus we look for q-analogs of mathematical objects with the particularity that when we consider the limit situation, of q tending to 1, we obtain the original object.

Surprisingly, q-calculus can be also identified with developments that took place centuries ago. In fact, its beginnings can be identified with works of Euler [19] at the beginning of the 18th century, e.g. with Euler's identities for q-exponential functions, as well as other relevant later discoveries by Gauss and Heine (e.g., Gauss's q-binomial formula and Heine's formula for a q-hypergeometric function). Thus, the history of q-calculus, its basic concepts and q-differential equations have already been reported in a substantial number of works that have been appearing over several years, but with some relevance and growing interest in recent years (cf. the monographs [8, 13, 16–18, 26]). Indeed, interest in q-calculus has grown given its relevance to several subareas of mathematics and its relationships with other sciences, such as quantum theory, relativity theory, calculus of variations, hypergeometric functions, orthogonal polynomials, number theory, combinatorics and fractal geometry.

Naturally, as expected, this has also had very high repercussions in the sub-area of differential equations and boundary value problems, given their high relevance in the modeling of various types of problems at the level of mechanics, physics and engineering.

The pioneering works of Al-Salam [7] and Agarwal [4] were the starting point for the fractional q-difference calculus, thus generalizing the notion of q-derivatives and q-integrals to non-integer orders. Such a generalization aroused the interest of several authors in considering new problems and possibilities of mathematical modeling that use these new concepts and objects of fractional q-difference calculus. This recent dynamic and interest in the fractional q-difference calculus can be seen, for example, in [1–3, 5, 9, 10, 12, 14, 15, 20–23, 28–33, 35–39] and the references therein.

Taking this context into account, the main objective of this work is to study the following nonlinear q-boundary value problem, for unknowns u, a given function f (to be detailed later), with two-term Riemann-Liouville fractional q-derivatives of orders α and $\alpha - 1$ (for $1 < \alpha \leq 2$), a real parameter $\lambda > 0$ and an integral condition:

$$\begin{cases} \left(D^{\alpha}_{q,0+}u\right)(t) - \lambda\left(D^{\alpha-1}_{q,0+}u\right)(t) + f\left(t, u(t)\right) = 0, \quad t \in [0,1], \\ u(0+) = 0, \quad u(1) = \lambda \int_0^1 u(s) d_q s. \end{cases} \tag{1.1}$$

Our primary goal in addressing this class of problems is to examine situations where the parameter dependence is present in both the integral form of the initial condition and the main differential equation. This presents a problem with more variations and interactions between the data than those previously discussed, in addition to having two-term Riemann-Liouville fractional q-derivatives.

At this point, it makes sense to make some observations about the role of λ in the structure of the problem under analysis here. It is noteworthy that the parameter λ appears in two different places in the problem, thus influencing both the dynamics of the differential operator and the nature of the boundary condition. In fact, first, λ appears as a coefficient of part of the differential operator that characterizes the first equation, affecting only the lower-order q-differentiation, thus verifying that, in this way, it weights the influence of the lower-order term. In this context, since $D_{q,0+}^{\alpha-1}$ is of lower-order, the term $-\lambda D_{q,0+}^{\alpha-1}u$ acts similarly to a fractional damping or deviation term. As a consequence, for small λ, the dominant term is $D_{q,0+}^{\alpha}u$, so that the operator behaves more like a pure fractional derivative problem of order α. For large λ, the lower-order q-derivative becomes more influential; the problem behaves more like a fractional equation of order $\alpha - 1$, or like a mixed structure between both orders. This has immediate mathematical consequences. That is, λ influences solvability, a priori bounds, and fixed point conditions used to prove the existence and multiplicity of solutions. A large λ can cause the operator to lose monotonicity or positivity properties, which can lead to bifurcations in the set of solutions, changes in stability, appearance/disappearance of solutions, and different compactness properties. Thus, λ acts as an "adjustable structural parameter" in the operator itself.

The presence of λ in the nonlocal boundary condition $u(1) = \lambda \int_0^1 u(s)d_q s$ allows us to incorporate a force that is also scaled by λ. That is, the value at the right endpoint $t = 1$ is constrained by the weighted global average of the solution on [0, 1], and the parameter λ controls the intensity with which $u(1)$ must correspond to this average. Therefore, for small λ, the condition is weak and $u(1)$ is almost unrestricted, leading to more flexibility in the forms of the solution. For "moderate" λ, the value $u(1)$ is linked to the "average behavior" of the solution, imposing a kind of nonlocal feedback. For large λ, the solution must satisfy a strong self-consistency relation. Thus, a high λ can force the solutions to be more "flat" or "balanced", can restrict oscillatory behavior, can create resonance-like effects depending on the Green's function of the problem, and can interact in a non-trivial way with the sign/size of $f(t, u)$.

If we put all this together and, at the same time, take into account what the parameter λ can contribute in terms of modeling, from the perspective of the potential applications of the problem under study here, we conclude that λ will be associated with memory strength, feedback, damping, transport, or accumulated effects, which can be considered through this problem in the model in question.

We now proceed to elucidate the remaining structure of the article. In Section 2, we introduce notations, definitions and preliminary facts that will allow us to we interpret the theoretical formulation of the fractional q-boundary value problem (1.1), and which will also be useful in the deductions to be made in later sections. In Section 3, we obtain the equivalence of Problem (1.1) to a q-integral equation whose Green's function is identified and studied. In Section 4, we will use a convenient cone setting and an upper and lower solutions method to identify the existence of one positive solution to Problem (1.1). Section 5 introduces certain Lipschitz conditions that allow us to justify the uniqueness of a continuous and positive solution of (1.1). In Section 6 we use the Leggett–Williams Theorem, a non-negative continuous concave functional over a cone, and compare the images it produces with the norm of the problem solutions, so that conditions are obtained to guarantee the existence of (at least) three positive continuous solutions to Problem (1.1).

Auxiliary Material

In the present section we will recall some elements of fractional q-calculus, just for the convenience of the reader, in order to make the text as complete as possible, and to have essential definitions and known results to be considered in later parts. There is already an extensive literature on the general properties of q-calculus where the elements of the present section can be found; for a more detailed knowledge and also a sequence of the corresponding consequences, we cite here the monographs [8, 13, 16–18, 26].

We shall be always assuming that

$$0 < q < 1, \qquad 0 \leq a < b < \infty.$$

Let $\alpha \in \mathbb{R}$. Then, a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1-q^\alpha}{1-q},$$

where $\lim\limits_{q\to 1} \frac{1-q^\alpha}{1-q} = \alpha$. Moreover, for $n \in \mathbb{N}$, it is also used

$$(a;q)_0 := 1,\ (a;q)_n := \prod_{k=0}^{n-1} \left(1-q^k a\right),\ (a;q)_\infty := \lim_{n\to\infty} (a;q)_n,\ (a;q)_\alpha := \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

With the last identities, it is possible to define a q-analogue of the power function $(x-b)^\alpha$ as

$$(x-b)_q^\alpha := x^\alpha \frac{(\frac{x}{b};q)_\infty}{(q^\alpha \frac{x}{b};q)_\infty}.$$

From the last identities, it follows that

$$\begin{aligned} (x-b)_q^\alpha &= x^\alpha \left(\frac{x}{b};q\right)_\alpha, \\ (a(t-s))_q^\alpha &= a^\alpha (t-s)_q^\alpha. \end{aligned}$$

Moreover, the q-analogue of the binomial coefficients is defined by

$$[n]_q! := \begin{cases} 1, & \text{if} \quad n = 0, \\ [1]_q \cdot [2]_q \cdot \dots \cdot [n]_q, & \text{if} \quad n \in \mathbb{N}. \end{cases}$$

The q-Gamma function $\Gamma_q(t)$ is defined by

$$\Gamma_q(t) := (1-q)^{1-t} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{k+t}} = \frac{(q;q)_\infty}{(q^t;q)_\infty}(1-q)^{1-t}, \qquad t > 0.$$

Notice that the previous definition yields a similar fundamental identity as in the classical case:

$$\Gamma_q(t)[t]_q = \Gamma_q(t+1).$$

The q-analogue differential operator D_q [24] is given by

$$D_q f(t) := \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0,$$

and $D_q f(0) := \lim_{t\to 0} D_q f(t)$, and the q-differential operators of higher integer orders D_q^n are defined inductively in the following natural way:

$$D_q^0(f(t)) := f(t), \quad D_q^n(f(t)) := D_q\left(D_q^{n-1} f(t)\right), \quad n = 1, 2, 3, \ldots. \tag{2.1}$$

In particular, we have

$$D_q\left[(t-b)_q^{\alpha}\right] = [\alpha]_q (t-b)_q^{\alpha-1}$$

and

$$D_q\left[(a-t)_q^{\alpha}\right] = -[\alpha]_q (a-qt)_q^{\alpha-1}.$$

For a function *f* defined in the interval [0, *b*], its q-integral or Jackson integral [25], denoted by $(I_{q,0+}f)(b)$ or $\int_0^b f(t) d_q t$, is defined by

$$(I_{q,0+}f)(b) = \int_0^b f(t) d_q t := (1-q) b \sum_{m=0}^{\infty} q^m f(bq^m) \tag{2.2}$$

(provided that the series at the right-hand side of (2.2) converges) and the Jackson integral of *f* from *a* to *b*, for some $a \in (0, b)$, is defined by

$$(I_{q,a+}f)(b) = \int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Notice that

$$\int_a^b D_q f(t)\, d_q t = f(b) - f(a)$$

and

$$\int_a^b \int_a^t f(t)g(s)\, d_q s\, d_q t = \int_a^b \int_{qs}^b f(t)g(s)\, d_q t\, d_q s.$$

Moreover, for $n = 2, 3, \ldots$, the multiple q-integral operator $I^n_{q,a+}$ is given by

$$\left(I^n_{q,a+} f\right)(t) = \frac{1}{\Gamma_q(n)} \int_a^t (t - qs)_q^{n-1} f(s) d_q s. \tag{2.3}$$

The identity (2.3) provides the main motivation for the following definition of Riemann-Liouville fractional q-integral [6, 7, 29] which englobes the remaining (fractional) exponents.

Definition 1 The Riemann-Liouville fractional q-integral $I^\alpha_{q,a+}$ of order $\alpha > 0$, of a function *f*, between $a \geq 0$ and $t \geq a$, is defined by

$$\left(I^\alpha_{q,a+} f\right)(t) := \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s) d_q s.$$

Notice that (cf. Lemma 6 in [29])

$$\left(I^\alpha_{q,a+} (s-a)_q^\lambda\right)(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (t-a)_q^{\alpha+\lambda}, \quad \lambda \in (-1, \infty).$$

In particular, for $\lambda = 0$, it follows

$$\left(I^\alpha_{q,a+} 1\right)(t) = \frac{1}{\Gamma_q(\alpha+1)} (t-a)_q^\alpha.$$

The last definition also yields a consequent formalization of a Riemann-Liouville fractional q-derivative, upon the use of $I^\beta_{q,a+}$, $\beta > 0$.

Definition 2 The Riemann-Liouville fractional q-derivative $D^\alpha_{q,a+} f$ of order $\alpha > 0$, of a function *f*, is defined by

$$\left(D^\alpha_{q,a+} f\right)(t) := \left(D_q^{[\alpha]} I^{[\alpha]-\alpha}_{q,a+} f\right)(t),$$

where $[\alpha]$ denotes the smallest integer greater or equal than α, and $D_q^{[\alpha]}$ is defined in (2.1).

Lemma 1 *(Cf. Theorem 2.4 in [20]) Let $m \in \mathbb{N}$ (and $\alpha > 0$), and f be a function defined on [0, b], $b > 0$, such that there exist $\left(D^k_{q,0+} f\right)(0)$, for all positive integers $k \leq m - 1$. Then, for $t \in [0, b]$, the following equality holds:*

$$\left(I^\alpha_{q,0+} D^m_{q,0+} f\right)(t) = \left(D^m_{q,0+} I^\alpha_{q,0+} f\right)(t) - \sum_{k=0}^{m-1} \frac{t^{\alpha-m+k}}{\Gamma_q(\alpha+k-m+1)} \left(D^k_{q,0+} f\right)(0).$$

For $1 \leq p < \infty$, the q-analogue of Lebesgue spaces, $L_q^p[a,b]$, is formalized by

$$L_q^p[a,b] := \left\{ f : [a,b] \to \mathbb{R} : \|f\|_{L_q^p[a,b]} := \left(\int_a^b |f(t)|^p d_q t \right)^{\frac{1}{p}} < \infty \right\}.$$

Lemma 2 *(Cf. Lemma 2.6 in [34]) Let $\alpha > 0$ and $1 \leq p < \infty$. The Riemann-Liouville fractional q-integral I_{a+}^{α}, of order $\alpha > 0$, is bounded in $L_q^p[a,b]$ and a constant for this boundedness is known in the following way:*

$$\|I_{q,a+}^{\alpha} f\|_{L_q^p[a,b]} \leq \frac{(b-qa)_q^{\alpha}}{\Gamma_q(\alpha+1)} \|f\|_{L_q^p[a,b]}, \quad f \in L_q^p[a,b].$$

Lemma 3 *(Cf. Theorem 5 and Lemma 9 in [29]) (i) Let $\alpha > 0$, $\beta > 0$ and $1 \leq p < \infty$. Then, the Riemann-Liouville fractional q-integral has the following semigroup property*

$$\left(I_{q,a+}^{\alpha} I_{q,a+}^{\beta} f \right)(t) = \left(I_{q,a+}^{\alpha+\beta} f \right)(t),$$

for all $t \in [a,b]$ and $f \in L_q^p[a,b]$.

(ii) Let $\alpha > \beta > 0$, $1 \leq p < \infty$ and $f \in L_q^p[a,b]$. Then the following equalities

$$\left(D_{q,a+}^{\alpha} I_{q,a+}^{\alpha} f \right)(t) = f(t), \quad \left(D_{q,a+}^{\beta} I_{q,a+}^{\alpha} f \right)(t) = \left(I_{q,a+}^{\alpha-\beta} f \right)(t),$$

hold for all $t \in [a,b]$.

On the Solutions of a Fractional *q*-boundary Value Problem Involving Riemann-Liouville Fractional *q*-derivatives and a *q*-integral Condition

To shorten the formulas, we will now use $a = 0$ and $b = 1$. Anyway, from a mathematical point of view, all the arguments also work for general *a* and *b* (from the previous section).

To begin with, we will simply be assuming that the given data *f* in the fractional q-boundary value problem (1.1) is in $L_q^1[0,1]$, but in the next sections we will assume that *f* is continuous, so we can also look for continuous solutions to this problem.

Equivalence between the Fractional *q*-boundary Value Problem (1.1) and a *q*-integral Equation The main objective of this subsection is to rewrite, under very global conditions, the fractional q-boundary value problem (1.1) in terms of an integral equation in which its kernel is explicitly identified. Once this is done, it will also be important and convenient to understand as much as possible what are the most relevant properties of the Green's function associated with the aforementioned integral equation.

Theorem 4 *Let $u \in L_q^1[0,1]$ and $f(\cdot,\cdot):[0,1]\times\mathcal{O}\to\mathbb{R}$ be a function such that $f(t,u(t))\in L_q^1[0,1]$, for some open set $\mathcal{O}$ in $\mathbb{R}$.*

Then, u(t) satisfies a.e. the fractional q-boundary value problem (1.1) if and only if u(t) satisfies a.e. the integral equation

$$u(t) :=\lambda\int_0^t u(s)d_qs+\int_0^1 G(t,qs)f(s,u(s))d_qs,$$

where

$$G(t,qs):=\begin{cases}\frac{t^{\alpha-1}(1-qs)_q^{\alpha-1}-(t-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)}, & 0\le qs\le t\le 1\\ \frac{t^{\alpha-1}(1-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)}, & 0\le t\le qs\le 1.\end{cases} \tag{3.1}$$

Proof *Necessity*. Let $u\in L_q^1[0,1]$ satisfy a.e. the identities in (1.1), for some $f(\cdot,u(\cdot))\in L_q^1[0,1]$. Applying the operator $I_{q,0+}^{\alpha}$ to both sides of the first equation in (1.1), we have

$$\left(I_{q,0+}^{\alpha}D_{q,0+}^{\alpha}u\right)(t)-\lambda\left(I_{q,0+}^{\alpha}D_{q,0+}^{\alpha-1}u\right)(t)+I_{q,0+}^{\alpha}f\left(t,u(t)\right)=0,$$

where each term in this equation is still belonging to $L_q^1[0,1]$; cf. Lemma 2. Now, from Definition 2, it follows that

$$\left(I_{q,0+}^{\alpha}D_{q,0+}^{2}I_{q,0+}^{2-\alpha}u\right)(t)-\lambda\left(I_{q,0+}^{\alpha}D_{q,0+}^{\alpha-1}u\right)(t)+I_{q,0+}^{\alpha}f\left(t,u(t)\right)=0. \tag{3.2}$$

Additionally, having in mind Lemma 1, we deduce that

$$\begin{aligned}\left(I_{q,0+}^{\alpha}D_{q,0+}^{2}\left(I_{q,0+}^{2-\alpha}u\right)\right)(t)=\left(D_{q,0+}^{2}I_{q,0+}^{\alpha}\left(I_{q,0+}^{2-\alpha}u\right)\right)(t)-\frac{t^{\alpha-2}}{\Gamma_q(\alpha-1)}I_{q,0+}^{2-\alpha}u(0)\\ -\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left(D_{q,0+}I_{q,0+}^{2-\alpha}u\right)(0).\end{aligned} \tag{3.3}$$

Inserting the last identity (3.3) in (3.2) allows us to arrive at

$$u(t)=ct^{\alpha-1}+\lambda\left(I_{q,0+}^{\alpha}D_{q,0+}^{\alpha-1}u\right)(t)-I_{q,0+}^{\alpha}f\left(t,u(t)\right),\quad t\in[0,1], \tag{3.4}$$

for a constant c which can be determined by using the integral boundary condition in (1.1). In fact, from that condition and (3.4) for $t=1$, it follows that

$$c=I_{q,0+}^{\alpha}f\left(1,u(1)\right).$$

Therefore, from (3.4), we obtain

$$u(t) = t^{\alpha-1} I^{\alpha}_{q,0+} f\left(1, u(1)\right) + \lambda \left(I^{\alpha}_{q,0+} D^{\alpha-1}_{q,0+} u\right)(t) - I^{\alpha}_{q,0+} f\left(t, u(t)\right). \tag{3.5}$$

Moreover, using Definition 2, Lemma 3 and Lemma 1 in (3.5), it follows

$$\begin{aligned} u(t) &= t^{\alpha-1} I^{\alpha}_{q,0+} f\left(1, u(1)\right) + \lambda \left(I_{q,0+} I^{\alpha-1}_{q,0+} D_{q,0+} I^{2-\alpha}_{q,0+} u\right)(t) - I^{\alpha}_{q,0+} f\left(t, u(t)\right) \\ &= t^{\alpha-1} I^{\alpha}_{q,0+} f\left(1, u(1)\right) + \lambda \left(I_{q,0+} D_{q,0+} I^{\alpha-1}_{q,0+} I^{2-\alpha}_{q,0+} u\right)(t) \\ &\quad - \lambda I_{q,0+} \left(\frac{t^{\alpha-2}}{\Gamma_q(\alpha-1)} \left(I^{2-\alpha}_{q,0+} u(0)\right)\right) - I^{\alpha}_{q,0+} f\left(t, u(t)\right) \\ &= t^{\alpha-1} I^{\alpha}_{q,0+} f\left(1, u(1)\right) + \lambda \left(I_{q,0+} u\right)(t) - I^{\alpha}_{q,0+} f\left(t, u(t)\right), \end{aligned} \tag{3.6}$$

which is precisely for the Green function G given in (3.1).

Sufficiency. Let $u \in L^1_q[0,1]$ and assume that it satisfies the identity. First, noticing that $G(0, qs) = G(1, qs) = 0$, the boundary conditions in (1.1),

$$u(0+) = 0, \quad u(1) = \lambda \int_0^1 u(s) d_q s,$$

are directly satisfied by a substitution of these particular values of the variable t in (3.1). Second, by applying the operator $D^{\alpha}_{q,0+}$ to both sides of (3.1) (and having in mind the deduction in (3.6)), we obtain

$$\begin{aligned} \left(D^{\alpha}_{q,0+} u\right)(t) = \left(D^{\alpha}_{q,0+} t^{\alpha-1}\right) I^{\alpha}_{q,0+} f\left(1, u(1)\right) + \lambda \left(D^{\alpha}_{q,0+} I^{\alpha}_{q,0+} D^{\alpha-1}_{q,0+} u\right)(t) \\ - D^{\alpha}_{q,0+} I^{\alpha}_{q,0+} f\left(t, u(t)\right) \end{aligned}$$

which is nothing else than the first identity in (1.1). □

Properties of the Green function

Let us now point out some of the properties of the Green function G that will be useful later on for further deductions.

Lemma 5 The function G, defined in (3.1), has the following properties:

(i) $G(0, qs) = G(1, qs) = G(t, 0) = G(t, 1) = 0, \quad t, qs \in [0, 1]$;
(ii) $G(t, qs) > 0, \quad t, qs \in (0, 1)$;
(iii) $G : [0, 1] \times [0, 1] \to [0, +\infty)$ is continuous;
(iv) $0 \leq t^{\alpha-1}(1 - qs)^{\alpha-1}_q - (t - qs)^{\alpha-1}_q \leq \Gamma_q(\alpha) G(t, qs) \leq t^{\alpha-1}(1 - qs)^{\alpha-1}_q$;
(v) for a given $qs \in (0, 1)$, $G(t, qs)$ is decreasing with respect to t, for $qs \leq t$, and increasing with respect to t, for $t \leq qs$;

(vi)

$$\Gamma_q(\alpha) \max_{t\in[0,1]} G(t,qs) = \{\Gamma_q(\alpha)G(qs,qs) : qs \neq 0\}$$
$$= (qs)^{\alpha-1}(1-qs)_q^{\alpha-1}, \quad qs \neq 0$$
$$= ((qs)(1-qs))_q^{\alpha-1}, \quad qs \neq 0$$
$$\leq \frac{1}{4^{\alpha-1}};$$

(vii)

$$\min_{1/4\leq t\leq 3/4} G(t,qs) = G(qs,qs)\,\mathcal{G}(qs), \quad qs \in (0,1),$$

where $\mathcal{G}$ is a positive and continuous function defined in (0, 1) by

$$\mathcal{G}(s) := \begin{cases} \frac{\left(\frac{3}{4}(1-s)\right)_q^{\alpha-1} - \left(\frac{3}{4}-s\right)_q^{\alpha-1}}{s^{\alpha-1}(1-s)_q^{\alpha-1}}, & 0 < s \leq r, \\ \frac{1}{(4s)^{\alpha-1}}, & r \leq s < 1, \end{cases} \tag{3.7}$$

with $r \in (1/4, 3/4)$ satisfying the equation

$$\left(\frac{3}{4}(1-x)\right)_q^{\alpha-1} - \left(\frac{3}{4}-x\right)_q^{\alpha-1} = \frac{(1-x)_q^{\alpha-1}}{4^{\alpha-1}}. \tag{3.8}$$

Proof The item (i) follows directly from the definition of *G*. For verifying (ii), being clear that $t^{\alpha-1}(1-qs)_q^{\alpha-1} > 0$, for $t, qs \in (0,1)$, let us focus on proving that

$$t^{\alpha-1}(1-qs)_q^{\alpha-1} - (t-qs)_q^{\alpha-1} > 0, \quad \text{for} \quad 0 < qs \leq t < 1. \tag{3.9}$$

From [20] we already know that if $\beta > 0$ and $a < b \leq t$, then $(t-a)_q^{\beta} > (t-b)_q^{\beta}$. Therefore, since $tqs < qs \leq t$, it follows

$$t^{\alpha-1}(1-qs)_q^{\alpha-1} = (t-tqs)_q^{\alpha-1} > (t-qs)_q^{\alpha-1}, \quad \text{for} \quad 0 < qs \leq t < 1,$$

which gives (3.9). Thus, we have $G(t,qs) > 0$, for $t, qs \in (0,1)$.

Propositions (iii), (iv) and (v) are clear (from the definition of *G*) and, therefore, we omit the respective justifications here.

For realizing (vi), we have only to use the definition of *G*, (iv) and the properties of the q-power functions (cf. (2.1)), and also to have in mind that $t = 1/2$ gives the maximum of $g(t) = t(1-t)$, $t \in [0,1]$.

Let us now turn to the proof of (vii). Recalling that

$$G(t,qs) = \begin{cases} G_1(t,qs), & 0 \leq qs \leq t \leq 1 \\ G_2(t,qs), & 0 \leq t \leq qs \leq 1, \end{cases}$$

for

$$
\begin{aligned}
G_1(t,qs) :=& \frac{t^{\alpha-1}(1-qs)_q^{\alpha-1} - (t-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)},\\
G_2(t,qs) :=& \frac{t^{\alpha-1}(1-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)},
\end{aligned}
$$

and taking into account (v), we have (for $r \in (1/4, 3/4)$ satisfying equation (3.8)):

$$
\begin{aligned}
\min_{1/4\le t\le 3/4} G(t,qs) =& \begin{cases} G_1(3/4,qs), & 0 < qs \le 1/4\\ \min\left\{G_1(3/4,qs), G_2(1/4,qs)\right\}, & 1/4 \le qs \le 3/4\\ G_2(1/4,qs), & 3/4 \le qs < 1,\end{cases}\\
=& \begin{cases} G_1(3/4,qs), & 0 < qs \le r\\ G_2(1/4,qs), & r \le qs < 1,\end{cases}\\
=& \begin{cases} \frac{(3/4)^{\alpha-1}(1-qs)_q^{\alpha-1} - ((3/4)-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)}, & 0 < qs \le r\\ \frac{(1-qs)_q^{\alpha-1}}{4^{\alpha-1}\Gamma_q(\alpha)}, & r \le qs < 1,\end{cases}\\
=& \begin{cases} \frac{(qs)^{\alpha-1}(1-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)} \frac{(3/4)^{\alpha-1}(1-qs)_q^{\alpha-1} - ((3/4)-qs)_q^{\alpha-1}}{(qs)^{\alpha-1}(1-qs)_q^{\alpha-1}}, & \\ \qquad\qquad 0 < qs \le r & \\ \frac{(qs)^{\alpha-1}(1-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)} \frac{1}{4^{\alpha-1}(qs)^{\alpha-1}}, & r \le qs < 1,\end{cases}\\
=& G(qs,qs)\,\mathcal{G}(qs), \quad qs \in (0,1),
\end{aligned}
$$

as desired. □

A Positive and Continuous Solution

The central objective of this section is to find conditions that guarantee the existence of a positive and continuous solution of the fractional q-boundary value problem (1.1). Thus, from now on we will consider that the given function f in (1.1) is continuous. Additionally, we will consider the Banach space $X := C[0,1]$ endowed with the norm

$$
\|u\| := \sup_{t\in[0,1]} |u(t)|,
$$

and will also consider the cone

$$
P := \{u \in X : \ u(t) > 0, \ t \in (0,1]; \ u(0) = 0\}.
$$

In what follows, we will refer that a solution u of the fractional q-boundary value problem (1.1) is positive if $u \in P$.

Having in mind Theorem 4 and looking for the use of fixed point arguments, we will consider and analyse the operator $\mathcal{T} : X \to X$ defined by

$$
\mathcal{T}u(t) := \lambda \int_0^t u(s)d_qs + \int_0^1 G(t,qs)f(s,u(s))d_qs, \quad t \in [0,1]. \tag{4.1}
$$

Lemma 6 *The operator $\mathcal{T} : X \to X$ defined in (4.1) is completely continuous.*

Proof The continuity of $\mathcal{T}$ follows from the continuity of u and f.

For $\eta > 0$, let us use $B_\eta := \{u \in X : \|u\| \leq \eta\}$. Then, for any $t \in [0,1]$, $u \in B_\eta$, there exists a constant K_1 such that $f(t, u(t)) \leq K_1$ and so, from Lemma 5, we have

$$\begin{aligned}\mathcal{T}u(t) =&\lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f(s, u(s)) d_q s \\ \leq&\lambda\eta \int_0^t d_q s + K_1 \int_0^1 G(qs, qs) d_q s \\ \leq&\lambda\eta + \frac{K_1}{4^{\alpha-1}\Gamma_q(\alpha)}.\end{aligned}$$

Thus, $\mathcal{T}(B_\eta)$ is uniformly bounded.

For all $u \in B_\eta$ and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ (without loss of generality), we have

$$\begin{aligned}|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)| =& \Bigg| \lambda \int_0^{t_2} u(s) d_q s - \lambda \int_0^{t_1} u(s) d_q s \\ &+ \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{\alpha-1} f(s, u(s)) d_q s \\ &- \frac{1}{\Gamma_q(\alpha)} \int_0^{t_2} (t_2 - qs)_q^{\alpha-1} f(s, u(s)) d_q s \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)_q^{\alpha-1} f(s, u(s)) d_q s \Bigg| \\ \leq& \Big| \lambda\eta |t_2 - t_1| + |t_2^{\alpha-1} - t_1^{\alpha-1}| (I_q^\alpha K_1)(1) \\ &- \frac{K_1}{\Gamma_q(\alpha)} \int_0^{t_2} \left[(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1} \right] d_q s \\ &+ \frac{K_1}{\Gamma_q(\alpha)} \int_{t_2}^{t_1} (t_1 - qs)_q^{\alpha-1} d_q s \Bigg|.\end{aligned}$$

Considering now $t_1 \to t_2$, we realize that the right-hand side of the last inequality goes to zero, and so we conclude that $\mathcal{T}(B_\eta)$ is equicontinuous, and so the proof is concluded. $\square$

Just for the purpose of introducing some definitions below, let us consider again $a, b \in \mathbb{R}_+$, with $b > a$. For any $y \in [a, b]$, we define the "upper control function"

$$\mathcal{U}(t, y) := \sup_{\eta \in [a,y]} f(t, \eta) \tag{4.2}$$

and the "lower control function"

$$\mathcal{L}(t,y) := \inf_{\eta\in[y,b]} f(t,\eta). \tag{4.3}$$

It is clear that $\mathcal{U}(t,y)$ and $\mathcal{L}(t,y)$ are nondecreasing on y, and

$$\mathcal{L}(t,y) \leq f(t,y) \leq \mathcal{U}(t,y). \tag{4.4}$$

Definition 3 Let $a, b \in \mathbb{R}_+$, with $b > a$. If $\underline{u}, \overline{u} \in P$, with

$$a \leq \underline{u}(t) \leq \overline{u}(t) \leq b,$$

satisfy

$$\begin{aligned}
\underline{u}(t) \leq& \lambda \int_0^t \underline{u}(s) d_q s + \int_0^1 G(t,qs)\mathcal{L}(s,\underline{u}(s)) d_q s, \quad t \in [0,1],\\
\overline{u}(t) \geq& \lambda \int_0^t \overline{u}(s) d_q s + \int_0^1 G(t,qs)\mathcal{U}(s,\overline{u}(s)) d_q s, \quad t \in [0,1],
\end{aligned}$$

then we call $(\overline{u}, \underline{u})$ a pair of upper and lower solutions of the Problem (1.1).
The last definition has been introduced here simply because we will be using the method of upper and lower solutions together with Schauder's fixed point theorem to achieve our goal of guaranteeing positive (continuous) solutions of (1.1).

Theorem 7 *Assume that $(\overline{u}, \underline{u})$ is a pair of upper and lower solutions of the Problem (1.1). Then, the Problem (1.1) has at least one positive solution $u \in P$, and*

$$\underline{u}(t) \leq u(t) \leq \overline{u}(t), \, t \in [0,1].$$

Proof Let us consider the set

$$\mathcal{K} := \{u \in P : \underline{u}(t) \leq u(t) \leq \overline{u}(t), \quad t \in [0,1]\}.$$

It is clear that $\|u\| \leq b$ and so $\mathcal{K} \subset X$ is bounded, convex and closed. From Lemma 6, we know already that the operator $\mathcal{T} : X \to X$ is completely continuous. Let us now verify that $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$. To this end, let us take $u \in \mathcal{K}$, and from the definition of the pair of upper and lower solutions, it follows

$$
\begin{aligned}
(\mathcal{T}u)(t) =& \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f(s, u(s)) d_q s \\
\leq& \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) \mathcal{U}(s, u(s)) d_q s \\
\leq& \lambda \int_0^t \overline{u}(s) d_q s + \int_0^1 G(t, qs) \mathcal{U}(s, \overline{u}(s)) d_q s \\
\leq& \overline{u}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{T}u)(t) =& \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f(s, u(s)) d_q s \\
\geq& \lambda \int_0^t \underline{u}(s) d_q s + \int_0^1 G(t, qs) \mathcal{L}(s, \underline{u}(s)) d_q s \\
\geq& \underline{u}(t).
\end{aligned}
$$

Therefore, $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$, and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is a compact operator. Thus, Schauder's fixed point theorem allows us to conclude that $\mathcal{T}$ has at least one fixed point $u \in \mathcal{K}$. This means that the Problem (1.1) has at least one positive solution $u \in \mathcal{K}$ with $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in [0, 1]$. □

Corollary 8 If there exist continuous functions f_1 and f_2 such that

$$
0 < f_1(t) \leq f(t, y) \leq f_2(t) < \infty, \; (t, y) \in [0, 1] \times [0, +\infty),
$$

then the fractional q-boundary value problem (1.1) has at least one positive solution $u \in P$, and

$$
\begin{aligned}
u(t) \geq& \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f_1(s) d_q s, \\
u(t) \leq& \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f_2(s) d_q s.
\end{aligned}
$$

Proof Let us consider the auxiliary q-boundary value problem

$$
\begin{cases}
\left(D_{q,0+}^{\alpha} v\right)(t) - \lambda \left(D_{q,0+}^{\alpha-1} v\right)(t) + f_2(t) = 0, \quad t \in [0, 1], \\
v(0+) = 0, \quad v(1) = \lambda \int_0^1 v(s) d_q s.
\end{cases}
\tag{4.5}
$$

Then, (4.5) has a positive solution

$$
v(t) = \lambda \int_0^t v(s) d_q s + \int_0^1 G(t, qs) f_2(s) d_q s. \tag{4.6}
$$

Taking into account the notion of control functions (4.2), (4.3), (4.4), we have

$$f_1(t) \leq \mathcal{L}(t, y) \leq \mathcal{U}(t, y) \leq f_2(t), \quad (t, y) \in [0, 1] \times [a, b]$$

where a, b are the minimum and maximum of $y = v(t)$ on [0, 1]. Therefore, it follows that for $v(t)$ in (4.5) we have

$$v(t) \geq \lambda \int_0^t v(s) d_q s + \int_0^1 G(t, qs) \mathcal{U}(s, v(s)) d_q s.$$

We then have that (4.6) is an upper solution to the fractional q-boundary value problem (1.1) and reasoning similar to the previous one also allows us to conclude that

$$w(t) = \lambda \int_0^t w(s) d_q s + \int_0^1 G(t, qs) f_1(s) d_q s$$

is a lower solution to the fractional q-boundary value problem (1.1).

Now, from Theorem 7, it follows that the fractional q-boundary value problem (1.1) has at least one (positive) solution $u \in P$ such that

$$w(t) \leq u(t) \leq v(t), \quad t \in [0, 1].$$

□

Corollary 9 Let $f(t, y) \geq f_1(t) \geq 0, t \in [0, 1]$, hold for some $f_1(t)$ not identically zero, and assume that $f(t, y)$ converges uniformly to $F(t)$, on [0, 1], as $y \to +\infty$. Then, the fractional q-boundary value problem (1.1) has at least one positive solution on P.

Proof From the hypothesis, there exist $K_1, K_2 > 0$ such that

$$|f(t, y) - F(t)| < K_1, \quad y > K_2, \quad t \in [0, 1],$$

and so,

$$0 \leq f(t, y) \leq F(t) + K_1, \quad y > K_2, \quad t \in [0, 1].$$

Thus,

$$f_1(t) \leq f(t, y) \leq F(t) + K_1 + K_3, \quad t \in [0, 1], \quad y \in [0, +\infty), \tag{4.7}$$

for $K_3 := \max_{0 \leq t \leq 1, 0 \leq y \leq K_2} f(t, y)$.

The inequalities of (4.7) and Corollary 8 allow us to conclude that the fractional q-boundary value problem (1.1) has at least one positive solution $u \in P$ and such that

$$
\begin{aligned}
u(t) \geq & \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs) f_1(s) d_q s, \\
u(t) \leq & \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs)[F(s) + K_1 + K_3] d_q s.
\end{aligned}
$$

□

Corollary 10 Suppose that

$$
\lim_{y \to +\infty} \max_{t \in [0,1]} \frac{f(t,y)}{y} < K_4 \in \mathbb{R}_+ \tag{4.8}
$$

with

$$
\begin{aligned}
\max_{t \in [0,1]} \left\{ \frac{(t^{\alpha-1} - t^{\alpha}) K_4}{\Gamma_q(\alpha+1)} + \lambda t \right\} < & 1, \\
f(t,y) \geq & K_5, \quad t \in [0,1], \quad y \in [0, +\infty)
\end{aligned} \tag{4.9}
$$

(for some $K_5 \geq 0$). Under these conditions, the fractional q-boundary value problem (1.1) has at least one positive solution $u \in P$.

Proof From (4.8) it follows that there exists a positive constant K_6 such that

$$
f(t,y) \leq K_4 y, \quad y > K_6, \quad t \in [0,1].
$$

Defining

$$
M_1 := \max_{t \in [0,1], y \in [0, K_6]} f(t,y),
$$

we may write

$$
f(t,y) \leq K_4 y + M_1, \quad t \in [0,1], \quad y \in [0, +\infty).
$$

Considering now the “adapted” q-boundary value problem

$$
\begin{cases} \left(D^{\alpha}_{q,0+} u\right)(t) - \lambda \left(D^{\alpha-1}_{q,0+} u\right)(t) + K_4 u(t) + M_1 = 0, \\ u(0+) = 0, \quad u(1) = \lambda \int_0^1 u(s) d_q s, \end{cases} \tag{4.10}
$$

from our previous knowledge it follows that u is a solution of (4.10) if and only if

$$
\begin{aligned}
u(t) := & \lambda \int_0^t u(s) d_q s + \int_0^1 G(t, qs)[K_4 u(s) + M_1] d_q s, \\
= & \lambda \int_0^t u(s) d_q s + K_4 \int_0^1 G(t, qs) u(s) d_q s + M_1 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)}.
\end{aligned}
$$

This leads us to consider the corresponding operator $\mathcal{T}_{M_1} : X \to X$, defined by

$$\left(\mathcal{T}_{M_1} u\right)(t) := \lambda \int_0^t u(s) d_q s + K_4 \int_0^1 G(t, qs) u(s) d_q s + M_1 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)},$$

which is also a completely continuous operator in X (cf. the reasoning of the proof of Lemma 6).

Let us now consider

$$\eta_\lambda := \max_{t\in[0,1]} \frac{M_1 \frac{t^{\alpha-1}-t^{\alpha}}{\Gamma_q(\alpha+1)}}{1 - \lambda t - K_4 \frac{t^{\alpha-1}-t^{\alpha}}{\Gamma_q(\alpha+1)}}$$

(recall that due to the hypothesis (4.9) the last quotient is well-defined). For $u \in B_{\eta_\lambda} = \{u \in X : \|u\| \le \eta_\lambda\}$, we have

$$\begin{aligned} \left|\left(\mathcal{T}_{M_1} u\right)(t)\right| \le & \lambda \|u\| t + K_4 \|u\| \int_0^1 G(t, qs) d_q s + M_1 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)} \\ = & \left[\lambda t + K_4 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)}\right] \|u\| + M_1 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)} \\ \le & \eta_\lambda, \end{aligned}$$

since by hypothesis $\max_{t\in[0,1]}\{\lambda t + K_4(t^{\alpha-1} - t^{\alpha})/\Gamma_q(\alpha+1)\} < 1$.

Therefore, the operator $\mathcal{T}_{M_1} : B_{\eta_\lambda} \to B_{\eta_\lambda}$ is compact. This allows us to use once more the Schauder fixed point theorem, now applied to this $\mathcal{T}_{M_1}$, and conclude that $\mathcal{T}_{M_1}$ has at least one fixed point in B_{η_λ}, and that the q-boundary value problem (4.10) has at least one positive solution $\overline{u}$ given in the form

$$\overline{u}(t) = \lambda \int_0^t \overline{u}(s) d_q s + K_4 \int_0^1 G(t, qs) \overline{u}(s) d_q s + M_1 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)}.$$

And so, from the definition of a control function, we get

$$\overline{u}(t) \ge \lambda \int_0^t \overline{u}(s) d_q s + K_4 \int_0^1 G(t, qs) \mathcal{U}(s, \overline{u}(s)) d_q s.$$

Thus, $\overline{u}$ is an upper positive solution of the q-boundary value problem (1.1).

Analogously,

$$\begin{aligned} \underline{u}(t) = & \lambda \int_0^t \underline{u}(s) d_q s + K_5 \int_0^1 G(t, qs) d_q s \\ = & \lambda \int_0^t \underline{u}(s) d_q s + K_5 \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha+1)} \end{aligned}$$

is a lower solution of (1.1).

Therefore, from Theorem 7, it follows that the fractional q-boundary value problem (1.1) has at least one positive solution $u \in P$ with $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in [0, 1]$. □
We will now present an example to illustrate that the conditions considered in the last result are not impossible to occur and, therefore, in this corollary, we are not proposing conditions that represent the void.

Example 1 Let us consider the continuous function

$$f(t, y) := 0.1 + t^5 + \frac{t^2 y}{20 + y}, \qquad t \in [0, 1], \quad y \in [0, +\infty). \tag{4.11}$$

In addition, we will choose the constants

$$\alpha = \frac{3}{2}, \quad K_4 = \frac{1}{10}, \quad K_5 = \frac{1}{100}.$$

With these elements, it is clear that one has $f(t, y) \geq K_5 = 1/100$ for all $t \in [0, 1]$, $y \in [0, +\infty)$. Moreover,

$$\lim_{y \to \infty} \max_{t \in [0,1]} \frac{f(t, y)}{y} = 0 < K_4,$$

and the (4.9) inequality

$$\max_{t \in [0,1]} \left\{ \frac{(t^{\alpha-1} - t^{\alpha}) K_4}{\Gamma_q(\alpha + 1)} + \lambda t \right\} < 1,$$

holds for every $\lambda < 1$ (and $q \in (0, 1)$).

Given the choices just made, here is a concrete example (or even a class of examples) that the hypotheses presented in the last result exhibit situations that are admissible to occur.

Uniqueness of a Positive Solution

We will now dedicate ourselves to identifying conditions that guarantee the uniqueness of positive solutions to the fractional q-boundary value problem (1.1). For this, we will simply use Banach's fixed point theorem.

Theorem 11 *Let us assume that $f \not\equiv 0$ and that there exists a constant L such that*

$$|f(t, y) - f(t, z)| \leq L |y - z|,$$

for all $t \in [0, 1]$ and $y, z \in [0, +\infty)$. If

$$\max_{t \in [0,1]} \left[\lambda t + \frac{t^{\alpha-1} - t^{\alpha}}{\Gamma_q(\alpha + 1)} L \right] < 1, \tag{5.1}$$

then the fractional q-boundary value problem (1.1) has a unique positive solution (on P).

Proof Under the present conditions, the operator $\mathcal{T}$, defined by (4.1), is a contraction on P. Indeed, being already clear that $\mathcal{T}u \in P$, for $u \in P$, we have

$$\begin{aligned}
|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| \leq & \lambda \int_0^t |u(s) - v(s)| d_q s \\
& + \int_0^1 G(t, qs) |f(s, u(s)) - f(s, v(s))| d_q s \\
\leq & \lambda t \|u - v\| + \frac{(t^{\alpha-1} - t^{\alpha}) L}{\Gamma_q(\alpha + 1)} \|u - v\| \\
= & \left[\lambda t + \frac{(t^{\alpha-1} - t^{\alpha}) L}{\Gamma_q(\alpha + 1)} \right] \|u - v\|,
\end{aligned}$$

for any $u, v \in P$.

Thus, from (5.1) and the Banach contraction principle, we conclude that $\mathcal{T}$ has a unique fixed point on P and so the fractional q-boundary value problem (1.1) has a unique positive solution on P. □

Remark 1 Note that the conditions of Corollary 10 imply the conditions of Theorem 11, and so now we can conclude that Corollary 10 also ensures the uniqueness of a positive solution of the fractional q-boundary value problem (1.1).

Example 2 Let

$$\alpha = \frac{3}{2}, \quad \lambda \in (0, 1), \quad f(t, y) = t^2 + \frac{ty}{10 + y}. \tag{5.2}$$

Then,

$$|f(t, y) - f(t, z)| = \left| \frac{ty}{10 + y} - \frac{tz}{10 + z} \right| \leq \frac{1}{10} |y - z|,$$

for all $t \in [0, 1]$ and $y, z \in [0, +\infty)$. Moreover, taking into account that $\Gamma_q(5/2) > 1$ for all $q \in (0, 1)$, we have

$$\max_{t \in [0,1]} \left[\lambda t + \frac{t^{1/2} - t^{3/2}}{10\,\Gamma_q(5/2)} \right] < 1$$

(for all $\lambda \in (0, 1)$).

Therefore, from Theorem 11 we have that (for any $q \in (0, 1)$) the fractional q-boundary value problem (1.1) with the data (5.2) has a unique positive solution.

Multiple Positive Solutions

The main objective now is to find new conditions that guarantee that the problem under analysis has more than one positive solution. For this, we will arrive at conditions that guarantee us at least three different solutions to the fractional q-boundary value problem (1.1).

To achieve that goal, we will use the Leggett–Williams Theorem, certain separations by regions and a continuous concave functional on a cone that we will now expose.

Definition 4 We say that the map ψ is a non-negative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0, +\infty)$ is continuous and

$$\psi(tu + (1-t)v) \geq t\psi(u) + (1-t)\psi(v),$$

for all $u, v \in P$ and $0 \leq t \leq 1$.
We shall also make use of

$$P(\psi, b, d) := \{u \in P : b \leq \psi(u),\ \|u\| \leq d\}.$$

Theorem 12 *(Leggett–Williams [27]) Let P be a cone in a real Banach space E, $\overline{P_c} = \{u \in P : \|u\| \leq c\}$, ψ be a non-negative continuous concave functional on P such that $\psi(u) \leq \|u\|$, for all $u \in \overline{P_c}$. Suppose $T : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that*

(A1) $\{u \in P(\psi, b, d) : \psi(u) > b\} \neq \emptyset$ *and* $\psi(Tu) > b$ *for* $u \in P(\psi, b, d)$*;*
(A2) $\|Tu\| < a$ *for* $u \in \overline{P_a}$*;*
(A3) $\psi(Tu) > b$ *for* $u \in P(\psi, b, c)$ *with* $\|Tu\| > d$*.*

Then T has at least three fixed points u_1, u_2 and u_3 such that $\|u_1\| < a$, $b < \psi(u_2)$, $\|u_3\| > a$ with $\psi(u_3) < b$.

Remark 2 If there holds $d = c$, then condition (A1) of Theorem 12 implies condition (A3) of Theorem 12.

Let us now identify conditions that guarantee the existence of at least three different positive solutions to the fractional q-boundary value problem (1.1).

For this, we will now make use of Theorem 12 and the non-negative continuous concave functional ψ on the cone P which is given by

$$\psi(u) := \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} |u(t)|.$$

From this definition it directly follows that

$$\psi(u) \leq \|u\|, \quad u \in P.$$

Theorem 13 *Let*

$$M := (1-\lambda)\left(\int_0^1 G(qs,qs)d_qs\right)^{-1},$$
$$N := \left(\int_{1/4}^{3/4} \mathcal{G}(qs)G(qs,qs)d_qs\right)^{-1},$$

and $0 < \lambda < 1$.

If $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ and there exist constants $\eta_3 > \eta_2 > \eta_1 > 0$ so that
(B1)$f(t,y) < M\eta_1$, *for* $(t,y) \in [0,1]\times[0,\eta_1]$,
(B2)$f(t,y) \geq N\eta_2$, *for* $(t,y) \in [1/4,3/4]\times[\eta_2,\eta_3]$,
(B3)$f(t,y) \leq M\eta_3$, *for* $(t,y) \in [0,1]\times[0,\eta_3]$,
then the fractional q-boundary value problem (1.1) has at least three positive solutions u^*, u^{**} and u^{***} with

$$\max_{0\leq t\leq 1}|u^*(t)| < \eta_1, \tag{6.1}$$

$$\begin{aligned}&\eta_2 < \min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u^{**}(t)| < \max_{0\leq t\leq 1}|u^{**}(t)| \leq \eta_3\\ &\eta_1 \leq \max_{0\leq t\leq 1}|u^{***}(t)| \leq \eta_3,\\ &\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u^{***}(t)| < \eta_2.\end{aligned} \tag{6.2}$$

Proof We will consider the framework of Theorem 12 to verify that we are in the conditions it requires and therefore we can also use its consequences.

For $u \in \overline{P_{\eta_3}}$ (cf. the definition of this set in the statement of Theorem 12), we have $\|u\| \leq \eta_3$, and by (B3) it follows

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\leq t\leq 1}\left|\lambda\int_0^t u(s)d_qs + \int_0^1 G(t,qs)f(s,u(s))d_qs\right|\\ &\leq \int_0^1 G(qs,qs)f(s,u(s))d_qs + \lambda\|u\|\\ &\leq M\eta_3\int_0^1 G(qs,qs)d_qs + \lambda\eta_3\\ &= \eta_3.\end{aligned}$$

Therefore, $\mathcal{T}$ maps $\overline{P_{\eta_3}}$ into $\overline{P_{\eta_3}}$ and from the arguments of Lemma 6 is completely continuous.

If we just choose now $u(t) = (\eta_2+\eta_3)/2$, $t \in [0,1]$, then it follows

$$\begin{aligned}u(t) = \frac{\eta_2+\eta_3}{2} &\in P(\psi,\eta_2,\eta_3),\\ \psi\left(\frac{\eta_2+\eta_3}{2}\right) &> \eta_2.\end{aligned}$$

Thus, $\{u \in P(\psi, \eta_2, \eta_3) : \psi(u) > \eta_2\} \neq \emptyset$. Considering $u \in P(\psi, \eta_2, \eta_3)$, we have $\eta_2 \leq u(t) \leq \eta_3$, for $t \in [1/4, 3/4]$. Then, the condition (B2) and proposition (vii) of Lemma 5 yield

$$\begin{aligned}
\psi\left(\mathcal{T}u\right) =& \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \left| \int_0^1 G(t, qs) f(s, u(s)) d_q s + \lambda \int_0^t u(s) d_q s \right| \\
\geq& \int_0^1 \mathcal{G}(qs) G(qs, qs) f(s, u(s)) d_q s \\
>& N\eta_2 \int_{1/4}^{3/4} \mathcal{G}(qs) G(qs, qs) d_q s \\
=& \eta_2,
\end{aligned} \tag{6.3}$$

where we recall that $\mathcal{G}$ is given by (3.7).

This means that $\psi(\mathcal{T}u) > \eta_2$, for $u \in P(\psi, \eta_2, \eta_3)$. Thus, we have already verified that condition (A1) of Theorem 12 holds true.

Let us now concentrate on condition (A2) of Theorem 12, that is $\|\mathcal{T}u\| < \eta_1$, for $u \in \overline{P_{\eta_1}}$.

From the definition of $\overline{P_{\eta_1}}$, it follows that $\|u\| \leq \eta_1$, for $u \in \overline{P_{\eta_1}}$. Moreover, from (B1) we have $f(t, y) < M\eta_1$, for $t \in [0, 1]$. Therefore,

$$\begin{aligned}
\|\mathcal{T}u\| =& \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, qs) f(s, u(s)) d_q s + \lambda \int_0^t u(s) d_q s \right| \\
<& \lambda \|u\| + M\eta_1 \int_0^1 G(qs, qs) d_q s \\
\leq& \eta_1 \left(\lambda + M \int_0^1 G(qs, qs) d_q s \right) \\
=& \eta_1,
\end{aligned}$$

as desired.

To conclude, we realize that in the present situation the condition (A3) of Theorem 12 is automatically satisfied (cf. Remark 2). Thus, from Theorem 12, we are able to conclude that the fractional q-boundary value problem (1.1) admits at least three (different) positive solutions u^*, u^{**} and u^{***} satisfying conditions (6.1)–(6.2). □

We will now illustrate, using the following two examples, that the conditions required in the last theorem can occur simultaneously.

Example 3 Let us consider the concrete case of a continuous and positive function $f : [0, 1] \times [0, \infty) \to \mathbb{R}$ defined by

$$f(t, y) := \begin{cases} 0.005 + 0.01\, t^2 + 206.71855\, y^3, & 0 \leq y < 1, \\ 204.72355 + 0.01\, t^2 + 2\, y, & y \geq 1, \end{cases}$$

and let

$$q = \frac{1}{2}, \qquad \alpha = \frac{9}{5}, \qquad \lambda = \frac{3}{4}.$$

Therefore, it is possible to compute $M \approx 2.1031117305969427, N \approx 206.72417465494004$, where the values shown are rounded to the last decimal place only.

Then, let us choose

$$\eta_1 := 0.01, \qquad \eta_2 := 1, \qquad \eta_3 := 1987.$$

Here, in addition to ensuring that the condition $\eta_3 > \eta_2 > \eta_1 > 0$ is satisfied, we deliberately choose η_1 small enough so that the cubic term given by *f* is quite small in $[0, \eta_1]$, we chose $\eta_2 = 1$ so that, in the verification of (B2), we only have to use the second branch of the definition of *f*, and finally, η_3 sufficiently large so that the upper bound presented in (B3) is valid

In fact, for $(t, y) \in [0, 1] \times [0, \eta_1]$ the maximum value of f is obviously reached when $t = 1$ and $y = \eta_1 = 0.01$ and under these circumstances we have $f(1, \eta_1) = 0.005 + 0.01 + 206.71855 \times 10^{-6}$, thus having that $f(t, y) < M\eta_1$, for all $(t, y) \in [0, 1] \times [0, \eta_1]$.

On the other hand, for $(t, y) \in [1/4, 3/4] \times [\eta_2, \eta_3] = [1/4, 3/4] \times [1, 1987]$, we are on the branch of $y \geq 1$ and therefore $f(t, y) = 204.72355 + 0.01t^2 + 2y$. It is therefore clear that the minimum of this function in $[1/4, 3/4] \times [1, 1987]$ occurs when $t = 1/4$ and $y = \eta_2 = 1$, being given by $f(1/4, 1) = 204.72355 + 0.01 \times 0.0625 + 2 = 206.724175$, thus guaranteeing that $f(t, y) \geq N\eta_2$, for all $(t, y) \in [1/4, 3/4] \times [\eta_2, \eta_3]$.

Finally, in $(t, y) \in [0, 1] \times [0, \eta_3] = [0, 1] \times [0, 1987]$, the maximum of f is reached at $t = 1$ and $y = \eta_3 = 1987$, given by $f(1, 1987) = 4178.73355$, thus ensuring that $f(t, y) \leq M\eta_3$ for all $(t, y) \in [0, 1] \times [0, \eta_3]$.

We are thus left with a concrete example that illustrates that the conditions imposed in Theorem 13 are not impossible to occur.

To conclude this work, we will construct one last example, quite different from the previous one, and which will exhibit a certain balance in the contribution of the values of *t* and *y* within the function *f*.

Example 4 The constructive reasoning we will propose in this example can be repeated for other parameters (and therefore, in a certain sense, a class of examples can be built based on the choice of parameters), but here we will exemplify it for *q*, α, and λ to be

$$q = \frac{1}{2}, \qquad \alpha = \frac{3}{2}, \qquad \lambda = \frac{1}{10}.$$

Taking these parameters into account, we have $M \approx 3.8022725362054013$ and $N \approx 204.8323994424567$ (where these values are rounded to the last decimal place only).

In this example, we will consider a continuous function

$$f(t, y) := K_7 + K_8\, \varphi(t)\, \phi(y), \qquad (t, y) \in [0, 1] \times [0, \infty),$$

where K_7 and K_8 are positive constants to be specified later, $\varphi : [0, 1] \to [0, 1]$ is a continuous function given by

$$\varphi(t) := \begin{cases} 4t, & 0 \le t \le \frac{1}{4}, \\ 1, & \frac{1}{4} \le t \le \frac{3}{4}, \\ 4(1-t), & \frac{3}{4} \le t \le 1. \end{cases}$$

and $\phi : [0, \infty) \to [0, 1]$ is a function given by

$$\phi(y) := \begin{cases} 0, & 0 \le y \le \frac{1}{100}, \\ 3\left(\frac{100\,y-1}{99}\right)^2 - 2\left(\frac{100\,y-1}{99}\right)^3, & \frac{1}{100} \le y \le 1 \\ 1, & y \ge 1 \end{cases}$$

Therefore, ϕ is continuous, $0 \le \phi \le 1$ (and we deliberately choose $\phi(y) = 1$ for $y \ge 1$, just to simplify the verification of (B3)).

With this in mind, we will choose

$$\eta_1 := 0.01, \qquad \eta_2 := 1, \qquad \eta_3 := 55.$$

Note that we are deliberately choosing η_3 to be slightly larger than *N*/*M*, so that

$$M\eta_3 > N. \tag{6.4}$$

Now we have some freedom in choosing the constants K_7 and K_8, and we choose

$$K_7 := \frac{M\eta_1}{2}, \qquad K_8 := N - K_7$$

(where we intentionally define K_7 as half of $M\eta_1$ to facilitate the occurrence of (B1)).

With this construction, it is now easy to see that (B1)–(B3) are valid. That is, in $[0, 1] \times [0, \eta_1]$, we have $\phi(y) = 0$ and therefore $f(t, y) = K_7$. Since we have $K_7 = \frac{1}{2}M\eta_1$, then:

$$f(t, y) = K_7 = \frac{1}{2}M\eta_1 < M\eta_1.$$

Thus, (B1) is verified.

Let us now analyze the situation in $[1/4, 3/4] \times [\eta_2, \eta_3]$, with $\eta_2 = 1$. For $t \in [1/4, 3/4]$, we have $\varphi(t) = 1$. For $y \in [1, \eta_3]$, it holds $\phi(y) = 1$. Thus, for these (t, y),

$$f(t, y) = K_7 + K_8 = N$$

by our choice of $K_8 = N - K_7$. Therefore, $f(t, y) = N \ge N\eta_2$. Thus, $(B2)$ is valid.

Finally, for any $(t, y) \in [0, 1] \times [0, \eta_3]$, we have $\varphi(t) \le 1$ and $0 \le \phi(y) \le 1$, then:

$$f(t, y) \le K_7 + K_8 \cdot 1 = N < M\eta_3$$

(due to (6.4)).

Thus, all the conditions of Theorem 13 are met.

Acknowledgements The author would like to thank the Referees for their careful reading of this article and for their comments and suggestions. This work was partially supported by CIDMA (https://ror.org/05pm2mw36) under the Portuguese Foundation for Science and Technology (FCT, https://ror.org/00snfqn58), Grants UID/04106/2025 (https://doi.org/10.54499/UID/04106/2025) and UID/PRR/04106/2025.

Funding Open access funding provided by FCT|FCCN (b-on).

References

1. Abdeljawad, T., Samei, M.E.: Applying quantum calculus for the existence of solution of q-integro-differential equations with three criteria. Discrete Contin. Dyn. Syst. Ser. S **14**(10), 3351–3386 (2021)
2. Alam, M., Zada, A.: Implementation of q-calculus on q-integro-differential equation involving anti-periodic boundary conditions with three criteria. Chaos Solitons Fractals **154**(111625), 32 (2022)
3. Ali, M.A.A., Thabet, S.T.M., Abdeljawad, T., Kedim, I.: New results on an impulsive differential equation involving a q-analogue of the ψ-Caputo fractional derivative. Bound. Value Probl. **2025**(139), 28 (2025)
4. Agarwal, R.P.: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365–370 (1969)
5. Alsaedi, A., Al-Hutami, H., Ahmad, B., Agarwal, R.P.: Existence results for a coupled system of nonlinear fractional q-integro-difference equations with q-integral-coupled boundary conditions. Fractals **30**(2240042), 19 (2022)
6. Al-Salam, W.A.: q-analogues of Cauchy's formulas. Proceedings of the American Mathematical Society **17**(3), 616–621 (1966)
7. Al-Salam, W.: Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **15**, 135–140 (1966/1967)
8. Annaby, M.H., Mansour, Z.S.: q-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Heidelberg (2012)
9. Bangerezako, G.: Variational q-calculus. J. Math. Anal. Appl. **289**, 650–665 (2004)
10. Bouzid, H., Abdelkader, B., Tabharit, L., Samei, M.E.: Existence of solutions to a fractional differential equation involving the Caputo q-derivative with boundary conditions in Banach spaces. J. Inequal. Appl. **2025**(56), 27 (2025)
11. Carmichael, R.D.: The general theory of linear q-difference equations. Amer. J. Math. **34**(2), 147–168 (1912)
12. Castro, L.P.: On the solutions to a Riemann-Liouville fractional q-derivative boundary value problem. Fract. Calc. Appl. Anal. **28**(5), 2302–2332 (2025)
13. Cheung, P., Kac, V.: Quantum Calculus. Edwards Brothers Inc, Ann Arbor, MI, USA (2000)
14. Cui, Y., Kang, S., Chen, H.: Uniqueness of solutions for an integral boundary value problem with fractional q-differences. J. Appl. Anal. Comput. **8**(2), 524–531 (2018)
15. Duc Phuong, N., Etemad, S., Rezapour, S.: On two structures of the fractional q-sequential integro-differential boundary value problems. Math. Methods Appl. Sci. **45**(2), 618–639 (2022)
16. Ernst, T.: The History of q-Calculus and a New Method. U.U.D.M. Report 2000 **16**, Uppsala, Department of Mathematics, Uppsala University (2000)
17. Ernst, T.: A New Method of q-Calculus. Doctoral Thesis, Uppsala university (2002)
18. Ernst, T.: A Comprehensive Treatment of q-Calculus. Birkhäuser/Springer, Basel AG, Basel (2012)
19. Euler, L.: Introduction in Analysin Infinitorum, vol. 1. Lausanne, Switzerland, Bousquet (1748)
20. Ferreira, R.A.C.: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **70**, 1–10 (2010)

21. Ferreira, R.A.C.: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**, 367–373 (2011)
22. Guo, C., Guo, J., Kang, S., Li, H.: Existence and uniqueness of positive solution for nonlinear fractional q-difference equation with integral boundary conditions. J. Appl. Anal. Comput. **10**(1), 153–164 (2020)
23. Guo, F., Kang, S.: Positive solutions for a class of fractional boundary value problem with q-derivatives. Mediterr. J. Math. **113**, 1–16 (2019)
24. Jackson, F.H.: On q-functions and a certain difference operator. Trans. Roy. Soc. Edin. **46**, 253–281 (1908)
25. Jackson, F.H.: On a q-definite integrals. Quart. J. Pure Appl. Math. **41**, 193–203 (1910)
26. Kac, V., Cheung, P.: Quantum Calculus. Universitext, Springer, New York (2002)
27. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. **28**, 673–688 (1979)
28. Ma, K., Li, X., Sun, S.: Boundary value problems of fractional q-difference equations on the half-line. Bound. Value Probl. **2019**(46), 16 (2019)
29. Rajković, P.M., Marinković, S.D., Stanković, M.S.: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**, 311–323 (2007)
30. Ren, J., Zhai, C.: A fractional q-difference equation with integral boundary conditions and comparison theorem. Int. J. Nonlinear Sci. Numer. Simul. **18**(7–8), 575–583 (2017)
31. Ren, J., Zhai, C.: Nonlocal q-fractional boundary value problem with Stieltjes integral conditions. Nonlinear Anal. Model. Control **24**(4), 582–602 (2019)
32. Sevinik Adıgüzel, R.: Uniqueness of solution for second order nonlinear q-difference equations with multi-point and integral boundary conditions. J. Nonlinear Convex Anal. **23**(8), 1663–1672 (2022)
33. Shaimardan, S., Persson, L.E.: Some new Hardy-type inequalities for Riemann-Liouville fractional q -integral operator. J. Inequal. Appl. **296**, 1–17 (2015)
34. Shaimardan, S., Persson, L.E., Tokmagambetov, N.S.: Existence and uniqueness of some Cauchy type problems in fractional q-difference calculus. Filomat **34**(13), 4429–4444 (2020)
35. Stanković, M.S., Rajković, P.M., Marinković, S.D.: On the fractional q-derivative of Caputo type. C. R. Acad. Bulg. Sci. **63**(2), 197–204 (2010)
36. Wang, G.: Twin iterative positive solutions of fractional q-difference Schrödinger equations. Appl. Math. Lett. **76**, 103–109 (2018)
37. Zada, A., Alam, M., Riaz, U.: Analysis of q-fractional implicit boundary value problems having Stieltjes integral conditions. Math. Methods Appl. Sci. **44**(6), 4381–4413 (2021)
38. Zhao, Y., Chen, H., Zhang, Q.: Existence results for fractional q-difference equations with nonlocal q -integral boundary conditions. Adv. Differ. Equ. **48**, 15 (2013)
39. Zhao, Y., Ye, G., Chen, H.: Multiple positive solutions of a singular semipositone integral boundary value problem for fractional q-derivatives equation. Abstr. Appl. Anal. **2013**(643571), 12 (2013)