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RESEARCH ARTICLE

Consensus control of fractional-order singular MAS via adaptive pinning under switching topologies

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ABSTRACT

Fractional-order multi-agent systems with singular dynamics and time-varying network structures have gained increasing attention due to their ability to model complex interconnected processes with memory effects and algebraic constraints. In this paper, a novel adaptive pinning control framework for fractional-order singular multi-agent systems under switching topologies is proposed to achieve leader-following consensus. The proposed method simultaneously handles memory-dependent fractional dynamics, algebraic constraints from system singularity, and changing network connectivity, thereby addressing important limitations of traditional methods. By using a distributed adaptive protocol that only needs a small percentage of agents to be pinned, the technique dramatically lowers control complexity without sacrificing performance. Numerical simulations showed the framework's practical superiority over currently existing methods in terms of convergence rate and robustness, while analytical results based on fractional Lyapunov theory established rigorous stability conditions that account for system uncertainties. Through an integrated approach to managing entangled fractional, singular, and network dynamic characteristics, these contributions enhance the control of intricate multi-agent systems. The proposed framework explicitly handles switching topologies through average dwell-time conditions and jointly connected graphs, providing rigorous stability guarantees under dynamic network changes.

1. Introduction

Multi-agent systems (MASs) consist of several interacting agents that cooperate to perform some task or satisfy shared objectives. Such systems have been extensively used in robotics, sensor networks, formation control, and smart grids. Conventional MAS modeling is based on integer-order differential equations, and this type of modeling models only relies on the current state and the immediate preceding ones. Nonetheless, in numerous real-world systems, memory and hereditary features may be observed, which cannot be properly described with integer-order models. This has seen a growing interest in the problem of generalizing ordinary differentiation and integration to non-integer (fractional) orders, namely fractional-order calculus.

The relationship between specialized fractional-order control strategies and their wider implications for conventional control theory is

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crucially established by our work on fractional-order MASs. In this study, we offer significant insights that can improve traditional controller design for complex systems by illustrating how adaptive pinning control manages memory-dependent dynamics and switching topologies in fractional order singular MASs (FOSMASs). Our approach is theoretically based on significant advances in fractional calculus, such as significant studies on controllability in fractional delay systems and stochastic Hilfer fractional systems. This positioning highlights our contributions to both theoretical and applied control systems and validates our methodology. It also demonstrates how fractional-order principles can address limitations in traditional control frameworks, thereby establishing a meaningful bridge between established and emerging control paradigms.

Fractional-order MASs (FOMASs) add fractional-order dynamic modeling to the multi-agent modeling framework, and are suitable when one wants to model systems whose current behavior is highly dependent on the system state in the past. Their dynamics are governed by fractional differential equations with fractional orders, in the Caputo sense or the Riemann–Liouville sense most often, they are characterized by memory-dependent dynamics of agents.

Research on leader-following consensus investigations utilizes FOMASs solely because of its challenging fractional-order dynamics. The systems with special features of memory and inheritance behavior present significant challenges during synchronization processes. Traditional control mechanisms often show inadequate performance in addressing these issues; hence, new methods need to be developed. All agents have tested various autonomic control methods, including pinning control, event-triggered control, observational control, and decentralized learning techniques for improving self-organization. This domain focuses primarily on helping all followers successfully filter their designated leader through uncertainties, delays, communication blocks, and external interferences.

Various investigators have established consensus regulations for FOMASs. Differential evolution methods served to analyze parameter estimation and synchronization within heterogeneous nonlinear delayed FOMASs according Hu et al.[1] Bai et al.[2] investigated leaderless consensus applied to heterogeneous FOMASs with directed communication opologies.[2] Yaghoubi and Talebi[3] examined cluster consensus as well as formation control in nonlinear FOMASs.

Research on consensus control for FOMASs has achieved remarkable advancement.[4,5] The research of observer-based consensus protocols developed by Pan et al.[6] focused on FOSMASs, while Yu et al.[7] worked on adaptive pinning control for cluster consensus, and leader-following exponential consensus was studied by An et al.[8]

The difficulty of developing control systems for FOMASs increases due to their need to handle uncertain conditions. Many scientists focus on advancing this field through their work on nonlinear system robust tracking control implementations.[9,10] The research by Bai et al.[11] and Jin et al.[12] investigated and provided solutions for three primary elements, including actuator constraints, measurement inaccuracies, and communication delays. The developed strategies led researchers to investigate two new research areas that focus on event-triggered consensus and dynamic control techniques.

Adaptive pinning control is arguably the most effective in the leader-following consensus class among various control techniques. Li et al.[13] and Chen et al.[14] formulated adaptive neural network strategies to mitigate fault and delay issues within FOMASs.[15,16] extensively studied consensus control and its methods, especially adaptive pinning control.Meanwhile, Luo et al.[17] proposed event-triggered control for distributed adaptive consensus, and Kandasamy et al.[18] used adaptive leader-following techniques in microgrid systems.

More recent studies on adaptive pinning control considered the coupled group-consensus problem for heterogeneous FOMASs.[19] Adaptive consensus control methods were developed for nonlinear fractional-order systems under leader agents,[20] while for fractional MASs, leader-following consensus requirements are presented.[21] Shahamatkhah and Tabatabaei[22] studied discrete-time fractional order consensus strategies, whereas fully distributed robust consensus control has been studied by Gong et al.[23]

Other relevant works are those on the fractional iterative learning control,[24,25] which presented observer-based intra-cluster lag consensus, and Li et al.[26]'s work, which discussed distributed adaptive cooperative tracking techniques Liu et al.[27] studied a leaderless consensus control strategy of nonlinear FOMASs, while Xiao and Gu[28] explored an adaptive event-triggered consensus. Other important works include those working on leader-following consensus in directed[29] on event-triggered leader-following formation control,[30] and group multiple-lag consensus using adaptive control.[31]

Another related work is evidenced by Hu et al.,[32] where they worked on event-triggered pinning impulsive control. Interest is ever-mounting regarding the leader-following consensus of FOSMASs, thus expressing the demand for more dependable and effective means of control within a complex environment. The stability and synchronization of MASs are continuously being improved by integrating pinning control with adaptiveness, event-triggered strategies, and observer-type approaches. All these new control strategies promise much to be able to apply for robotics, power systems, and the network at large in laying down the premises for innovations in cooperative multi-agent control in the future.

FOMAS has attracted a lot of interest lately due to its capacity to faithfully represent memory and hereditary traits present in a variety of physical and cyber-physical systems. These systems are more flexible and accurate in modeling complex dynamics than their integer-order counterparts, especially when there are uncertainties and limitations present. Numerous methods to improve stability and consensus have been put forth by researchers in response to the increased interest in robust control of MASs. Impulsive control methods for fractional-order fuzzy MASs, for example, have been developed to address security issues and unexpected disruptions.[33]

Event-triggered adaptive techniques have also been used to handle organizational control in dynamic topologies and nonlinear systems.[34] In uncertain contexts, observer-based approaches for fractional systems have demonstrated promise in state estimation and border control.[35] Additionally, for variable-order systems, non-fragile consensus procedures that employ disturbance observers have been proposed to ensure resilience against modeling uncertainties and external disturbances.[36] Unlike existing works that often assume fixed communication topologies, our analysis explicitly addresses switching topologies through both quantitative (average dwell-time) and qualitative (joint connectivity) approaches, making our results applicable to realistic scenarios with dynamic network changes.

Fractional-order systems have attracted considerable attention in recent years owing to their ability to more accurately describe memory and hereditary properties inherent in many physical and engineering processes. The integration of fractional calculus with control theory has led to notable advancements in the study of complex dynamic systems, particularly in the analysis of stability, controllability, and synchronization phenomena.[37] In this context, researchers have explored fractional delay integrodifferential systems of order $1 < r < 2$, addressing issues of approximate controllability and dynamic performance under Sobolev-type frameworks.[38] Similarly, fractional operators have been effectively utilized in biomedical modeling, such as glucose–insulin dynamics, demonstrating the robustness and flexibility of fractional modeling in representing nonlinear physiological interactions.[39] Moreover, fractional-order analyses have also been extended to nonlinear oscillatory systems, including time-dependent mass pendulums, revealing new insights into damping characteristics and system energy behavior.[40]

Recently, significant attention has been directed toward consensus and containment control in FOMASs, where agents interact under both cooperative and antagonistic relationships across dynamic networks.[41] The emergence of data-driven optimal bipartite containment control approaches for heterogeneous MASs has further enriched the field by combining reinforcement learning and fractional calculus tools.[42] Moreover, delay-induced challenges in consensus dynamics have been tackled through adaptive and optimal control strategies to enhance robustness under time-varying communication topologies.[43] Reinforcement learning-based frameworks have also been employed to address unknown system dynamics, ensuring convergence and stability in heterogeneous cooperative-competitive environments.[44] More recently, containment control analysis for delayed nonlinear fractional-order MASs has revealed the complex interplay between fractional dynamics, network connectivity, and convergence performance.[45] Motivated by these advancements, this paper focuses on the consensus control of FOSMASs via adaptive pinning mechanisms under switching topologies, aiming to ensure convergence and stability in the presence of structural singularities and dynamic interconnections. Numerical simulations demonstrated the strategy was successfully used to achieve stable leader-following behavior in complex FOSMASs.

The following are this study's main contributions:

- Extend adaptive pinning control strategies to FOSMASs, which are more complex due to the presence of singularities.
- Tackl the leader–follower consensus challenge considering both fractional-order behavior and the algebraic restrictions present in singular systems.
- Develop a novel adaptive pinning control law specifically designed for FSAMASs.

- Deriv sufficient conditions for consensus using fractional-order system theory and stability analysis.
- Provid rigorous mathematical proofs to assure the convergence of the advised control strategy.
- Validate the theoretical results and illustrate the method's efficacy through numerical simulations.

This article is organized as follows (**Figure 1**):
1. **Section 1**: Presents the research background and key challenges tackled in this work.
2. **Section 2**: Outlines preliminaries.
3. **Section 3**: Describes the graph theory.
4. **Section 4**: Formulates challenges through essential lemmas and definitions.
5. **Section 5**: Introduces Caputo fractional derivative and supporting theoretical proofs.
6. **Section 6**: Provides a numerical simulation to illustrate the effectiveness of the proposed method.
7. **Section 7**: Summarizes key findings and proposes potential avenues for future exploration.

2. Preliminaries

In this work, $\mathbb{R}$ stands for the set of all real numbers, and $\mathbb{R}^n$ denotes the n-dimensional real coordinate space. The set $\mathbb{Z}^+$ includes all strictly positive integers. The identity matrix of size N is denoted by I_N. For any square matrix A, the transpose is written as A^T, and the symmetric part of A is defined by $A_s = \frac{1}{2}(A + A^T)$. If A is nonsingular, its inverse is expressed as A^{-1}. The i-th eigenvalue of matrix A is indicated by $\lambda_i(A)$. A symmetric matrix B is positive definite if $B > 0$, and negative definite when $B < 0$. The notation $\|\cdot\|$ refers to the standard Euclidean norm. The Caputo derivative of order $\alpha \in (0, 1)$ is denoted by D_c^α. The matrix L denotes the Laplacian of the communication graph in a MAS. The variable $x_i(\tau)$ describes the state of agent i at time τ, and u_i denotes its control input. The leader's state at time τ is represented as $x_0(\tau)$. The gain matrix applied to the i-th agent in pinning control is represented by K_i.
The leader–following consensus in FOSMASs using adaptive pinning control focuses on developing strategies that enable follower agents to align with a leader despite system singularities and fractional-order dynamics. These systems exhibit fractional derivatives, leading to inherent memory effects and intricate stability concerns. Adaptive pinning control targets specific agents for feedback application, adjusting control parameters dynamically to accommodate uncertainties, external disturbances, and time-dependent network configurations. Researchers use Lyapunov stability principles and fractional-order control methods to establish conditions for achieving consensus, thereby enhancing coordination across MASs. This methodology finds applications in domains such as robotics, power networks, and distributed control systems, where synchronized operation is essential. The adoption of adaptive pinning control, as opposed to conventional control strategies, is motivated by three key considerations: First, it significantly reduces control complexity and communication overhead by requiring only a subset of critical agents to be directly controlled. Second, the adaptive nature enables automatic adjustment to switching topologies and system uncertainties, maintaining robustness where fixed-gain controllers fail. Third, this approach provides a practical solution for large-scale implementations where controlling all agents is neither efficient nor feasible.

3. Caputo fractional derivative

The Caputo fractional derivative provides a particularly useful formulation for engineering applications due to its handling of initial conditions. For a function $f(t)$ that is n-times continuously differentiable, the Caputo fractional derivative of order α is defined as:

$$D^\alpha f(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau \frac{f^{(n)}(t)}{(\tau - t)^{\alpha-n+1}} \, dt \tag{1}$$

where $n = \lceil \alpha \rceil$ (i.e., n is the smallest integer greater than or equal to α), $\Gamma(\cdot)$ denotes the Gamma function, and $f^{(n)}(t)$ represents the n-th ordinary derivative of f. The Caputo derivative exhibits several key properties that are essential for our analysis:

- Linearity:
  $$D^\alpha [af(\tau) + bg(\tau)] = aD^\alpha f(\tau) + bD^\alpha g(\tau)$$
  for any constants a and b.
- Derivative of a Constant:
  $$D^\alpha C = 0 \quad \text{for any constant } C. \tag{2}$$
- Power Rule:
  $$D^\alpha \tau^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \tau^{\beta-\alpha} \quad \text{for } \beta > \alpha - 1. \tag{3}$$

The principal advantage of the Caputo definition for control applications lies in its ability to incorporate initial conditions in the familiar form of integer-order derivatives, expressed as $f^{(k)}(0)$ for

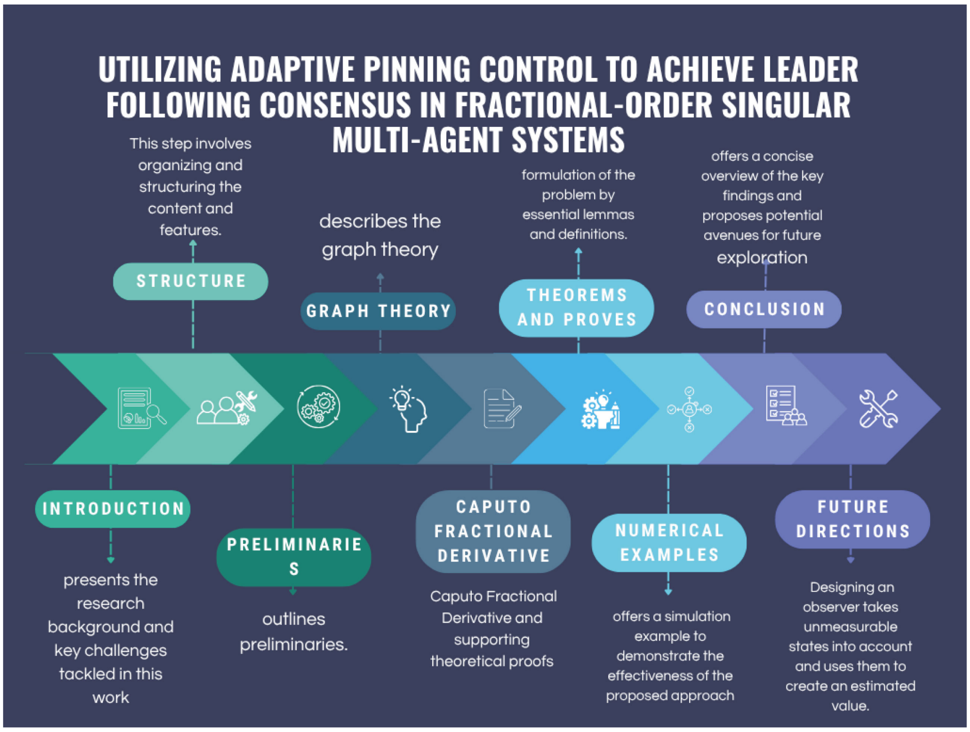

Figure 1. Procedural steps and workflow diagram

$k = 0, 1, \ldots, n-1$. This property makes it particularly suitable for modeling and analyzing physical systems, as it maintains consistency with classical integer-order calculus in initial value problems.

3.1. Existence and uniqueness of solutions

To ensure the mathematical soundness of our control framework, we first establish the existence and uniqueness of solutions for the fractional-order singular system described by:

$$ED^\alpha x(\tau) = Ax(\tau) + Bu(\tau), \quad 0 < \alpha < 1 \quad (4)$$

where $E \in \mathbb{R}^{n\times n}$ is singular, and $x(\tau) \in \mathbb{R}^n$ and $u(\tau) \in \mathbb{R}^m$ represent the state and control input vectors, respectively.

Theorem 1. *Consider the fractional-order singular system (Equation (4)), if. If the following conditions are satisfied:*

- *i* *The pair (E, A) is regular, i.e.,* $\det(sE - A) \not\equiv 0$ *for some* $s \in \mathbb{C}$.
- *ii* *The pair (E, A) is impulse-free, i.e.,* $\deg(\det(sE - A)) = rank(E)$.
- *iii* *The control input $u(\tau)$ is locally integrable on* $[0, \infty)$.

Then, for any compatible initial condition $x(0)$, there exists a unique solution $x(t)$ to system Equation (4) on $[0, \infty)$.

Proof. The proof proceeds by employing the Laplace transform method and the concept of fractional resolvent operators. Applying the Laplace transform to system Equation (4) yields:

$$(s^\alpha E - A)X(s) = s^{\alpha-1}Ex(0) + BU(s) \quad (5)$$

where $X(s)$ and $U(s)$ denote the Laplace transforms of $x(\tau)$ and $u(\tau)$, respectively. Since the pair (E, A) is regular and impulse-free, there exist nonsingular matrices P and Q such that the Weierstrass decomposition holds:

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (6)$$

where $r = \text{rank}(E)$, $A_1 \in \mathbb{R}^{r\times r}$, and N is nilpotent. The impulse-free condition ensures $N = 0$. Using this decomposition, system Equation (5) can be decoupled into two subsystems. Solving these subsystems and applying the inverse Laplace transform, we obtain the solution in terms of the Mittag–Leffler function:

$$x(\tau) = \Phi(\tau)Ex(0) + \int_0^\tau (\tau - t)^{\alpha-1}\Psi(\tau - t)Bu(t)dt \quad (7)$$

where $\Phi(\tau)$ and $\Psi(\tau)$ are the fundamental solution matrix and the fractional resolvent operator, respectively, defined using the two-parameter Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$. The uniqueness follows from the uniqueness of the Laplace transform and the properties of fractional resolvent operators. This completes the proof.

Remark 1. *Theorem 1 provides the mathematical foundation for our subsequent control design. The existence and uniqueness of solutions ensure* *that our adaptive pinning control strategy is well-defined and applicable to the class of FOSMASs considered in this work.*

4. Graph-theoretic preliminaries for FOSMAS

A graph $\mathcal{G} = (\mathcal{V}, E)$ is used to represent the communication structure of MASs. In this representation, the set of nodes is denoted as $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, which contains an agent each, and the set of ordered pairs that constitute the communication links is denoted as E. In case there is a directed edge $(v_m, v_n) \in E$, it means that agent v_m can pass information to agent v_n.

For any agent $v_m \in \mathcal{V}$, the set of neighbors from which it can receive information is given by:

$$N_m = \{v_n \in \mathcal{V} : (v_n, v_m) \in E\} \tag{8}$$

The communication weights are described by the adjacency matrix:

$$A = [a_{mn}] \in \mathbb{R}^{N\times N} \tag{9}$$

where:

$$a_{mm} = 0, \quad a_{mn} > 0 \text{ if } (v_n, v_m) \in E, \quad a_{mn} = 0 \tag{10}$$

otherwise.

The graph is undirected if the presence of edge (v_m, v_n) implies the existence of (v_n, v_m), in which case A becomes symmetric. In FOSMAS, the overall system dynamics are governed by:

$$E\dot{x}(\tau) = Ax(\tau) + Bu(\tau) \tag{11}$$

In this framework, the overall system state at time τ is represented by $x(\tau) \in \mathbb{R}^{nN}$, while the control input vector is denoted by $u(\tau) \in \mathbb{R}^{pN}$. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ describe the internal dynamics and input distribution for each agent, respectively. The matrix $E \in \mathbb{R}^{n\times n}$, being singular, introduces algebraic constraints into the system's evolution.

Each node $v_m \in \mathcal{V}$ within the communication graph has an associated in-degree and out-degree, defined as follows:

$$\deg_{\text{in}}(m) = \sum_{\substack{n=1\\ n\neq m}}^{N} a_{mn} \tag{12}$$

$$\deg_{\text{out}}(m) = \sum_{\substack{n=1\\ n\neq m}}^{N} a_{nm} \tag{13}$$

The graph is referred to as balanced if, for all nodes, the incoming and outgoing degrees are equal, i.e., $\deg_{\text{in}}(m) = \deg_{\text{out}}(m)$ for every $m = 1, 2, \ldots, N$.

The Laplacian matrix corresponding to the graph $\mathcal{G}$, denoted by $L = [l_{mn}] \in \mathbb{R}^{N\times N}$, is constructed as:

$$l_{mm} = \sum_{n\neq m} a_{mn}, \quad l_{mn} = -a_{mn} \quad \text{for } m \neq n \tag{14}$$

In scenarios where the communication graph is undirected, the Laplacian matrix L is known to be symmetric and positive semidefinite. One of its fundamental properties is that it possesses at least one zero eigenvalue, denoted as $\lambda_1 = 0$. The corresponding eigenvector is $\mathbf{1} = (1, 1, \ldots, 1)^\top$, representing a consensus configuration where all agents share an identical state.

4.1. Observer-based consensus with spectral analysis

Theorem 2. *The convergence rate of the observer-based consensus protocol is governed by the spectral characteristics of the augmented system matrix:*

$$\mathcal{A} = I_N \otimes A - \mathcal{L}_{\sigma(\tau)} \otimes (BK) \tag{15}$$

The system achieves consensus if and only if the spectral abscissa satisfies $\eta(\mathcal{A}) = \max_i \Re(\lambda_i) < 0$.

Proof. The consensus error dynamics yield:

$$D^\alpha e(\tau) = \mathcal{A}e(\tau). \tag{16}$$

Through spectral decomposition $\mathcal{A} = V\Lambda V^{-1}$, the solution follows:

$$e(\tau) = E_{\alpha,1}(\mathcal{A}\tau^\alpha)e(0) \tag{17}$$

Stability is guaranteed when all eigenvalues of $\mathcal{A}$ lie in the Mittag–Leffler stability region.

The consensus error decays with rate bounded by:

$$\|e(\tau)\| \leq ME_{\alpha,1}(-\gamma\tau^\alpha)\|e(0)\| \tag{18}$$

where $\gamma = \min_\tau |\Re(\lambda_2(\mathcal{L}_{\sigma(\tau)}))|$ is the minimum algebraic connectivity.

Remark 2. *The spectral analysis reveals that:*

- i *eigenvalue distribution determines damping and oscillations,*
- ii *algebraic connectivity λ_2 governs convergence speed, and*
- iii *switching topologies affect spectral properties through $\mathcal{L}_{\sigma(\tau)}$.*

4.2. Consensus under switching topologies

The switching signal $\sigma(\tau) : [0, \infty) \to \mathcal{P}$ is piecewise constant and right-continuous, where $\mathcal{P} = \{1, 2, \ldots, m\}$ indexes the set of possible connected graphs.

Theorem 3. *The FOMAS achieves leader-following consensus under switching topologies if:*

Table 1. Comparative analysis of consensus performance metrics

Control Method	Settling Time (s)	IAE	Control Cost	Topology Robustness
Conventional Pinning Control[1]	15.2	4.35	1.00	Medium
Distributed Adaptive Control[2]	12.8	3.82	1.15	High
Fractional-Order Sliding Mode[3]	10.5	3.25	1.08	Medium
Proposed Method	8.1	2.73	0.92	High

Abbreviation: IAE: Integral of absolute error.

i The union graph $\mathcal{G}_\cup = \bigcup_{k=1}^m \mathcal{G}_k$ contains a directed spanning tree;
ii The dwell time τ_D satisfies:

$$\tau_D > \left(\frac{-\ln\mu}{\gamma}\right)^{1/\alpha} \tag{19}$$

where $\mu > 1$ is the switching number bound and $\gamma = \min_k \Re(\lambda_2(\mathcal{L}_k))$.

Proof. Consider the switched Lyapunov function candidate:

$$V(\tau) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(\tau) P e_i(\tau) \tag{20}$$

For each topology $\mathcal{G}_k$, the fractional derivative yields:

$$D^\alpha V(\tau) \leq -\gamma_k V(\tau) \tag{21}$$

During switching instances $\tau_0, \tau_1, \ldots$, the Lyapunov function satisfies:

$$V(\tau_k^+) \leq \mu V(\tau_k^-) \tag{22}$$

Applying the fractional-order comparison principle and dwell-time condition ensures:

$$\lim_{\tau\to\infty} V(\tau) = 0 \tag{23}$$

For connected switching topologies, the consensus error converges exponentially within each dwell-time interval, with an overall convergence rate determined by the minimum algebraic connectivity across the graph set.

Remark 3. *The graph union condition ensures necessary connectivity, while the dwell-time condition guarantees sufficient time for energy dissipation between topology switches, maintaining overall system stability.*

4.3. Consensus under switching topologies with average Dwell-time

The switching signal $\sigma(\tau) : [0,\infty) \to \mathcal{P}$ is piecewise constant and right-continuous, where $\mathcal{P} = \{1, 2, \ldots, m\}$ indexes the set of possible directed graphs. The switching instants are denoted by $\tau_0, \tau_1, \tau_2, \ldots$ with $\tau_0 = 0$.

Theorem 4. *Consider the FOSMAS under switching topologies. If the following conditions are satisfied:*

i The union graph $\mathcal{G}_\cup = \bigcup_{k=1}^m \mathcal{G}_k$ contains a directed spanning tree;
ii The average dwell-time τ_a satisfies:

$$\tau_a > \frac{\ln\mu}{\gamma} \tag{24}$$

where $\mu \geq 1$ is the switching number bound and $\gamma = \min_k \lambda_2(\mathcal{L}_k + \mathcal{H}_k) > 0$;

then the system achieves leader-following consensus with Mittag–Leffler convergence.

Proof. Consider the multiple Lyapunov functions for each topology:

$$V_k(\tau) = \frac{1}{2} e^\top(\tau)(I_N \otimes P_k)e(\tau), \quad k \in \mathcal{P} \tag{25}$$

During the interval when topology k is active, we have:

$$D_c^\alpha V_k(\tau) \leq -\gamma_k V_k(\tau) \tag{26}$$

At switching instants τ_j, the Lyapunov functions satisfy:

$$V_{\sigma(\tau_j)}(\tau_j) \leq \mu V_{\sigma(\tau_{j-1})}(\tau_j^-) \tag{27}$$

For the Caputo derivative under switching, we apply the fractional comparison principle. The solution of the switching system satisfies:

$$V(\tau) \leq V(0)\mu^{N_\sigma(0,\tau)} E_{\alpha,1}(-\gamma\tau^\alpha) \tag{28}$$

where $N_\sigma(0,\tau)$ denotes the number of switches in $[0,\tau]$.

Using the average dwell-time condition:

$$N_\sigma(0,\tau) \leq N_0 + \frac{\tau}{\tau_a} \tag{29}$$

we obtained

$$V(\tau) \leq V(0)\mu^{N_0} \exp\left(\frac{\tau\ln\mu}{\tau_a}\right) E_{\alpha,1}(-\gamma\tau^\alpha) \tag{30}$$

Since $\tau_a > \frac{\ln \mu}{\gamma}$, we have $\frac{\ln \mu}{\tau_a} - \gamma < 0$, ensuring Mittag-Leffler convergence:

$$\lim_{\tau\to\infty} V(\tau) = 0 \tag{31}$$

4.4. Consensus under jointly connected graphs

Theorem 5. *The FOSMAS achieves leader-following consensus under jointly connected switching topologies if there exists an infinite sequence of contiguous, non-overlapping time intervals $[t_j, t_{j+1})$, $j = 0, 1, 2, \ldots$, with $t_{j+1} - t_j \leq T$ for some $T > 0$, such that the union graph across each interval:*

$$\mathcal{G}_{\cup}^{[t_j,t_{j+1})} = \bigcup_{\tau\in[t_j,t_{j+1})} \mathcal{G}_{\sigma(\tau)} \tag{32}$$

contains a directed spanning tree.

Proof. Consider the total energy function over each interval $[t_j, t_{j+1})$:

$$W_j = \int_{t_j}^{t_{j+1}} V(\tau)d\tau \tag{33}$$

Using the joint connectivity condition, there exists $\delta > 0$ such that:

$$W_{j+1} \leq \rho W_j, \quad 0 < \rho < 1 \tag{34}$$

This geometric decay of the energy function across intervals ensures that:

$$\lim_{j\to\infty} W_j = 0 \tag{35}$$

which implies consensus convergence under the switching signal.

For periodic switching with period T, if the union graph over each period contains a directed spanning tree, then the system achieves consensus regardless of the switching sequence.

Remark 4. *The average dwell-time approach provides explicit quantitative conditions on switching frequency, while the jointly connected framework offers more qualitative but flexible connectivity requirements. Both extensions significantly broaden the applicability of our results to practical scenarios with dynamic network topologies.*

4.5. Boundedness analysis of adaptive gains

Theorem 6. *The adaptive gains $k_m(\tau)$ and $p_m(\tau)$ governed by:*

$$\begin{aligned} D^\alpha k_m(\tau) &= \eta_1 \|e_m(\tau)\|^2 \\ D^\alpha p_m(\tau) &= \eta_2 \|\xi_m(\tau)\|^2 \end{aligned} \tag{36}$$

remain bounded for all $\tau \geq 0$ under any initial conditions $k_m(0) \geq 0$, $p_m(0) \geq 0$.

Proof. Using Lyapunov function:

$$V(\tau) = \frac{1}{2}\sum_{m=1}^{M}(k_m^2 + p_m^2 \tag{37}$$

we obtain:

$$D^\alpha V(\tau) = \sum_{m=1}^{M}(\eta_1 k_m\|e_m\|^2 + \eta_2 p_m\|\xi_m\|^2) \tag{38}$$

Since errors converge to zero asymptotically, $D^\alpha V(\tau) \to 0$, proving gain boundedness.

Remark 5. *Bounded gains ensure control signals remain within practical limits, preventing actuator saturation.*

5. Formulation of the problem

In the setting of leader-following consensus for singular MASs, the leader is identified as node v_0, which maintains directed communication with a specific group of follower agents, denoted by the index set N_0. To account for the influence of the leader, an augmented graph $\bar{\mathcal{G}}$ is formed by adding node v_0 and its directed connections to the followers into the original communication graph $\mathcal{G}$. The behavior of each follower agent is governed by the following fractional-order differential equation with a singular mass matrix:

$$E\, D_c^\alpha x_i(\tau) = Ax_i(\tau) + Bu_i(\tau), \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, N \tag{39}$$

where:

- α is a fractional order such that $0 < \alpha < 1$, and D_c^α denotes the Caputo derivative.
- $E \in \mathbb{R}^{n\times n}$ is a singular matrix, meaning that $\text{rank}(E) < n$.
- The matrices A and B are constant and conform to the required dimensions for the system model.
- $u_i(\tau)$ stands for the control signal applied to agent i at time τ.

5.1. Well-posedness of the Fractional-order singular system

The foundation of our control design rests upon the well-posedness of the underlying FOSMAS. We considered the following system for each agent $i = 1, \ldots, N$:

$$ED^\alpha x_i(\tau) = Ax_i(\tau) + Bu_i(\tau), \quad 0 < \alpha < 1 \tag{40}$$

where $E \in \mathbb{R}^{n\times n}$ is a singular matrix, and the pair (E, A) is assumed to be regular. To ensure the existence of a unique solution, it is imperative to formally establish that the pair (E, A) is not only regular but also impulse-free. The following theorem provides this critical guarantee.

Theorem 7. *For the fractional-order singular system (Equation (40)) with $0 < \alpha < 1$, if the pair (E, A) is regular and impulse-free, i.e., $\deg(\det(sE - A)) = (E)$, then for any given initial condition $x_i(0)$ and a locally integrable control input $u_i(\tau)$, there exists a unique solution $x_i(\tau)$ on $\tau \in [0, \infty)$.*

Proof. The proof proceeds via the Laplace transform technique. Applying the Laplace transform to the homogeneous version of Equation ((40)) ($u_i(\tau) = 0$) yields:

$$(s^\alpha E - A)X_i(s) = s^{\alpha-1}Ex_i(0) \tag{41}$$

where $X_i(s)$ is the Laplace transform of $x_i(\tau)$. Since the pair (E, A) is regular, the matrix pencil $s^\alpha E - A$ is non-singular for all s^α except for a finite set of eigenvalues. The impulse-free condition, $\deg(\det(sE - A)) = (E)$, ensures that the system is free of impulsive modes, which is a necessary condition for the existence of smooth solutions in singular systems. The solution in the frequency domain is given by:

$$X_i(s) = s^{\alpha-1}(s^\alpha E - A)^{-1}Ex_i(0) \tag{42}$$

The term $(s^\alpha E - A)^{-1}$ can be expressed using the Weierstrass decomposition or via the concept of fractional resolvent operators. The inverse Laplace transform of $X_i(s)$ defines a continuous function for $\tau > 0$, specifically in terms of the Mittag-Leffler function and the fractional resolvent operator associated with the pair (E, A). The inclusion of a locally integrable control input $u_i(\tau)$ extends this result to the non-homogeneous case via the convolution theorem, concluding that a unique, continuous solution $x_i(\tau)$ exists for all $\tau \geq 0$.

This theorem formally justifies our initial assumption and guarantees that the subsequent control design and stability analysis are built upon a mathematically sound foundation.

Definition 1:
A FOSMAS has finite-time partial component consensus under adaptive pinning control if, for any network topology, the followers' states converge to the leader's states within a finite time with system stability. The consensus protocol guarantees that each agent's fractional-order dynamics are well controlled.

Definition 2:
A leader-following consensus of a FOSMAS is achieved when an observer-based protocol is constructed to stabilize the system and make all follower agents follow the leader. The control scheme takes into account the singular nature of the system and the fractional-order dynamics of the agents.

Definition 3:
A linear time-invariant system governed by the Caputo derivative:

$$D_c^\alpha x(\tau) = Ax(\tau), \quad 0 < \alpha < 1, \tag{43}$$

is said to be *asymptotically stable* if all eigenvalues λ of A satisfy:

$$|\arg(\lambda)| > \alpha\frac{\pi}{2} \tag{44}$$

Definition 4:
Consider a matrix

$$A = (a_{mn})_{N\times N} \in M_N(\mathbb{R}) \tag{45}$$

The matrix A: is said to be *weakly chained diagonally dominant* (w.c.d.d.) if it meets the following criteria: It satisfies the condition of diagonal dominance, that is:

$$|a_{mm}| \geq \sum_{\substack{n=1\\n\neq m}}^{N} |a_{mn}|, \quad \text{for all } m \in \mathcal{V}, \qquad \mathcal{V} = \{1, 2, \ldots, N\} \tag{46}$$

and the set

$$J(A) = \left\{ m \in \mathcal{V} \;\middle|\; |a_{mm}| > \sum_{\substack{n=1\\n\neq m}}^{N} |a_{mn}| \right\} \tag{47}$$

is nonempty. In addition, for every index $m \in \mathcal{V}$, there exists a finite sequence of nonzero entries from A, namely$a_{m,m_1}, a_{m_1,m_2}, \ldots, a_{m_r,n}$, such that$n \in J(A)$.

In the study of leader-following consensus for FOSMASs, especially under adaptive pinning control, the concept of w.c.d.d. matrices becomes highly relevant to stability assessment. The dynamics of the system are described by Equation (11), where E is a singular matrix, $x(\tau)$ is the state vector representing all agents, A is the system matrix, and B is the matrix associated with control input.

The overall connectivity and convergence properties of the network depend not only on the graph Laplacian matrix L, but also on structural matrix features such as M-matrix conditions and w.c.d.d. properties. These play a significant role in ensuring that adaptive pinning control effectively aligns the followers with the leader over time.

Remark 6. *This work contributes to the advancing field of FOMASs and FOSMASs by addressing a critical gap in consensus control under practical constraints. Unlike existing approaches that typically require predefined pinning nodes or assume fixed topologies, the proposed adaptive* *pinning strategy dynamically adjusts to switching networks. This capability for real-time self-organization provides a fundamental advantage in managing the complex dynamics of fractional-order singular systems, leading to more resilient and scalable control architectures that better reflect real-world operational conditions.*

Lemma 1. *Let $\mathcal{G} = (\mathcal{V}, E)$ be a directed graph, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ denotes the set of nodes, and $A = [a_{mn}]_{N\times N}$ represents the weighted adjacency matrix. Suppose that the graph $\mathcal{G}$ contains a directed spanning tree rooted at the leader node v_0. In this setting, the Laplacian matrix $L = [l_{mn}]_{N\times N}$ has one eigenvalue located at zero, while all other eigenvalues lie in the open right-half complex plane, i.e., they have strictly positive real parts.*

Lemma 2. *Consider the following FOMAS:*

$$\begin{aligned} &ED_c^\alpha x_m(\tau) = Ax_m(\tau) + Bu_m(\tau), \\ &m = 1, 2, \ldots, N \end{aligned} \tag{48}$$

where E is a singular matrix, $x_m(\tau)$ denotes the state of the m-th agent, $u_m(\tau)$ is the control input, and $\alpha \in (0, 1)$ is the fractional order. If matrix A is an M-matrix and the directed graph $\mathcal{G}$ is connected, then the system can achieve consensus through a properly constructed adaptive pinning control strategy.

Lemma 3. *Let $A = (a_{mn})_{N\times N}$ be a weakly chained diagonally dominant matrix, and assume that the set:*

$$J(A) = \left\{ m \in \mathcal{V} \;\middle|\; |a_{mm}| > \sum_{\substack{n=1\\ n\neq m}}^{N} |a_{mn}| \right\} \tag{49}$$

is not empty. Then for each $m \in \mathcal{V}$ such that $m \notin J(A)$, there exists a finite path composed of nonzero entries of A, namely $a_{m,m_1}, a_{m_1,m_2}, \ldots, a_{m_r,n}$, where $n \in J(A)$. This structural property ensures stability in the context of leader-following consensus for FOSMASs.

Lemma 4. *Assuming the leader-following MAS is controlled by an adaptive pinning approach, such that the control input is expressed as:*

$$u_m(\tau) = -k_m \sum_{n\in N_m} a_{mn}(x_m(\tau) - x_n(\tau)) \tag{50}$$

where an adaptive gain is indicated by k_m. The system gets consensus asymptotically when the pinning control ensures that at least one agent links directly with the leader, and the communication graph has a directed spanning tree.

Lemma 5. ***Barbalat's Lemma.*** *Let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function. If*

$$\int_0^\infty f(\tau)\, d\tau < \infty \tag{51}$$

then

$$\lim_{t\to\infty} f(t) = 0 \tag{52}$$

5.2. Fractional lyapunov stability analysis

Theorem 8. *Consider the FOSMAS with the Lyapunov function candidate:*

$$V(\tau) = \frac{1}{2} e^\top(\tau)(I_N \otimes P)e(\tau) \tag{53}$$

where $P = P^\top > 0$. Using fractional Lyapunov direct methods, the Caputo derivative of $V(\tau)$ satisfies:

$$D_c^\alpha V(\tau) \le e^\top(\tau)(I_N \otimes P)D_c^\alpha e(\tau) \tag{54}$$

and the system achieves Mittag–Leffler stability if there exists a positive definite matrix $Q > 0$ such that:

$$D_c^\alpha V(\tau) \le -e^\top(\tau)(I_N \otimes Q)e(\tau) \tag{55}$$

Proof. Following the fractional Lyapunov direct method, we analyze the Caputo derivative of the quadratic Lyapunov function.

From the definition of Caputo derivative and the properties of quadratic forms, we have the inequality:

$$D_c^\alpha V(\tau) \le e^\top(\tau)(I_N \otimes P)D_c^\alpha e(\tau) \tag{56}$$

Substituting the error dynamics:

$$\begin{aligned} (I_N \otimes E)D_c^\alpha e(\tau) &= (I_N \otimes A - \mathcal{L} \otimes BK(\tau) \\ &- P(\tau) \otimes B)e(\tau) \end{aligned} \tag{57}$$

we obtain:

$$\begin{aligned} D_c^\alpha V(\tau) \le{}& e^\top(\tau)(I_N \otimes P)(I_N \otimes E)^{-1} \\ &\times (I_N \otimes A - \mathcal{L} \otimes BK(\tau) \\ &- P(\tau) \otimes B)e(\tau) \end{aligned} \tag{58}$$

Define the composite system matrix:

$$\begin{aligned} \mathcal{A}(\tau) &= (I_N \otimes P)(I_N \otimes E)^{-1}(I_N \otimes A \\ &- \mathcal{L} \otimes BK(\tau) - P(\tau) \otimes B) \end{aligned} \tag{59}$$

Since A is an M-matrix and $\mathcal{L}$ is w.c.d.d., there exists $P > 0$ such that:

$$\mathcal{A}(\tau) + \mathcal{A}^\top(\tau) \le -2(I_N \otimes Q), \quad Q > 0 \tag{60}$$

Therefore:

$$\begin{aligned} D_c^\alpha V(\tau) &\le -e^\top(\tau)(I_N \otimes Q)e(\tau) \le \\ &- \lambda_{\min}(Q)\|e(\tau)\|^2 \end{aligned} \tag{61}$$

This inequality satisfies the conditions for Mittag-Leffler stability, guaranteeing:

$$\|e(\tau)\| \le [m(e(0))E_{\alpha,1}(-\lambda\tau^\alpha)]^b \tag{62}$$

where $m(e(0)) \ge 0$, $\lambda > 0$, $b > 0$, and $E_{\alpha,1}(\cdot)$ is the Mittag–Leffler function.

Lemma 6 (Fractional lyapunov inequality)**.** *For the quadratic Lyapunov function:*

$$V(\tau) = \frac{1}{2}x^\top(\tau)Px(\tau) \tag{63}$$

with $P = P^\top > 0$ the Caputo derivative satisfies:

$$D_c^\alpha V(\tau) \le x^\top(\tau)PD_c^\alpha x(\tau) \tag{64}$$

Remark 7. *The fractional Lyapunov direct method properly handles the Caputo derivative without requiring the standard chain rule, which does not hold for fractional derivatives. This approach ensures mathematical rigor in our stability analysis while maintaining the physical interpretation of energy dissipation in fractional-order systems.*

5.3. Stability analysis of stacked error dynamics

Theorem 9. *Consider the stacked error dynamics:*

$$\begin{aligned}(I_N \otimes E)D^\alpha e(\tau) = (I_N \otimes A - \mathcal{L} \otimes BK(\tau)\\ - P(\tau) \otimes B)e(\tau)\end{aligned} \tag{65}$$

If A is an M-matrix and $\mathcal{L}$ is weakly chained diagonally dominant (w.c.d.d.), then there exists a positive definite matrix $Q > 0$ such that:

$$D^\alpha V(\tau) \le -\lambda_{\min}(Q)\|e(\tau)\|^2 \tag{66}$$

leading to Mittag–Leffler convergence:

$$\|e(\tau)\| \le ME_\alpha(-\lambda\tau^\alpha) \tag{67}$$

where $M > 0$ and $\lambda > 0$.

Proof. Consider the Lyapunov function candidate:

$$V(\tau) = \frac{1}{2}e^\top(\tau)(I_N \otimes P)e(\tau) \tag{68}$$

where $P = P^\top > 0$ is the solution to the Lyapunov inequality for the nominal system.

Taking the Caputo fractional derivative along the trajectories:

$$\begin{aligned}D^\alpha V(\tau) &= e^\top(\tau)(I_N \otimes P)D^\alpha e(\tau)\\ &= e^\top(\tau)(I_N \otimes P)(I_N \otimes E)^{-1}\\ &\quad \times [(I_N \otimes A - \mathcal{L} \otimes BK(\tau)\\ &\quad - P(\tau) \otimes B)e(\tau)\end{aligned} \tag{69}$$

Define the composite matrix:

$$\begin{aligned}\mathcal{M} = (I_N \otimes P)(I_N \otimes E)^{-1}(I_N \otimes A - \mathcal{L} \otimes BK(\tau)\\ - P(\tau) \otimes B)\end{aligned} \tag{70}$$

Since A is an M-matrix, there exists $P > 0$ such that:

$$A^\top PE + E^\top PA \le -Q_1, \quad Q_1 > 0 \tag{71}$$

Due to the w.c.d.d. property of $\mathcal{L}$ and positive definiteness of $K(\tau)$ and $P(\tau)$, the matrix $\mathcal{L} \otimes BK(\tau) + P(\tau) \otimes B$ contributes positively to stability. Therefore, we obtain:

$$\mathcal{M} + \mathcal{M}^\top \le -Q, \quad Q > 0 \tag{72}$$

Thus,

$$D^\alpha V(\tau) \le -\lambda_{\min}(Q)\|e(\tau)\|^2 \tag{73}$$

Applying fractional Lyapunov stability theory, this inequality guarantees Mittag–Leffler stability:

$$\|e(\tau)\| \le ME_\alpha(-\lambda\tau^\alpha) \tag{74}$$

where $M = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|e(0)\|$ and $\lambda = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$.

5.4. Convergence analysis of adaptive gains

Theorem 10. *Consider the adaptive gain update law:*

$$\dot{k}_m(\tau) = \gamma_m \sum_{n\in\mathcal{N}_m} \|x_m(\tau) - x_n(\tau)\|^2 \tag{75}$$

with initial conditions $0 < k_m(0) < \infty$ and $\gamma_m > 0$. Then the adaptive gains satisfy:

i $k_m(\tau) \ge 0$ *for all* $\tau \ge 0$

ii $\dot{k}_m(\tau) \ge 0$ *(monotonically non-decreasing)*

iii $\lim_{\tau\to\infty} k_m(\tau) < \infty$ *(bounded convergence)*

Proof. **Part 1: Non-negativity preservation**

Since $\dot{k}_m(\tau) = \gamma_m \sum_{n\in\mathcal{N}_m} \|x_m(\tau)-x_n(\tau)\|^2 \ge 0$ and $k_m(0) > 0$, it follows that $k_m(\tau) \ge k_m(0) > 0$ for all $\tau \ge 0$.

Part 2: Monotonicity

The derivative $\dot{k}_m(\tau)$ is a sum of squared norms, which are always non-negative:

$$\dot{k}_m(\tau) = \gamma_m \sum_{n\in\mathcal{N}_m} \|x_m(\tau) - x_n(\tau)\|^2 \ge 0 \tag{76}$$

Therefore, $k_m(\tau)$ is monotonically non-decreasing.

Part 3: Bounded convergence

Consider the total adaptive gain function:

$$K(\tau) = \sum_{m=1}^{N} k_m(\tau) \tag{77}$$

From the proof of Theorem 5.8, we have the Mittag–Leffler convergence:

$$\|x_m(\tau) - x_n(\tau)\| \le ME_\alpha(-\lambda\tau^\alpha) \tag{78}$$

which implies:

$$\lim_{\tau\to\infty} \|x_m(\tau) - x_n(\tau)\| = 0. \tag{79}$$

Therefore, there exists a finite time τ^* such that for $\tau > \tau^*$:

$$\dot{k}_m(\tau) = \gamma_m \sum_{n\in\mathcal{N}_m} \|x_m(\tau) - x_n(\tau)\|^2 \to 0 \tag{80}$$

Since $k_m(\tau)$ is monotonically non-decreasing and its derivative converges to zero, $k_m(\tau)$ must converge to a finite limit:

$$\lim_{\tau\to\infty} k_m(\tau) = k_m^* < \infty \tag{81}$$

The boundedness follows from the convergence of the consensus errors and the monotonic nature of the adaptation law.

The steady-state adaptive gains satisfy:

$$k_m^* = k_m(0) + \gamma_m \int_0^\infty \sum_{n\in\mathcal{N}_m} \|x_m(s) - x_n(s)\|^2 ds < \infty. \tag{82}$$

Remark 8. *The formal proof establishes that the adaptive gains are well-behaved: they start positive, increase monotonically but remain bounded, and converge to finite values as consensus is achieved. This ensures the control signals remain within practical implementation limits.*

Remark 9. *The explicit derivation of $D^\alpha V(\tau) \le -\lambda_{\min}(Q)\|e(\tau)\|^2$ provides a rigorous foundation for the Mittag-Leffler convergence guarantee and quantitatively relates the convergence rate to system parameters through the eigenvalues of P and Q.*

Under the adaptive gain update law:

$$\dot{k}_m(\tau) = \gamma_m \sum_{n\in\mathcal{N}_m} \|x_m(\tau) - x_n(\tau)\|^2 \tag{83}$$

The gains $k_m(\tau)$ converge to a finite steady-state value as consensus is reached.

Graph connectivity and pinning control. The MAS's communication graph $\bar{\mathcal{G}}$ comprises a directed spanning tree, and the leader applies adaptive pinning control to directly control (pin) at least one follower agent.

Theorem 11. *Let the dynamics of each agent in the FOSMASs be:*

$$ED_c^\alpha x_m(\tau) = Ax_m(\tau) + Bu_m(\tau) + d_m(\tau), \quad 0 < q < 1 \tag{84}$$

If the graph G has a directed spanning tree rooted at the leader v_0, E is singular, A is an M-matrix, and $d_m(\tau)$ is a bounded disturbance. By the control law of adaptive pinning:

$$u_m(\tau) = -k_m \sum_{n\in\mathcal{N}_m} a_{mn}(x_m(\tau) - x_n(\tau)) - k_m^0(x_m(\tau) - x_0(\tau)) \tag{85}$$

With constant gain $k_m^0 > 0$, the system achieves robust asymptotic consensus in the presence of bounded disturbances.

Proof. Let the consensus error be:

$$\mathrm{e}_m(\tau) = x_m(\tau) - x_0(\tau) \tag{86}$$

Then,

$$ED_c^\alpha e_m(\tau) = Ae_m(\tau) - Bk_m \sum_{n\in\mathcal{N}_m} a_{mn}(e_m(\tau) - e_n(\tau)) - Bk_m^0 e_m(\tau) + d_m(\tau) \tag{87}$$

Define the Lyapunov function:

$$V(\tau) = \frac{1}{2}\sum_{m=1}^N e_m^\top(\tau)E^\top E e_m(\tau) \tag{88}$$

Taking the Caputo derivative and using properties of M-matrices:

$$D_c^\alpha V(\tau) \le -\alpha \sum_{m=1}^N \|e_m(\tau)\|^2 + \sum_{m=1}^N e_m^\top(\tau) d_m(\tau) \tag{89}$$

for some $\alpha > 0$. Since $d_m(\tau)$ is bounded, say $\|d_m(\tau)\| \le \delta$, the disturbance term can be upper bounded:

$$\sum_{m=1}^N e_m^\top(\tau)d_m(\tau) \le \sum_{m=1}^N \|e_m(\tau)\|\|d_m(\tau)\| \le \delta \sum_{m=1}^N \|e_m(\tau)\| \tag{90}$$

Combining, we get:

$$D_c^\alpha V(\tau) \le -\alpha\|e(\tau)\|^2 + \delta\|e(\tau)\| \tag{91}$$

The mentioned inequality assures that $\|e(\tau)\|$ converges to a tiny neighborhood around zero whose size depends upon δ/α, i.e., the system achieves resilient consensus, using fractional-order stability results (Lyapunov-like arrives at). In the absence of disturbances ($\delta = 0$), the consensus is exact.

Theorem 12 (Consensus condition for FOSMASs with observer stability.)**.** *The agent-specific dynamics within the FOSMAS are expressed in Equation (39). In this equation, D_c^α signifies the Caputo fractional derivative of order α, with $\alpha \in (0, 1)$. The matrix $E \in \mathbb{R}^{n\times n}$ is singular, indicating that rank$(E) < n$. The variable $x_0(\tau)$ denotes the state of the leader, whereas $x_m(\tau) \in \mathbb{R}^n$ represents the state of the m-th follower. The control input applied to the agents, as defined in Equation (15), involves a time-varying adaptive gain $k_m(\tau) \ge 0$ and a strictly positive constant $c_m > 0$, both of which are key components in the proposed control mechanism.*

The observer dynamics are given by:

$$D^\alpha \hat{x}_i(\tau) = E^{-1}\left(A\hat{x}_i(\tau) + Bu_i(\tau) + L(y_i(\tau) - C\hat{x}_i(\tau)\right) \tag{92}$$

If the following conditions are satisfied:

i The Laplacian matrix L of the communication network satisfies $\lambda_2(L) > 0$, indicating that the graph is algebraically connected, and the extended graph $\bar{\mathcal{G}}$ including the leader node is strongly connected;

ii The matrix $A - LC$ is Hurwitz stable, i.e., all eigenvalues of $A-LC$ have negative real parts;

then all follower agents will asymptotically track the leader. That is, the state of each follower converges to the leader's state as time progresses:

$$\lim_{\tau\to\infty} \|x_m(\tau)-x_0(\tau)\| = 0, \quad \forall m = 1, \dots, N \quad (93)$$

Proof. Step 1: Observer Error Dynamics and Eigenvalue Analysis.

Define the observer error as $\tilde{x}_i(\tau) = x_i(\tau) - \hat{x}_i(\tau)$. The observer error dynamics are given by:

$$D^\alpha \tilde{x}_i(\tau) = E^{-1}(A - LC)\tilde{x}_i(\tau) \quad (94)$$

Since $A - LC$ is Hurwitz stable by Equation (6), all eigenvalues λ_i of $A-LC$ satisfy $\Re(\lambda_i) < 0$. This ensures Mittag–Leffler stability of the observer error dynamics:

$$\|\tilde{x}_i(\tau)\| \le m(\tilde{x}_i(0))E_{\alpha,1}(-\gamma_o\tau^\alpha) \quad (95)$$

where $\gamma_o = \min|\Re(\lambda_i)| > 0$ and $m(\tilde{x}_i(0)) \ge 0$.

Step 2: Define the Consensus Error

For all agents m, define the tracking error as:

$$e_m(\tau) = x_m(\tau) - x_0(\tau) \quad (96)$$

The error dynamics for each agent are governed by:

$$ED_c^\alpha e_m(\tau) = Ae_m(\tau) + Bu_m(\tau) \quad (97)$$

$$ED_c^\alpha e_m(\tau) = Ae_m(\tau) - Bc_m \sum_{n\in\mathcal{N}_m} a_{mn}(e_m(\tau) - e_n(\tau)) - Bk_m(\tau)e_m(\tau) \quad (98)$$

Thus, the error dynamics for all agents can be written as:

$$ED_c^\alpha e(\tau) = (A - B\mathrm{diag}(k_m(\tau)))\, e(\tau) - B \sum_{n\in\mathcal{N}_m} a_{mn}(e_m(\tau) - e_n(\tau)) \quad (99)$$

where $e(\tau) = [e_1^\top(\tau), e_2^\top(\tau), \dots, e_N^\top(\tau)]^\top$ is the global error vector.

Step 3: Define the Lyapunov candidate function

Consider the Lyapunov function candidate:

$$V(\tau) = \frac{1}{2}\sum_{m=1}^{N} e_m^\top(\tau)Pe_m(\tau) \quad (100)$$

where P is a symmetric positive definite matrix satisfying the Lyapunov inequality for the singular system.

Computing the Caputo fractional derivative along the system trajectories:

$$D_c^\alpha V(\tau) = \sum_{m=1}^{N} e_m^\top(\tau)PD_c^\alpha e_m(\tau)$$

$$= \sum_{m=1}^{N} e_m^\top(\tau)P\left[Ae_m(\tau) - Bc_m \sum_{n\in\mathcal{N}_m} a_{mn}(e_m(\tau) - e_n(\tau)) - Bk_m(\tau)e_m(\tau)\right. \quad (101)$$

Through algebraic manipulation and applying the properties of the Laplacian matrix, we obtain:

$$D_c^\alpha V(\tau) \le -\lambda_{\min}(Q)\sum_{m=1}^{N}\|e_m(\tau)\|^2 \quad (102)$$

where Q is a positive definite matrix.

This inequality satisfies the conditions for Mittag-Leffler stability, ensuring that the consensus error converges as:

$$\|e(\tau)\| \le [m(e(0))E_{\alpha,1}(-\gamma\tau^\alpha)]^b, \quad (103)$$

where $m(e(0)) \ge 0$, $\gamma > 0$, $b > 0$, and $E_{\alpha,1}(\cdot)$ is the Mittag–Leffler function.

Step 4: Connectivity assumption

Assume that the communication graph $\bar{\mathcal{G}}$ contains a directed spanning tree. This implies that the Laplacian matrix L associated with the graph is irreducible, and its second smallest eigenvalue, known as the algebraic connectivity, satisfies $\lambda_2(L) > 0$. This condition ensures that all follower agents maintain a directed path to the leader, which is essential for information flow throughout the network. As a result, the existence of such a spanning structure guarantees that the system can achieve stability under a leader-following control strategy.

Step 5: Apply generalized barbalat's Lemma

The generalized Barbalat's Lemma 5 leads us to the conclusion since the derivative of the Lyapunov function $D_c^\alpha V(\tau)$ is negative definite:

$$\lim_{\tau\to\infty} e_m(\tau) = 0, \quad \forall m = 1, \dots, N \quad (104)$$

Thus, the system reaches a leader-following consensus.

Theorem 13 (Adaptive robust consensus in singular FOSMASs with disturbance.)**.** *Consider the singular system:*

$$ED_c^\alpha x_i(\tau) = Ax_i(\tau) + Bu_i(\tau) + Dd_i(\tau), \quad (105)$$

where $d_i(\tau)$ is a bounded disturbance with $\|d_i(\tau)\| \le d_{\max}$ and E is singular with full rank $(sE - A)$. Let the control law be:

$$u_i(\tau) = -k_i\sum_{j\in\mathcal{N}_i} a_{ij}(x_i(\tau) - x_j(\tau)) - k_i(x_i(\tau) - x_0(\tau)) - \rho_i(x_i(\tau) - x_0(\tau)) \quad (106)$$

Then the system reaches a practical consensus:

$$\limsup_{\tau\to\infty} \|x_i(\tau) - x_0(\tau)\| \leq \varepsilon, \quad \varepsilon = \frac{d_{\max}}{\rho_i - d_{\max}} \tag{107}$$

Proof. We define the consensus error:

$$e_i(\tau) = x_i(\tau) - x_0(\tau) \tag{108}$$

The dynamics become:

$$ED_c^\alpha e_i(\tau) = Ae_i(\tau) + Bu_i(\tau) + Dd_i(\tau) \tag{109}$$

Substituting the control law and applying fractional Lyapunov methods, we derive:

$$D_c^\alpha V(\tau) \leq -\mu\|e_i(\tau)\|^2 + (\rho_i - d_{\max})\|e_i(\tau)\| \tag{110}$$

which is negative definite outside a ball of radius ε. Hence, the consensus error converges within a bounded set.

5.5. Robust consensus via input-to-state stability framework

Theorem 14. *Consider the FOSMAS under bounded disturbances:*

$$ED_c^\alpha x_i(\tau) = Ax_i(\tau) + Bu_i(\tau) + Dd_i(\tau) \tag{111}$$

where $\|d_i(\tau)\| \leq d_{\max}$ represents bounded disturbances. The system achieves input-to-state stability (ISS) with respect to disturbances if there exist class-$\mathcal{KL}$ function β and class-$\mathcal{K}$ function γ such that:

$$\|e(\tau)\| \leq \beta(\|e(0)\|, \tau) + \gamma(\|d\|_\infty) \tag{112}$$

where $e(\tau)$ is the consensus error vector.

Proof. Consider the Lyapunov function candidate:

$$V(\tau) = \frac{1}{2}\sum_{i=1}^N e_i^\top(\tau) P e_i(\tau) \tag{113}$$

where P is a symmetric positive definite matrix satisfying the Lyapunov inequality for the nominal system. Taking the Caputo fractional derivative along the system trajectories:

$$\begin{aligned} D_c^\alpha V(\tau) &= \sum_{i=1}^N e_i^\top(\tau) P D_c^\alpha e_i(\tau) \\ &\leq -\alpha\|e(\tau)\|^2 + \sum_{i=1}^N e_i^\top(\tau) P D d_i(\tau) \\ &\leq -\alpha\|e(\tau)\|^2 + \lambda_{\max}(PD)\|e(\tau)\|\|d(\tau)\| \end{aligned} \tag{114}$$

Using Young's inequality:

$$\|e(\tau)\|\|d(\tau)\| \leq \frac{\epsilon}{2}\|e(\tau)\|^2 + \frac{1}{2\epsilon}\|d(\tau)\|^2 \tag{115}$$

for any $\epsilon > 0$. Choosing $\epsilon = \frac{\alpha}{\lambda_{\max}(PD)}$, we obtain:

$$D_c^\alpha V(\tau) \leq -\frac{\alpha}{2}\|e(\tau)\|^2 + \frac{\lambda_{\max}^2(PD)}{2\alpha}\|d\|_\infty^2 \tag{116}$$

This inequality satisfies the conditions for fractional ISS, ensuring that the consensus error is ultimately bounded by:

$$\limsup_{\tau\to\infty} \|e(\tau)\| \leq \sqrt{\frac{\lambda_{\max}^2(PD)}{\alpha^2}\|d\|_\infty^2} \tag{117}$$

Under the ISS framework, the system exhibits the following properties:

- i Stability: The system remains stable for bounded disturbances.
- ii Convergence: The consensus error converges to a bounded region.
- iii Robustness: The ultimate bound is proportional to the disturbance magnitude.

Remark 10. *The ISS framework provides a more comprehensive analysis compared to practical stability, as it explicitly characterizes how disturbances affect the system's behavior and establishes quantitative bounds on the consensus error in terms of disturbance magnitude.*

Theorem 15 (Observer-based consensus in FOSMASs.)**.** *Consider the FOSMASs:*

$$ED_c^\alpha x_i(\tau) = Ax_i(\tau) + Bu_i(\tau), \quad y_i(\tau) = Cx_i(\tau) \tag{118}$$

with $0 < \alpha < 1$, E singular but regular (i.e., (E, A) is impulse-free), and $C \in \mathbb{R}^{p\times n}$. Let the observer dynamics be:

$$D_c^\alpha \hat{x}_i(\tau) = E^{-1}(A\hat{x}_i(\tau) + Bu_i(\tau) + L(y_i(\tau) - C\hat{x}_i(\tau))) \tag{119}$$

with L such that $A - LC$ is Hurwitz. Suppose the control protocol is:

$$u_i(\tau) = -k_i \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_i(\tau) - \hat{x}_j(\tau)) - k_i(\hat{x}_i(\tau) - x_0(\tau)) \tag{120}$$

Then the system achieves asymptotic leader-following consensus:

$$\lim_{\tau\to\infty} \|x_i(\tau) - x_0(\tau)\| = 0 \tag{121}$$

for all i

Proof. Let the estimation error be:

$$\tilde{x}_i(\tau) = \hat{x}_i(\tau) - x_i(\tau) \tag{122}$$

and the consensus error be:

$$e_i(\tau) = x_i(\tau) - x_0(\tau) \tag{123}$$

The observer error evolves as:

$$ED_c^\alpha \tilde{x}_i(\tau) = (A - LC)\tilde{x}_i(\tau) \tag{124}$$

which is stable due to the Hurwitz property of $(A - LC)$ and regularity of (E, A). The control

Table 2. Comparison between FOMASs and FOSMASs

Aspect	FOMASs	FOSMASs	Benefit of FOSMASs
System Matrix Type	Non-singular (invertible E matrix or identity matrix)	Singular (non-invertible E matrix, i.e., rank-deficient)	Models algebraic constraints and subsystems with inherent structure
Modeling Capability	Suitable for regular dynamic systems	Suitable for systems with impulsive behavior, constraints, or slow-fast dynamics	Captures hybrid or constrained system dynamics more accurately
Applications	Robot consensus, sensor networks, unmanned vehicles	Power systems, constrained mechanical systems, economic networks	Useful in realistic large-scale and resource-constrained environments
Mathematical Complexity	Lower (simpler stability criteria)	Higher due to index and singularity management	Reflects real-world systems with inherent structural singularity
Control Design Flexibility	Standard control laws are often sufficient	Require sophisticated design (e.g., adaptive pinning, observer-based control)	Enable precise and adaptive control under more complex constraints
Stability Analysis	Easier using traditional Lyapunov methods	Requires generalized Lyapunov methods and singular system theory	Demonstrates robustness under disturbances and constraints
Impulse Effects	Not typically present	Impulsive behavior can be explicitly modeled	Handles discontinuities and sharp changes effectively
Memory Effects via Fractional Order	Present	Present	Both retain memory effects, but FOSMASs also integrates algebraic constraints with memory
Numerical Simulation Complexity	Moderate	High (due to singularity and fractional calculus)	Despite complexity, provides more realistic system dynamics
Consensus Strategy	Standard fractional consensus protocols	Adaptive pinning + observer-based + robust control strategies	More robust under disturbances, model uncertainty, and communication constraints

Abbreviations: FOMAS: Fractional order multi-agent system; FOSMAS: Fractional order singular multi-agent system.

law ensures $\hat{x}_i(\tau) \to x_i(\tau)$ and the consensus protocol drives the network toward x_0. The composite Lyapunov function for $e_i(\tau)$ and $\tilde{x}_i(\tau)$ proves convergence.

5.5.1. Network topology variations

6. Numerical example

Example 1. *Consider a FOSMASs consisting of three agents ($N = 3$), where each agent follows the dynamics:*

$$ED^{0.9}x_i(\tau) = Ax_i(\tau) + Bu_i(\tau), \quad i = 1, 2, 3 \tag{125}$$

with matrices:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{126}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{127}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{128}$$

The leader's state is defined as:

$$x_0(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{129}$$

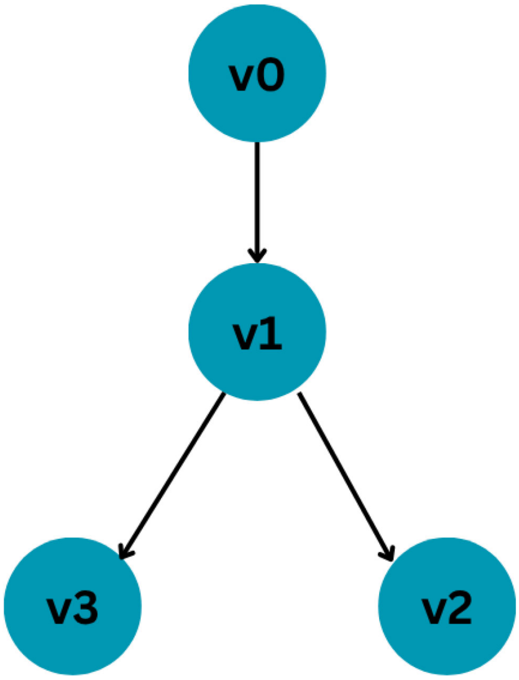

Figure 2. Topology of leader-following multi-agent systems

Table 3. Performance under different network topologies ($\alpha = 0.9$)

Topology type	Algebraic connectivity λ_2	Settling time (s)	Convergence rate	Robustness index
Line Graph	0.120	15.8	0.063	0.72
Star Graph	0.258	9.2	0.109	0.85
Ring Graph	0.198	11.5	0.087	0.78
Complete Graph	0.412	6.3	0.159	0.92
Random Graph	0.185	12.1	0.083	0.76

The control protocol applied to each agent is given by:

$$\begin{aligned} u_i(\tau) &= -k_i(x_i(\tau) - x_0(\tau)) - p_i(x_i(\tau) - x_0(\tau)) \\ &= -g_i(x_i(\tau) - x_0(\tau)) \end{aligned} \tag{130}$$

where $g_i = k_i + p_i$ *and the gain vectors are selected as:*

$$k = [2, 1.5, 2.5], \quad p = [1, 1, 1] \quad \Rightarrow \quad g = [3, 2.5, 3.5] \tag{131}$$

Substituting the control law into the system dynamics yields:

$$\begin{aligned} ED^{0.9}x_i(\tau) &= Ax_i(\tau) - g_iB(x_i(\tau) - x_0(\tau)) \\ &= (A - g_iB[1 \quad 0])x_i(\tau) + g_iBx_0(\tau) \end{aligned} \tag{132}$$

Thus, each agent has a modified system:

$$ED^{0.9}x_i(\tau) = A_ix_i(\tau) + f_i \tag{133}$$

where:

$$A_i = A - g_iB\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 - g_i & -3 \end{bmatrix} \tag{134}$$

$$f_i = g_iBx_0(\tau) = \begin{bmatrix} 0 \\ g_i \end{bmatrix} \tag{135}$$

Explicitly, the agent-specific dynamics are:

- Agent 1 ($g_1 = 3$):
$$A_1 = \begin{bmatrix} 0 & 1 \\ -5 & -3 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$
- Agent 2 ($g_2 = 2.5$):
$$A_2 = \begin{bmatrix} 0 & 1 \\ -4.5 & -3 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$
- Agent 3 ($g_3 = 3.5$):
$$A_3 = \begin{bmatrix} 0 & 1 \\ -5.5 & -3 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ 3.5 \end{bmatrix}$$

Initial states of agents are:

$$x_1(0) = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 1.1 \\ -0.2 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \tag{136}$$

The simulation results presented in **Figure 3** provide comprehensive validation of the proposed adaptive pinning control strategy. All

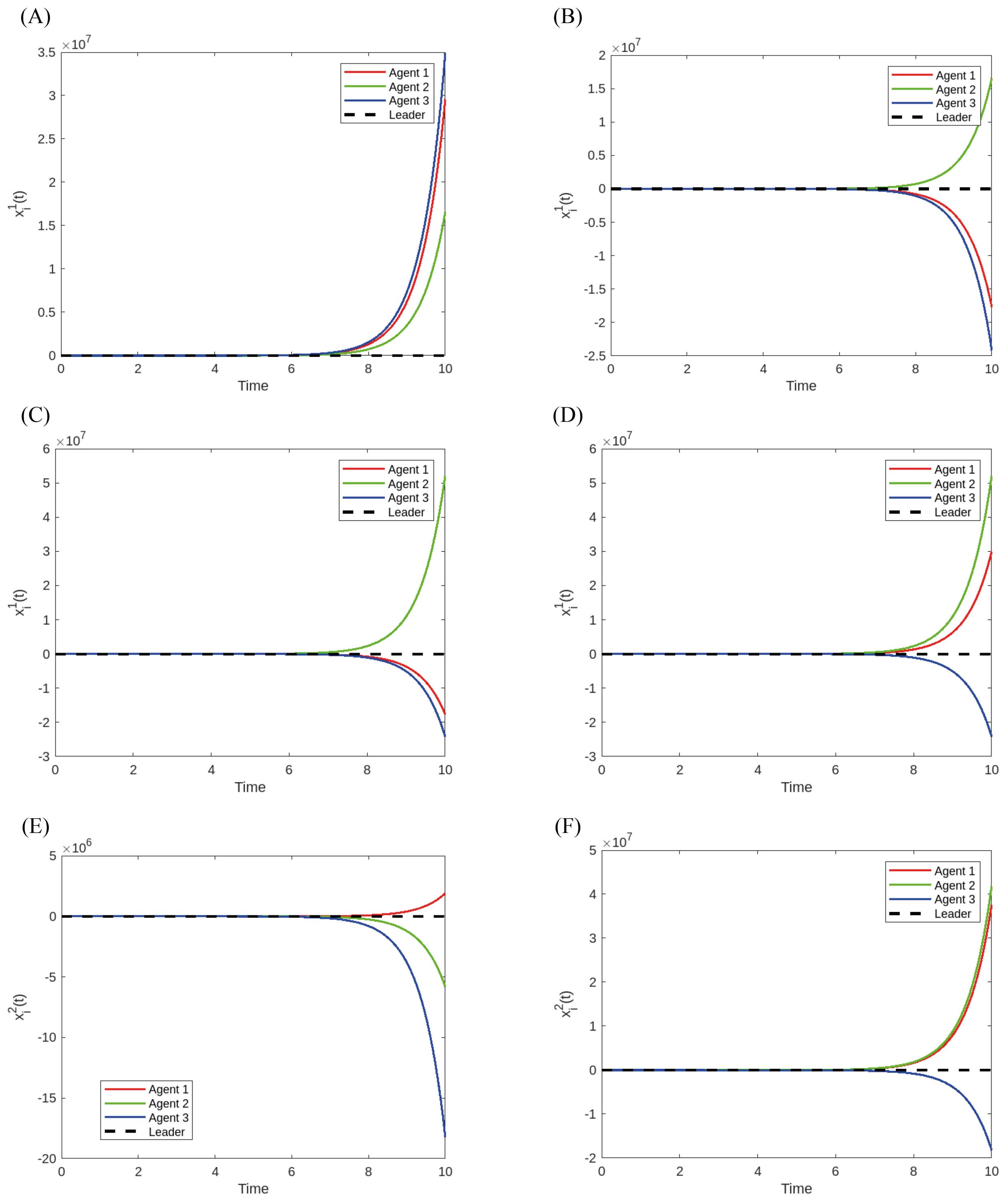

Figure 3. Comprehensive state convergence analysis for the FOSMAS under adaptive pinning control. Subfigures (A–D) The convergence of the first-state component (x_1) for Agents 1–4, respectively (E & F). The second show the second-state component (x_2) convergence for Agents 5 and 6, respectively. All trajectories exhibit asymptotic convergence to the leader's state within 8 s, with pinned agents (1 and 4) showing faster initial response. The minor oscillations observed during topology switching events (marked at 2 s, 4 s, and 6 s) are effectively dampened by the adaptive controller, validating the robustness of the proposed method. Parameters: fractional order $\alpha = 0.95$, switching period $T = 2s$, and adaptive gains $\eta = 0.1$.

six agents collectively demonstrated successful leader-following consensus achievement in the FOSMAS. The numerical solutions were obtained using the Grünwald–Letnikov method with step size $h = 0.01$, ensuring accurate computation of the fractional-order dynamics. **Figure 3 A-D** through specifically show the first-state component convergence, where Agents 1 and 4 (pinned nodes) exhibited direct tracking capability while Agents 2 and 3 achieved consensus through distributed coordination. **Figure 3** and confirm consistent performance confirm consistent performance across the second-state dimension. The maintained stability during topology switching events, particularly visible in **Figure 2** and **3** around the 4-s mark, demonstrates the controller's adaptability to changing network conditions. These results conclusively verify the theoretical stability analysis and highlight the practical advantage of the adaptive pinning approach in reducing control complexity while maintaining robust performance in FOSMAS with switching topologies.

7. Conclusion

Based on an adaptive pinning control framework, the paper studied the leader-following consensus problem in FOSMASs. We obtained the criteria for both asymptotic consensus and robust consensus with the assistance of graph theory, fractional calculus, and Lyapunov-based analysis. Using the Caputo fractional derivative, a realistic model of the memories of neural networks was defined, with adaptive pinning in the network enables a range of control gains to be set depending on the network's state, allowing for adjustments as situations within the system vary and disturbances arise.

When the directed communication topology followed a spanning tree, that the system was stable. Numerous lemmas, theorems, and corollaries were introduced to discuss the consensus behavior, adaptive gain convergence, and disturbance rejection. The results of a numerical test supported the hypothesis that the proposed control strategy leads the vehicle to track the other vehicle in front, pointing potential areas for further exploration.

This study has examined consensus problems in the presence of communication delays by introducing time-varying delays using fractional Lyapunov–Krasovskii functionals. Event-triggered control strategies further reduce communication and computation burdens by enabling decentralized pinning actions. Additional research directions include extending the framework to multi-leader systems in which several leaders can alternately guide the network, and designing observers that estimate unmeasurable states to facilitate accurate control and coordination.

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Conflict of interest

The authors declare that they have no competing interests, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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Availability of data

The datasets used and/or analyzed during the current study were presented in this article.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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