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Stability sets for fractional differential equations with two delays

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Abstract. The main subject of this study is a linear fractional-order differential equation with two delayed terms. By applying the D-decomposition method to the characteristic equation, we present a stability region as a necessary and sufficient condition for the asymptotic stability of the zero solution. Given the relationship between the Caputo fractional derivative and the integer-order derivative, we compare the conditions in this study with the results of the corresponding first-order delay differential equation to demonstrate the validity of the extension.

Keywords: fractional-order differential equation, delay differential equation, two delays, asymptotic stability, stability region.

2020 Mathematics Subject Classification: 34K37, 34K25, 34K20.

1 Introduction

Fractional calculus is a generalization concept of the classical calculus of integer order, including integrals and derivatives of any arbitrary real or complex order. There are several different families of fractional integrals and fractional derivatives, and the potentially useful properties of these definitions have been identified (see [7, 10, 12, 18] and references therein). As the basic theory of fractional calculus improves, its advantages in describing the memory and hereditary properties of processes has been gradually acknowledged. For example, empirical analyses of the transmission dynamics of COVID-19 (see [16, 22]) reveal that models incorporating Caputo fractional-order derivatives demonstrate superior performance in data fitting accuracy and epidemic trend prediction capabilities over their integer-order counterparts. By now, fractional differential equations have become one of the best tools in the areas of fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, and biophysics. Because stability is an important characteristic of the normal operation of a practical system, the stability theory of fractional differential equations have attracted the attention of mathematicians and engineers.

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In engineering and natural systems, the time delay in the process of material exchange and information transmission cannot be ignored. The existence of a time delay is usually the source of the instability of the system, which makes the discussion essential on the stability of fractional differential equations with time delay. Moreover, several factors can cause time delays in large or complex systems. In general, secondary factors are often overlooked for the convenience of processing; thus, the system is simplified into a single time-delay system with only the main time-delay parameter. However, for a complex system in which it is difficult to select the factors that cause a time delay, a model with multiple delay parameters should be considered.

The research on fractional differential equations with multiple delays has increased in the last double decades; see, e.g., [1,6,9,20,21,23,24] and references therein. The basic criterion of these studies are based on [6] in which an n-dimensional system of linear fractional differential equations with n^2 different delays was considered. In 2007, Deng et al. [6] established that the zero solution of the system is asymptotically stable if all the roots of the associated characteristic equation have negative real parts. It should be pointed out that the characteristic equation of a linear delay differential equation is a transcendental equation that cannot be solved using algebraic tools, and there are practical difficulties in condition verification.

This study focuses on a linear fractional differential equation with two delays

$$\begin{aligned} {}^{C}\mathrm{D}_0^{\alpha} x(t) &= a x(t-\tau_1) + b x(t-\tau_2), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2\}. \end{aligned} \tag{1.1}$$

Here, $\alpha \in (0,1)$ is a constant and ${}^{C}\mathrm{D}_0^{\alpha}$ is the Caputo fractional differentiation operator of order α defined by

$${}^{C}\mathrm{D}_0^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(r)}{(t-r)^{\alpha}} dr, \quad t > 0.$$

Parameters a and b are real numbers and τ_1 and τ_2 are positive real numbers. The initial function ϕ is continuous on the interval $[-\tau, 0]$. The main purpose of this study is to investigate the asymptotic behavior of the solutions to (1.1) and present the stability set of (a, b) for the zero solution to be asymptotically stable.

The study on (1.1) with $b = 0$ can be found in [5,13] and references therein. In 2011, Krol [13] determined the range of parameter a where the zero solution is asymptotically stable.

Theorem A. Let $b = 0$ and $\tau_1 = \tau$. Then, the zero solution to (1.1) is asymptotically stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} < a < 0.$$

This is stable but not asymptotically stable if and only if

$$a = -\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha}.$$

Stability criteria for (1.1) with $\tau_1 = 0$ was studied in [2,3,11] and references therein. Especially, Čermák et al. [3] gave the stability set of (a, b) for (1.1) with $\tau_1 = 0$ and $0 < \alpha < 1$, which has been extended for the case $\alpha > 1$ by Čermák et al. [4]. By virtue of their work, we have the following result.

Theorem B. Let $\alpha > 0$, $\tau_1 = 0$, and $\tau_2 = \tau$.

(i) The zero solution to (1.1) is asymptotically stable if and only if $(a, b) \in D$ where D is an open region enclosed by the line $a + b = 0$ from above and by the parametric curve

$$a = \frac{\omega^\alpha \sin(\tau\omega + \alpha\pi/2)}{\sin(\tau\omega)}, \quad b = -\frac{\omega^\alpha \sin(\alpha\pi/2)}{\sin(\tau\omega)} \tag{1.2}$$

for $\omega \in ((\pi - \alpha\pi)/\tau, \pi/\tau)$ from below.

(ii) The zero solution to (1.1) is stable but not asymptotically stable if and only if either

$$a + b = 0, \quad a \leq \frac{(\pi(1-\alpha))^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)},$$

or a and b satisfy (1.2) for an admissible value ω.

Letting $\alpha \to 1^-$, (1.1) becomes the first-order delay differential equation

$$x'(t) = ax(t - \tau_1) + bx(t - \tau_2), \quad t \geq 0. \tag{1.3}$$

When $b = 0$ or $\tau_1 = 0$ in (1.3), relevant classical results exist for the asymptotic stability of the zero solution. A simple calculation shows that the asymptotic stability condition in Theorems A and B with $\alpha = 1$ coincides with the result for the corresponding case in (1.3).

Stability criteria for (1.3) can be found in [14, 15, 19, 25]. In [19], Sakata discussed the stability set of (a, b) for (1.3) when $\tau_2 = n\tau_1$, $n = 2, 3$. Here, we present the result for $\tau_2 = 2\tau_1$.

Theorem C. Let $\tau_2 = 2\tau_1 = 2\tau$. Then, the zero solution to (1.3) is asymptotically stable if and only if $(a, b) \in D_1$ where D_1 is an open region enclosed by the line $a + b = 0$ from the right and by the parametric curve

$$a = \frac{\omega \cos(2\tau\omega)}{\sin(\tau\omega)}, \quad b = -\frac{\omega \cos(\tau\omega)}{\sin(\tau\omega)}, \quad 0 < \omega < \frac{2\pi}{3\tau}$$

from the left.

In this study, we consider the special case of (1.1) in which $\tau_2 = 2\tau_1$. For simplicity, we denote τ_1 as τ and τ_2 as 2τ. Subsequently, (1.1) is reduced to

$$\begin{aligned} {}^{C}\mathrm{D}_0^\alpha x(t) &= ax(t - \tau) + bx(t - 2\tau), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-2\tau, 0]. \end{aligned} \tag{1.4}$$

The following theorem is the main result of this study.

Theorem 1.1. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *The zero solution to (1.4) is asymptotically stable if and only if $(a, b) \in D_{0,0}$ where $D_{0,0}$ is an open region enclosed by the line $a + b = 0$ from the right and by the parametric curve*

$$a = \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)} \tag{1.5}$$

for

$$\rho \in \left(\frac{(1-\alpha)\pi}{3\tau}, \frac{(3-\alpha)\pi}{3\tau}\right)$$

from the left.

(ii) *The zero solution to* (1.4) *is stable but not asymptotically stable if and only if* $a + b = 0$ *or* a *and* b *satisfy* (1.5) *for*

$$\rho = \frac{(2-\alpha)\pi}{4\tau} \quad \text{or} \quad \rho \in \left[\frac{(2-\alpha)\pi}{2\tau}, \frac{(3-\alpha)\pi}{3\tau}\right).$$

Remark 1.2. We observe that the conditions for parameter a in Theorem 1.1 are the same as those in Theorem A by letting $b = 0$, that is, $\tau\rho = (2-\alpha)\pi/2$ in (1.5).

Remark 1.3. As previously mentioned, (1.4) corresponds to (1.3) with $\tau_1 = \tau$ and $\tau_2 = 2\tau$ as $\alpha \to 1^-$. Comparing the conditions of Theorems C and 1.1 (i), the regions for the asymptotic stability of the zero solution are the same if $\alpha = 1$.

Remark 1.4. As $\alpha \to 1^+$, (1.4) does not become (1.3) with $\tau_1 = \tau$ and $\tau_2 = 2\tau$ if $x'(0) \neq 0$. Therefore, the case of $\alpha > 1$ is out of the scope of the extension of Theorem C that we aim to achieve.

The set of the pair (a, b) that guarantees the asymptotic stability of the zero solution when the fractional order takes different values with a fixed delay is shown in Figure 1.1. Notably, these regions have no inclusion relationships with each other.

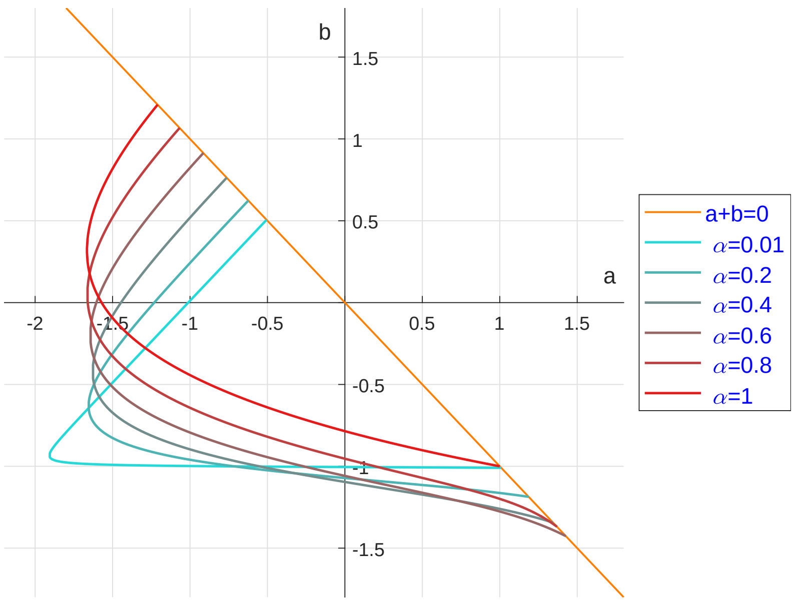

Figure 1.1: Parametric curve (1.5) for $\alpha = 0.2, 0.4, 0.6, 0.8, 1$, and $\tau = 1$.

Remark 1.5. According to [12, 18], as $\alpha \to 0^+$, (1.4) is reduced to

$$x(t) - \phi(0) = ax(t-\tau) + bx(t-2\tau), \quad t \geq 0.$$

Figure 1.1 indicates that as $\tau = 1$ and $\alpha \to 0^+$, $D_{0,0}$ given in Theorem 1.1 becomes the open triangle region D_0 enclosed by lines $a + b = 0$, $b = a + 1$, and $b = -1$. We note that D_0 can be regarded as the stability set for the equilibrium $\phi(0)/(1 - a - b)$ of the second-order difference equation

$$x(k) - ax(k-1) - bx(k-2) = \phi(0), \quad k = 0, 1, 2, \ldots,$$

where a, b, and $\phi(0)$ are real numbers with $a + b < 0$; see, e.g., [8].

The structure of this article is arranged as follows: In the next section, we present the expression for the solutions using the Laplace transform. In Section 3, we provide a detailed analysis of the characteristic equation to present a set of (a, b) in which all the characteristic roots are located in the left half of the complex plane. The proof of the main theorem is completed in Section 4. Finally, we summarize our results in Section 5 and compare them with those of the corresponding first-order differential equation with two delays. The scope for future directions is also provided.

2 Preliminaries

In this section, we introduce some definitions and properties as preliminary for investigating the asymptotic behavior of the solutions to (1.1).

First, we introduce the Laplace transform; see, e.g. [7, 10, 12, 18].

Definition 2.1. The Laplace transform of the function $f : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\mathcal{L}[f(t)](s) = \mathcal{L}_f(s) := \int_0^\infty e^{-st} f(t)dt, \quad s \in \mathbb{C}.$$

Suppose that $f \in L^1_{loc}(\mathbb{R}_+)$ and there exists $r \geq 0$ such that $|f(t)| \leq ce^{rt}$ for a large t. Then, $\mathcal{L}_f(s)$ exists and is analytic for $\operatorname{Re}(s) > r$. Moreover, if the integral converges at point $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \operatorname{Re}(s_0)$. The infimum R_f of the real part of s for which the Laplace integral of $f(t)$ converges, is called the abscissa of convergence. Clearly, the Laplace transform is linear. Next, we provide common rules and formulae for the Laplace transform.

Lemma 2.2.

(i) $\mathcal{L}[t^k](s) = \Gamma(k+1)/s^{k+1}$ *for* $k > -1$ *and* $s > 0$.

(ii) $\mathcal{L}\left[\int_0^t f(t-u)g(u)du\right](s) = \mathcal{L}_f(s)\mathcal{L}_g(s)$.

(iii) $\mathcal{L}[f^{(n)}(t)](s) = s^n\mathcal{L}_f(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(s)$.

(iv) *For* $c \in \mathbb{R}$, *let*

$$f_c(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c. \end{cases}$$

Then, $\mathcal{L}[f_c(t)](s) = e^{-cs}\mathcal{L}_f(s)$.

Definition 2.3. The inverse formula for the Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[\mathcal{L}_f(s)](t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{ts}\mathcal{L}_f(s)ds.$$

Here, the integral is along the vertical line $\operatorname{Re}(s) = r$ in $\mathbb{C}$ such that $r > R_f$, that is, it is greater than the real part of all the singularities of $\mathcal{L}_f$, and $\mathcal{L}_f$ is bounded on the line.

Next, we establish a representation formula for the solution $x(t)$ to (1.1). Let $X(s) = \mathcal{L}[x(t)](s)$. Then, the Laplace transform of (1.1) becomes

$$s^\alpha X(s) - s^{\alpha-1}\phi(0) = ae^{-\tau_1 s}X(s) + a\int_{-\tau_1}^0 \phi(u)e^{-s(\tau_1+u)}du + be^{-\tau_2 s}X(s) + b\int_{-\tau_2}^0 \phi(u)e^{-s(\tau_2+u)}du.$$

This implies that

$$X(s) = \frac{s^{\alpha-1}\phi(0) + a\int_{-\tau_1}^{0}\phi(u)e^{-s(\tau_1+u)}du + b\int_{-\tau_2}^{0}\phi(u)e^{-s(\tau_2+u)}du}{s^\alpha - ae^{-\tau_1 s} - be^{-\tau_2 s}}.$$

Let

$$\Phi^{\alpha,\beta}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^\alpha - ae^{-\tau_1 s} - be^{-\tau_2 s}}\right)(t).$$

Subsequently, from Lemma 2.2, it follows that

$$\begin{aligned} X(s) &= \phi(0)\mathcal{L}\left(\Phi^{\alpha,1}(t)\right)(s) + a\mathcal{L}\left(\Phi^{\alpha,\alpha}(t)\right)(s)\int_{-\tau_1}^{0}\phi(u)e^{-s(\tau_1+u)}du \\ &\quad + b\mathcal{L}\left(\Phi^{\alpha,\alpha}(t)\right)(s)\int_{-\tau_2}^{0}\phi(u)e^{-s(\tau_2+u)}du \\ &= \phi(0)\mathcal{L}\left(\Phi^{\alpha,1}(t)\right)(s) + a\int_{-\tau_1}^{0}\phi(u)\mathcal{L}\left({\Phi^{\alpha,\alpha}}_{\tau_1+u}(t)\right)(s)du \\ &\quad + b\int_{-\tau_2}^{0}\phi(u)\mathcal{L}\left({\Phi^{\alpha,\alpha}}_{\tau_2+u}(t)\right)(s)du. \end{aligned}$$

Using the inverse Laplace transform of $X(s)$, we obtain the following representation formula for the solution to (1.1):

$$x(t) = \phi(0)\Phi^{\alpha,1}(t) + a\int_{-\tau_1}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau_1+u}(t)du + b\int_{-\tau_2}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau_2+u}(t)du.$$

In particular, we obtain the representation formula for the solution to (1.4):

$$x(t) = \phi(0)\Phi^{\alpha,1}(t) + a\int_{-\tau}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau+u}(t)du + b\int_{-2\tau}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{2\tau+u}(t)du, \tag{2.1}$$

where

$$\Phi^{\alpha,1}(t) = \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}\right](t), \tag{2.2}$$

$${\Phi^{\alpha,\alpha}}_c(t) = \mathcal{L}^{-1}\left[\frac{e^{-cs}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}\right](t). \tag{2.3}$$

3 Analysis of characteristic equation

The representation formula (2.1) shows that

$$P(s) := s^\alpha - ae^{-\tau s} - be^{-2\tau s} = 0 \tag{3.1}$$

is a characteristic equation associated with (1.4). Therefore, it is essential to analyze the roots of (3.1) in detail. First, we note that $P(0) = -a - b$ and $P(s) > 0$ for s real and sufficiently large. Thus, if $a + b > 0$, then (3.1) has at least one positive root. Moreover, as $\overline{e^z} = e^{\bar{z}}$ for any $z \in \mathbb{C}$, we observe that

$$\overline{P(s)} = \overline{s^\alpha} - \overline{ae^{-\tau s}} - \overline{be^{-2\tau s}} = \bar{s}^\alpha - ae^{-\tau\bar{s}} - be^{-2\tau\bar{s}} = P(\bar{s}).$$

This implies that for any $s \in \mathbb{C}$, $P(s) = 0$ if and only if $P(\bar{s}) = 0$. Based on this property, we consider (3.1) only in the upper half of the complex plane in the following discussion.

Proposition 3.1. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *(3.1) has the root 0 if and only if $a + b = 0$.*

(ii) *(3.1) has a finite number of roots in the domain of $\{s \in \mathbb{C} \mid |\arg(s)| \leq \sigma\}$ for any given $\sigma \in (\pi/2, \pi)$.*

(iii) *(3.1) has a root ρi with $\rho > 0$ if and only if a and b satisfy*

$$a = \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)}, \quad \rho \neq \frac{m\pi}{\tau}, \quad m \in \mathbb{N}. \tag{3.2}$$

(iv) *(3.1) has no quadruple roots. Moreover, if $a \neq 0$ and $b > 0$, then (3.1) has no double roots on the imaginary axis.*

Proof. (i) This part is obvious.

(ii) Let $s = \rho e^{\theta i}$ be a root of (3.1), where $\rho \geq 0$ and $\theta \in [0, \pi]$. The real and imaginary parts of $P(\rho e^{\theta i}) = 0$ are separated to obtain

$$\rho^\alpha \cos(\alpha\theta) = a\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b\cos(2\tau\rho\sin\theta)e^{-2\tau\rho\cos\theta}, \tag{3.3}$$

$$\rho^\alpha \sin(\alpha\theta) = -a\sin(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} - b\sin(2\tau\rho\sin\theta)e^{-2\tau\rho\cos\theta}. \tag{3.4}$$

The sum of the squares of (3.3) and (3.4) is

$$\rho^{2\alpha} = e^{-2\tau\rho\cos\theta}\left[a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right]. \tag{3.5}$$

If $\theta \in [0, \pi/2]$, then

$$e^{-2\tau\rho\cos\theta}\left(a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right) \leq (|a| + |b|)^2.$$

This implies that (3.5) is impossible if $\rho > (|a| + |b|)^{1/\alpha}$. Thus, (3.1) has no roots in the region of

$$\{s \in \mathbb{C} \mid |s| > (|a| + |b|)^{1/\alpha}, \arg(s) \in [0, \pi/2]\}.$$

Similarly, if $\theta \in (\pi/2, \pi)$, it follows that

$$e^{-2\tau\rho\cos\theta}\left(a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right) \geq e^{-2\tau\rho\cos\theta}\left(|a| - |b|e^{-\tau\rho\cos\theta}\right)^2.$$

Because there exists a $\rho_1 > 0$ such that

$$\rho^{2\alpha} < e^{-2\tau\rho\cos\theta}\left(|a| - |b|e^{-\tau\rho\cos\theta}\right)^2$$

for all $\rho > \rho_1$, (3.1) has no root in the region of

$$\{s \in \mathbb{C} \mid |s| > \rho_1, \arg(s) \in (\pi/2, \pi)\}.$$

Therefore, there exists a $\overline{\rho}$ such that (3.1) has no root with $|s| > \overline{\rho}$ and $|\arg(s)| \in (\pi/2, \pi)$. Furthermore, if $a + b \neq 0$, then (3.3) does not hold as $\rho \to 0$. Hence, there exists $\underline{\rho} > 0$ such that (3.1) has no roots with $|s| < \underline{\rho}$. Let

$$\Omega = \{s \in \mathbb{C} \mid \underline{\rho} < |s| < \overline{\rho}, \ |\arg(s)| \in [0, \sigma]\}.$$

Because the function P is analytic on Ω, (3.1) has finite number of roots in Ω.

(iii) Let $P(\rho i) = 0$ where $\rho > 0$. Then, (3.3) and (3.4) with $\theta = \pi/2$ become

$$\rho^\alpha \cos(\alpha\pi/2) = a\cos(\tau\rho) + b\cos(2\tau\rho),$$
$$\rho^\alpha \sin(\alpha\pi/2) = -a\sin(\tau\rho) - b\sin(2\tau\rho),$$

namely,

$$a\sin(\tau\rho) = \rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho), \quad b\sin(\tau\rho) = -\rho^\alpha \sin(\alpha\pi/2 + \tau\rho). \tag{3.6}$$

Therefore, $P(\rho i) = 0$ is equivalent to (3.6). If $\sin(\tau\rho) = 0$, then (3.6) yields $\rho = 0$, which is a contradiction. Thus, $\sin(\tau\rho) \neq 0$, that is, $\tau\rho \neq m\pi$ for $m \in \mathbb{N}$; hence, $P(\rho i) = 0$ is equivalent to (3.2).

(iv) Finally, we consider the multiplicity of the roots of (3.1). Let $P'(s) = 0$. Then,

$$\frac{dP}{ds} = \alpha s^{\alpha-1} + a\tau e^{-\tau s} + 2b\tau e^{-2\tau s} = 0. \tag{3.7}$$

Clearly, $s = 0$ is not the root of (3.7). From Part (ii), we observe that if $s = \rho e^{\theta i}$ is a root of (3.7), then $\bar{s} = \rho e^{-\theta i}$ is also a root of (3.7). Eliminating s^α in (3.1) and (3.7) yields

$$a(\tau s + \alpha) + b(2\tau s + \alpha)e^{-\tau s} = 0.$$

Eliminating $e^{-2\tau s}$ in (3.1) and (3.7) yields

$$2\tau s^\alpha + \alpha s^{\alpha-1} - a\tau e^{-\tau s} = 0.$$

Then, we obtain

$$\frac{2\tau s + \alpha}{a\tau s^{1-\alpha}} = e^{-\tau s} = -\frac{a(\tau s + \alpha)}{b(2\tau s + \alpha)}. \tag{3.8}$$

This implies that

$$b(2\tau s + \alpha)^2 = -a^2\tau s^{1-\alpha}(\tau s + \alpha). \tag{3.9}$$

We observe that (3.9) has no positive real root s if $b > 0$. Let $s = \rho e^{\varphi i}$ where $\rho > 0$, $\varphi \in (0, 2\pi)$. Then, we have

$$b(4\tau^2\rho^2\cos 2\varphi + 4\alpha\tau\rho\cos\varphi + \alpha^2) = -a^2\tau\rho^{1-\alpha}\left(\tau\rho\cos((2-\alpha)\varphi) + \alpha\cos((1-\alpha)\varphi)\right),$$
$$b(4\tau^2\rho^2\sin 2\varphi + 4\alpha\tau\rho\sin\varphi) = -a^2\tau\rho^{1-\alpha}\left(\tau\rho\sin((2-\alpha)\varphi) + \alpha\sin((1-\alpha)\varphi)\right).$$

If $\varphi = \pi$, then ρ satisfies

$$b(2\tau\rho - \alpha)^2 = a^2\tau\rho^{1-\alpha}(\alpha - \tau\rho)\cos\alpha\pi,$$
$$0 = a^2\tau\rho^{1-\alpha}(\tau\rho - \alpha)\sin\alpha\pi.$$

The second equation implies that $\rho = 0$ or $\rho = \alpha/\tau$; however, none satisfies the first equation. Therefore, (3.9) has no negative real root.

Suppose that (3.9) has a root $s = \rho i$. Then, ρ satisfies

$$b(\alpha^2 - 4\tau^2\rho^2) = -a^2\tau\rho^{1-\alpha}\left(\alpha\sin(\alpha\pi/2) - \tau\rho\cos(\alpha\pi/2)\right),$$
$$4b\alpha\tau\rho = -a^2\tau\rho^{1-\alpha}\left(\tau\rho\sin(\alpha\pi/2) + \alpha\cos(\alpha\pi/2)\right).$$

If $b > 0$, then the second equation does not hold for any $\rho > 0$; hence, (3.9) has no root on the imaginary axis. If $b < 0$, we divide the two equations to obtain

$$\frac{\alpha^2 - 4\tau^2\rho^2}{4\alpha\tau\rho} = \frac{\alpha\sin(\alpha\pi/2) - \tau\rho\cos(\alpha\pi/2)}{\tau\rho\sin(\alpha\pi/2) + \alpha\cos(\alpha\pi/2)}.$$

This implies that

$$F(\rho) := \rho^3 + \frac{3\alpha^2}{4\tau^2}\rho - \frac{\alpha^3}{4\tau^3}\cot\frac{\alpha\pi}{2} = 0.$$

As $F(0) < 0$ and $F'(\rho) > 0$ for $\rho \geq 0$, there is a unique positive root ρ^* for this equation. Therefore, (3.9) contains only one pair of roots $s = \pm\rho^* i$ if $b < 0$.

In summary, (a) if $a \neq 0$ and $b > 0$, (3.1) has no real or purely imaginary double roots; (b) if $a \neq 0$ and $b < 0$, (3.1) has a positive real double root and one pair of purely imaginary double roots, but has no negative real double roots.

Let $P''(s) = 0$. Then,

$$\frac{d^2P}{ds^2} = \alpha(1-\alpha)s^{\alpha-2} + a\tau^2 e^{-\tau s} + 4b\tau^2 e^{-2\tau s} = 0. \tag{3.10}$$

From (3.7) and (3.10), we obtain

$$e^{-\tau s} = -\frac{a(\tau s + \alpha - 1)}{2b(2\tau s + \alpha - 1)}. \tag{3.11}$$

From (3.8) and (3.11), we obtain

$$\alpha = (2\tau s + \alpha)(\tau s + \alpha). \tag{3.12}$$

Note that (3.12) has only one positive real root and one negative real root. Therefore, (3.1) has a positive real triple root if $b < 0$.

Let $P'''(s) = 0$. Then,

$$\frac{d^3P}{ds^3} = \alpha(1-\alpha)(2-\alpha)s^{\alpha-3} + a\tau^3 e^{-\tau s} + 8b\tau^3 e^{-2\tau s} = 0. \tag{3.13}$$

From (3.10) and (3.13), we obtain

$$e^{-\tau s} = -\frac{a(\tau s + \alpha - 2)}{4b(2\tau s + \alpha - 2)}. \tag{3.14}$$

From (3.11) and (3.14), we obtain

$$3\tau s + 3\alpha - 2 = (2\tau s + \alpha)(\tau s + \alpha). \tag{3.15}$$

Note that (3.12) and (3.15) have only one common root, $s^* = (2 - 2\alpha)/(3\tau)$. Substituting s^* into (3.12), we obtain $\alpha^2 - 7\alpha - 8 = 0$. This yields $\alpha = 1$ or $\alpha = -8$. This result is in conflict with $\alpha \in (0,1)$. Therefore, (3.1) has no quadruple root. □

For simplicity, we define the family of parametric curves as follows:

$$\begin{aligned} a = a_m(\rho) &:= \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = b_m(\rho) := -\frac{\rho^\alpha \sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)}, \\ \Gamma_m &: (a_m(\rho), b_m(\rho)), \quad \frac{m\pi}{\tau} < \rho < \frac{(m+1)\pi}{\tau}, \quad m = 0,1,2,\ldots \end{aligned} \tag{3.16}$$

Proposition 3.1 (ii) indicates that conjugate complex numbers $\pm\rho i$ with $\rho > 0$ are roots of (3.1) if and only if the pair (a, b) satisfies (3.16). The parametric curve in (1.5) corresponds to a part of the Γ_0.

Lemma 3.2. *Let $m, n \in \mathbb{N}$.*

(i) *Each curve Γ_m intersects with the a-axis at $((-1)^m \rho^\alpha, 0)$ when $\tau\rho = (2-\alpha)\pi/2 + m\pi$; intersects with the b-axis at two points, which are $(0, -\rho^\alpha)$ when $\tau\rho = (2-\alpha)\pi/4 + m\pi$ and $(0, \rho^\alpha)$ when $\tau\rho = (4-\alpha)\pi/4 + m\pi$.*

(ii) *Each curve Γ_{2n} intersects with the line $a+b=0$ when $\tau\rho = (1-\alpha)\pi/3 + 2n\pi$ and $\tau\rho = (3-\alpha)\pi/3 + 2n\pi$; intersects with the line $a=b$ when $\tau\rho = (2-\alpha)\pi/3 + 2n\pi$.*

(iii) *Each curve Γ_{2n+1} intersects with the line $a+b=0$ when $\tau\rho = (2-\alpha)\pi/3 + (2n+1)\pi$; intersects with the line $a=b$ when $\tau\rho = (1-\alpha)\pi/3 + (2n+1)\pi$ and $\tau\rho = (3-\alpha)\pi/3 + (2n+1)\pi$.*

(iv) *If $\tau\rho$ tends to $2n\pi$, curves Γ_{2n-1} and Γ_{2n} approach their common asymptotes*

$$l_{2n} : b = -a + \left(\frac{2n\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

(v) *If $\tau\rho$ tends to $(2n+1)\pi$, curves Γ_{2n} and Γ_{2n+1} approach their common asymptotes*

$$l_{2n+1} : b = a + \left(\frac{(2n+1)\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Proof. Parts (i), (ii), and (iii) are clear. A direct calculation reveals that

$$\begin{aligned} a_m(\rho) + \cos(\tau\rho) b_m(\rho) &= \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho) - \rho^\alpha \cos(\tau\rho)\sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)} \\ &= \rho^\alpha \cos(\alpha\pi/2 + \tau\rho). \end{aligned}$$

Then, parts (iv) and (v) are obvious. □

Proposition 3.1 indicates that the range of $\tau\rho$ for the curve $\Gamma_{2n}(\rho)$ to lie in the region $\overline{D} := \{(a,b) \in \mathbb{R}^2 \mid a+b \le 0\}$ is

$$\tau\rho \in \left(\frac{(1-\alpha)\pi}{3} + 2n\pi, \frac{(3-\alpha)\pi}{3} + 2n\pi\right) =: J_{2n}$$

and the range of $\tau\rho$ for the curve $\Gamma_{2n+1}(\rho)$ to lie in the region $\overline{D}$ is

$$\tau\rho \in \left((2n+1)\pi, \frac{(2-\alpha)\pi}{3} + (2n+1)\pi\right)$$

for $n \in \mathbb{N}$.

Proposition 3.3. *Let $\alpha \in (0,1)$, $a+b \le 0$, and m, $n \in \mathbb{N}$. Suppose the point (a,b) lies on the curves Γ_m at $\tau = \tau^*$. Then, for $\tau^*\rho \in J_{2n} \cup J_{2n+1}$, the root ρi of (3.1) enters the right half of the complex plane as τ increases from τ^*, where*

$$J_{2n+1} := \left(\frac{(1-\alpha)\pi}{3} + (2n+1)\pi, \frac{(2-\alpha)\pi}{3} + (2n+1)\pi\right).$$

Proof. It suffices to demonstrate that

$$\operatorname{Re}\left(\frac{ds}{d\tau}\right)\Bigg|_{\substack{s=\rho i \\ \tau=\tau^*}} > 0 \quad \text{for} \quad \tau^*\rho \in J_{2n} \cup J_{2n+1}.$$

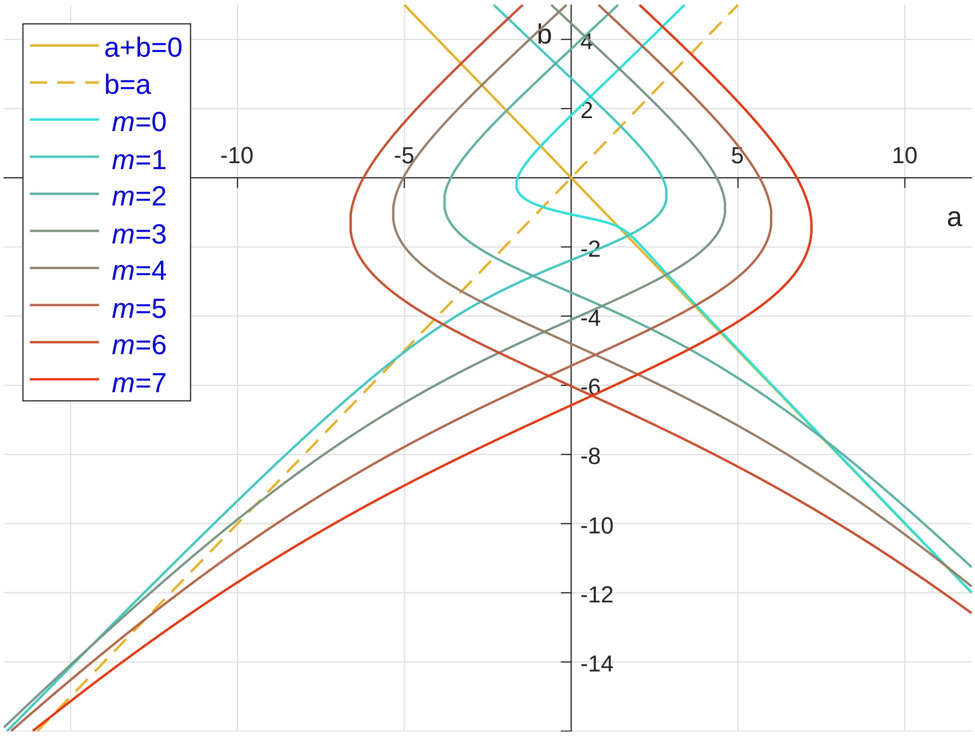

Figure 3.1: The graph of Γ_m for $m \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ with $\alpha = 0.6$ and $\tau = 1$.

From (3.1), we have

$$\frac{ds}{d\tau} = -\frac{s(ae^{-\tau s} + 2be^{-2\tau s})}{\alpha s^{\alpha-1} + a\tau e^{-\tau s} + 2b\tau e^{-2\tau s}}. \tag{3.17}$$

Substituting $s = \rho i$, $\tau = \tau^*$, $a = a_m(\rho)$, $b = b_m(\rho)$ into (3.17), we obtain

$$\left.\frac{ds}{d\tau}\right|_{\substack{s=\rho i \\ \tau=\tau^*}} = -\frac{A + Bi}{C + Di}, \tag{3.18}$$

where

$$\begin{aligned}
A &= \rho a_m(\rho)\sin(\tau^*\rho) + 2\rho b_m(\rho)\sin(2\tau^*\rho),\\
B &= \rho a_m(\rho)\cos(\tau^*\rho) + 2\rho b_m(\rho)\cos(2\tau^*\rho),\\
C &= \alpha\rho^{\alpha-1}\cos\left(\frac{(\alpha-1)\pi}{2}\right) + \tau^* a_m(\rho)\cos(\tau^*\rho) + 2\tau^* b_m(\rho)\cos(2\tau^*\rho),\\
D &= \alpha\rho^{\alpha-1}\sin\left(\frac{(\alpha-1)\pi}{2}\right) - \tau^* a_m(\rho)\sin(\tau^*\rho) - 2\tau^* b_m(\rho)\sin(2\tau^*\rho).
\end{aligned}$$

A direct calculation reveals that

$$\begin{aligned}
\operatorname{Re}\left(\frac{ds}{d\tau}\right)\bigg|_{\substack{s=\rho i \\ \tau=\tau^*}} \cdot (C^2 + D^2) &= -AC - BD\\
&= \alpha\rho^{\alpha} a_m(\rho)\left(\cos(\tau^*\rho)\cos\left(\frac{\alpha\pi}{2}\right) - \sin(\tau^*\rho)\sin\left(\frac{\alpha\pi}{2}\right)\right)\\
&\quad + 2\alpha\rho^{\alpha} b_m(\rho)\left(\cos(2\tau^*\rho)\cos\left(\frac{\alpha\pi}{2}\right) - \sin(2\tau^*\rho)\sin\left(\frac{\alpha\pi}{2}\right)\right)\\
&= \frac{\alpha\rho^{2\alpha}}{2}\left(3 - \frac{\sin(\alpha\pi + 3\tau^*\rho)}{\sin(\tau^*\rho)}\right).
\end{aligned}$$

Therefore, if we show that

$$f(x) = 3 - \frac{\sin(\alpha\pi + 3x)}{\sin x} > 0 \quad \text{for } x \in J_{2n} \cup J_{2n+1}, \tag{3.19}$$

then the proof will be complete. We only need to consider the behavior of $f(x)$ on the interval $(0, \pi)$ because $f(x)$ is a periodic function with period π and satisfies $\lim_{x\to k\pi^-} f(x) = +\infty$ and $\lim_{x\to k\pi^+} f(x) = -\infty$ for any integer k. First, we obtain $f((r-\alpha)\pi/3) = 3$ for $r = 1, 2, 3$, and

$$\begin{aligned} f'(x) &= \frac{\sin(\alpha\pi + 3x)\cos(x) - 3\cos(\alpha\pi + 3x)\sin(x)}{\sin^2(x)} \\ &= \frac{2\sin(\alpha\pi + 2x) - \sin(\alpha\pi + 4x)}{\sin^2(x)} =: \frac{g(x)}{\sin^2(x)}. \end{aligned}$$

Note that $g((1-\alpha)\pi/3) = 3\sin((1-\alpha)\pi/3) > 0$, $g((2-\alpha)\pi/3) = -3\sin((2-\alpha)\pi/3) < 0$, $g((3-\alpha)\pi/3) = 3\sin((3-\alpha)\pi/3) > 0$, and $g'(x) = 8\sin(x)\sin(\alpha\pi + 3x) = 0$ only at $x = (r-\alpha)\pi/3$ for $r = 1, 2, 3$. Therefore, there only exist two numbers x_1^* and x_2^* with

$$\frac{(1-\alpha)\pi}{3} < x_1^* < \frac{(2-\alpha)\pi}{3} < x_2^* < \frac{(3-\alpha)\pi}{3}$$

such that $g(x_1^*) = g(x_2^*) = 0$. Then we see that the minimum of $f(x)$ on the interval $((1-\alpha)\pi/3, (3-\alpha)\pi/3)$ is $f(x_2^*)$. Next, we will show that $f(x_2^*) > 0$. From $f'(x_2^*) = 0$, we have

$$\sin(\alpha\pi + 3x_2^*)\cos(x_2^*) = 3\cos(\alpha\pi + 3x_2^*)\sin(x_2^*).$$

This yields

$$\sin^2(\alpha\pi + 3x_2^*)\left(1 - \sin^2(x_2^*)\right) = 9\left(1 - \sin^2(\alpha\pi + 3x_2^*)\right)\sin^2(x_2^*).$$

Thus,

$$\left|\frac{\sin(\alpha\pi + 3x_2^*)}{\sin(x_2^*)}\right| = \frac{3}{\sqrt{1 + 8\sin^2(x_2^*)}} < 3,$$

which implies that $f(x_2^*) > 0$. Therefore, we obtain

$$f(x) > 0 \quad \text{for} \quad x \in \left(\frac{(1-\alpha)\pi}{3} + k\pi, \frac{(3-\alpha)\pi}{3} + k\pi\right), \ k \in \mathbb{N}.$$

In other words, (3.19) is satisfied. □

Proposition 3.4. *Let $\alpha \in (0,1)$, a, b and $\tau > 0$ be constants. Then, all roots of (3.1) are located in the left half of the complex plane if and only if $(a, b) \in D_{0,0}$, where the open region $D_{0,0}$ is bounded by the line $a + b = 0$ and the curve Γ_0 with $\rho \in ((1-\alpha)\pi/3\tau, (3-\alpha)\pi/3\tau)$.*

Proof. The region $\overline{D}$ is divided into an infinite number of regions by the curves Γ_m for $m \in \mathbb{N}$. As shown in Figure 3.2, the open regions that are enclosed by the line $a + b = 0$ and the curves Γ_m are denoted by $D_{h,k}$ with nonnegative integers h and k. It is well established that the number of roots with positive real part of the characteristic equation changes only if the pure imaginary root appears. Then the number of roots of (3.1) with positive real parts does not change in these regions $D_{h,k}$ if the time delay is fixed. Let $N(D_{h,k})$ denote the number of

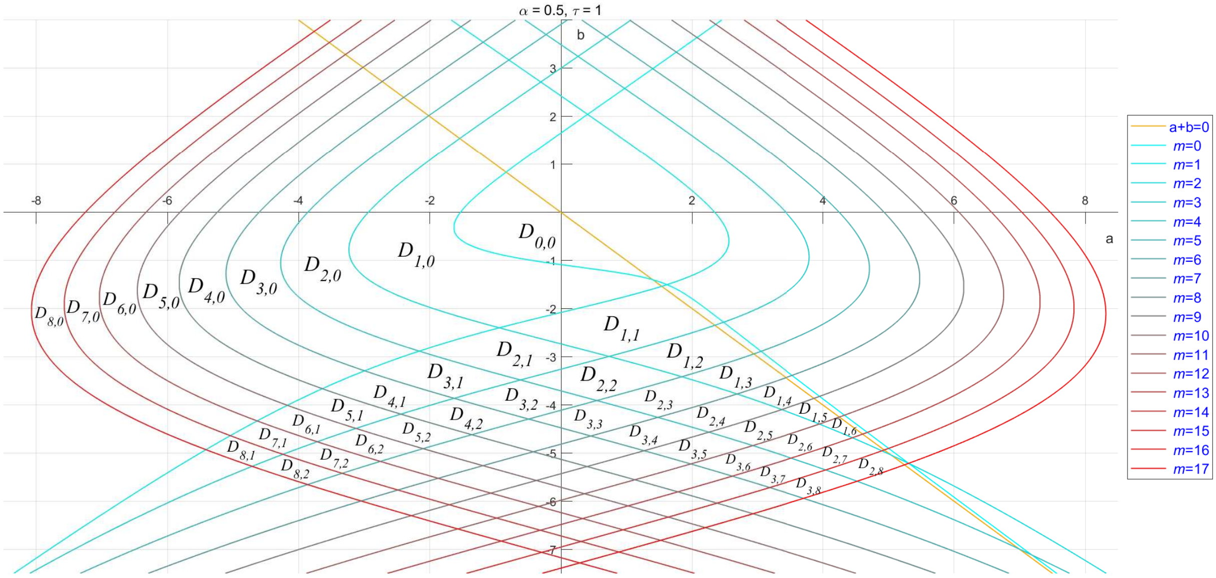

Figure 3.2: The graph of $\varGamma_m$ for $\alpha = 0.5$, $\tau = 1$.

roots of (3.1) with positive real parts for $(a, b) \in D_{h,k}$. From Theorem A and Lemma 3.2 (i), we see that $N(D_{0,0}) = 0$ and $N(D_{h,k}) > 0$ for $D_{h,k}$ which intersects the a-axis or the b-axis.

Suppose there exists a point (a, b) lies on the common boundary of $D_{1,0}$ and $D_{1,1}$, which is on the curve $\varGamma_1$. Especially, we suppose the point (a, b) is close to the common point of $\varGamma_1$ and the line $a + b = 0$, which guarantees that $\tau^*\rho \in J_1$. From Lemma 3.2 (i), the point (a, b) moves from $D_{1,0}$ into $D_{1,1}$ as τ grows near τ^*. Then, Proposition 3.3 shows that the number of roots of (3.1) with positive real parts increases as the point (a, b) moves from $D_{1,0}$ into $D_{1,1}$. Therefore, $N(D_{1,1}) > N(D_{1,0})$. Similarly, we can prove that $N(D_{1,k+1}) > N(D_{1,k}) > 0$ for any nonnegative integer k.

Suppose there exists a point (a, b) lies on the common boundary of $D_{1,0}$ and $D_{2,0}$, which is on the curve $\varGamma_2$. From Lemma 3.2 (i), the point (a, b) moves from $D_{1,0}$ into $D_{2,0}$ as τ grows near τ^*. Then, Proposition 3.3 shows that the number of roots of (3.1) with positive real parts increases as the point (a, b) moves from $D_{1,0}$ into $D_{2,0}$. Therefore, $N(D_{2,0}) > N(D_{1,0})$. Similarly, we can prove that $N(D_{2,k}) > N(D_{1,k})$ for any nonnegative integer k.

By repeating the same argument, we conclude that for any $D_{h,k}$ with $h = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$, (3.1) has at least one pair of conjugate roots with positive real part. This completes the proof. □

4 Proof of main theorem

To prove that the zero solution to (1.4) is asymptotically stable or stable, we must examine the behavior of the solution to (1.4) given in (2.1) as $t \to \infty$. Because the representation formula (2.1) comprises functions (2.2) and (2.3), the problem is reduced to studying the behavior of the family of functions Φ as $t \to \infty$.

Because function Φ is defined by the inverse Laplace transform, we introduce an integral curve that is commonly used in inverse Laplace transforms. For brevity, denote by $\gamma(\varepsilon, \vartheta)$ the contour comprising the following parts:

$$\{s \in \mathbb{C} \mid |s| = \varepsilon, |\arg(s)| \le \vartheta\}, \quad \{s \in \mathbb{C} \mid |s| \ge \varepsilon, |\arg(s)| = \vartheta\},$$

where $\varepsilon > 0$ and $0 < \vartheta \le \pi$.

The following lemma is utilized to compute the inverse Laplace transform of Φ.

Lemma 4.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1]$, $\varepsilon > 0$, $\sigma \in (\pi/2, \pi)$, and $\ell = 0$ or $\ell = \alpha$. Define*

$$w(t) = \frac{1}{2\pi\alpha i} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \frac{z^{\frac{1-\beta}{\alpha}} \exp\left(z^{\frac{1}{\alpha}}(1 + 2\tau/t)\right)}{z \exp\left((2\tau/t) z^{\frac{1}{\alpha}}\right) - at^\alpha \exp\left((\tau/t) z^{\frac{1}{\alpha}}\right) - bt^\alpha} dz.$$

If there exists $K > 0$ such that

$$\left| z \exp\left(\frac{2\tau}{t} z^{\frac{1}{\alpha}}\right) - at^\alpha \exp\left(\frac{\tau}{t} z^{\frac{1}{\alpha}}\right) - bt^\alpha \right| \ge Kt^\ell \left| \exp\left(\frac{2\tau}{t} z^{\frac{1}{\alpha}}\right) \right| \tag{4.1}$$

for all $z \in \gamma(\varepsilon^\alpha, \alpha\sigma)$, then $w(t) = O(t^{-\ell})$ as $t \to \infty$.

Proof. From (4.1), we observe that:

$$\begin{aligned} |w(t)| &\le \frac{1}{2\pi\alpha} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \frac{\left|z^{\frac{1-\beta}{\alpha}}\right| \left|\exp\left(z^{\frac{1}{\alpha}}(1 + 2\tau/t)\right)\right|}{Kt^\ell \left|\exp\left((2\tau/t) z^{\frac{1}{\alpha}}\right)\right|} |dz| \\ &= \frac{t^{-\ell}}{2\pi\alpha K} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \left|z^{\frac{1-\beta}{\alpha}}\right| \left|\exp\left(z^{\frac{1}{\alpha}}\right)\right| |dz| \\ &= \frac{t^{-\ell}}{2\pi\alpha K} \left(\varepsilon^{1+\alpha-\beta} \int_{-\alpha\sigma}^{\alpha\sigma} \exp\left(\varepsilon \cos \frac{\theta}{\alpha}\right) d\theta + 2 \int_{\varepsilon^\alpha}^{\infty} r^{\frac{1-\beta}{\alpha}} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) dr \right). \end{aligned}$$

From $\sigma \in (\pi/2, \pi)$, it follows that

$$\begin{aligned} \int_{\varepsilon^\alpha}^{\infty} r^{\frac{1-\beta}{\alpha}} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) dr &= \frac{\alpha}{(-\cos\sigma)^{1+\alpha-\beta}} \int_{\varepsilon^\alpha}^{\infty} \left(-\cos\sigma r^{\frac{1}{\alpha}}\right)^{\alpha-\beta} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) d\left(-\cos\sigma r^{\frac{1}{\alpha}}\right) \\ &< \frac{\alpha\Gamma(1+\alpha-\beta)}{(-\cos\sigma)^{1+\alpha-\beta}}. \end{aligned}$$

Therefore, we obtain

$$|w(t)| < \frac{t^{-\ell}}{\pi\alpha K} \left(\alpha\sigma\varepsilon^{1+\alpha-\beta} e^\varepsilon + \frac{\alpha\Gamma(1+\alpha-\beta)}{(-\cos\sigma)^{1+\alpha-\beta}} \right) = O\left(t^{-\ell}\right) \quad \text{as } t \to \infty,$$

as desired. □

Proposition 4.2. *Let $0 < \alpha < 1$, $0 < \beta \le 1$.*

(i) *If all the roots of (3.1) are located in the left half-plane, then $\Phi^{\alpha,\beta}(t) = O(t^{\beta-\alpha-1})$ as $t \to \infty$.*

(ii) *If there exists the root 0 of (3.1) and other roots of (3.1) are located in the left half-plane, then $\Phi^{\alpha,\beta}(t) = O(t^{\beta-1})$ as $t \to \infty$.*

(iii) *If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those of the roots on the imaginary axis are simple, then $\Phi^{\alpha,\beta}(t)$ is bounded.*

(iv) *If all the roots of* (3.1) *are located in the left half-plane or on the imaginary axis, and those of the roots on the imaginary axis are double, then*

$$\limsup_{t\to\infty}\left|\frac{\Phi^{\alpha,\beta}(t)}{t}\right| = \text{const} > 0. \tag{4.2}$$

(v) *If there are some roots s_i of* (3.1) *in the right half-plane, then*

$$\limsup_{t\to\infty}\left|\frac{\Phi^{\alpha,\beta}(t)}{t^2 e^{Mt}}\right| = \text{const} > 0, \tag{4.3}$$

where $M = \max_{s_i}\{\mathrm{Re}(s_i)\}$.

Proof. Proposition 3.1 (i) implies that only the finite number of roots of (3.1) satisfy $|\arg(s)| < \sigma$ for $\sigma \in (\pi/2, \pi)$. Then, for $t > 1$, there exist $R > 0$ and $\varepsilon > 0$ with $R > \varepsilon$ such that all the roots of (3.1) are located to the left of $\gamma(R,\sigma)$ and the roots of (3.1) with $|\arg(s)| \le \sigma$ are located to the right of $\gamma(\varepsilon/t,\, \sigma)$. Subsequently, $\Phi^{\alpha,\beta}(t)$ is expressed as

$$\begin{aligned}\Phi^{\alpha,\beta}(t) &= \frac{1}{2\pi i}\int_{\gamma(R,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds\\ &= \frac{1}{2\pi i}\int_{\gamma(R,\sigma)-\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds + \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds.\end{aligned}$$

Using the residue theorem, we obtain

$$I_1(t) := \frac{1}{2\pi i}\int_{\gamma(R,\sigma)-\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds = \sum_{k=1}^{N}\mathrm{Res}\left[\frac{s^{\alpha-\beta}e^{ts}}{P(s)}; s_k\right].$$

Here, s_k, $k = 1, 2, \ldots, N$, represent the roots of (3.1) in the area enclosed by $\gamma(R,\sigma) - \gamma(\varepsilon/t,\, \sigma)$. Proposition 3.1 (iii) implies that

$$\begin{aligned}\frac{s^{\alpha-\beta}}{P(s)} &= a^k_{-3}(s-s_k)^{-3} + a^k_{-2}(s-s_k)^{-2} + a^k_{-1}(s-s_k)^{-1} + a^k_0 + \cdots,\\ e^{ts} &= e^{ts_k}\left(1 + t(s-s_k) + t^2(s-s_k)^2/2! + \cdots\right),\end{aligned}$$

where a^k_j, $j = -3, -2, \ldots$, are complex constants that are not all zero. Hence,

$$I_1(t) = \sum_{k=1}^{N}\left(a^k_{-1} + a^k_{-2}t + \frac{a^k_{-3}}{2!}t^2\right)e^{ts_k}.$$

By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\begin{aligned}I_2(t) &:= \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds\\ &= \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}ds\\ &= \frac{t^{\beta-1}}{2\pi\alpha i}\int_{\gamma(\varepsilon^\alpha,\alpha\sigma)}\frac{z^{\frac{1-\beta}{\alpha}}\exp\left(z^{\frac{1}{\alpha}}(1+2\tau/t)\right)}{z\exp\left((2\tau/t)z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left((\tau/t)z^{\frac{1}{\alpha}}\right) - bt^\alpha}dz\\ &= t^{\beta-1}w_{\frac{1-\beta}{\alpha},2}(t).\end{aligned}$$

If $a+b=0$, then (3.1) has the root 0. Note that the points in

$$\{s \in \mathbb{C} \mid s = (\varepsilon/t)e^{\theta i},\, \theta \in [-\sigma, \sigma]\}$$

are the closest points of $\gamma(\varepsilon/t, \sigma)$ with respect to the root 0. Thus, we have

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \left| \left(\frac{\varepsilon}{t}\right)^\alpha e^{\alpha\theta i} - a\exp\left(-\frac{\varepsilon\tau}{t}e^{\theta i}\right) - b\exp\left(-\frac{2\varepsilon\tau}{t}e^{\theta i}\right)\right|.$$

Because

$$\lim_{t\to\infty} \left| \left(\frac{\varepsilon}{t}\right)^\alpha e^{\alpha\theta i} - a\exp\left(-\frac{\varepsilon\tau}{t}e^{\theta i}\right) - b\exp\left(-\frac{2\varepsilon\tau}{t}e^{\theta i}\right)\right| t^\alpha = \varepsilon^\alpha,$$

we observe that there exist $\eta_0 > 0$ and $T > 0$ such that

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \eta_0 t^{-\alpha} \quad \text{for } t > T.$$

By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\left| \frac{z}{t^\alpha} - a\exp\left(-\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - b\exp\left(-\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right| \geq \eta_0 t^{-\alpha} \quad \text{for } t > T,$$

namely,

$$\left| z\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left(\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - bt^\alpha\right| \geq \eta_0 \left|\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right| \quad \text{for } t > T.$$

Then, from Lemma 4.1 with $\ell = 0$, we obtain

$$I_2(t) = t^{\beta-1} w_{\frac{1-\beta}{\alpha},2}(t) = t^{\beta-1}O(1) = O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

If $a+b \neq 0$, then (3.1) contains only nonzero roots. Because contour $\gamma(\varepsilon/t, \sigma)$ does not contain any root of (3.1), there exists $\eta_1 > 0$ such that

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \eta_1$$

for all $s \in \gamma(\varepsilon/t, \sigma)$. By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\left| z\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left(\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - bt^\alpha\right| \geq \eta_1 t^\alpha \left|\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right|.$$

Then, from Lemma 4.1 with $\ell = \alpha$, it follows that

$$I_2(t) = t^{\beta-1} w_{\frac{1-\beta}{\alpha},2}(t) = t^{\beta-1}O(t^{-\alpha}) = O\big(t^{\beta-\alpha-1}\big) \quad \text{as } t \to \infty.$$

(i) If all the roots of (3.1) are located in the left half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_2(t) = O\big(t^{\beta-\alpha-1}\big) \quad \text{as } t \to \infty.$$

(ii) If there exists the root 0 of (3.1) and the other roots of (3.1) are located in the left half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_2(t) = O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

(iii) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are simple, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N_1} a_{-1}^k e^{\pm\rho_k it} + O\big(t^{\beta-1}\big),$$

which implies that $\Phi^{\alpha,\beta}(t)$ is bounded.

(iv) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are double, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N_1} (a_{-1}^k + a_{-2}^k t) e^{\pm\rho_k it} + O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

This yields (4.2).

(v) If there are some roots s_i of (3.1) in the right half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N} (a_{-1}^k + a_{-2}^k t + a_{-3}^k t^2) e^{ts_k} + O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

This implies (4.3). □

Remark 4.3. The asymptotic behaviors in Proposition 4.2 are fully characterized by Matignon's Theorem 1 in [17]: the algebraic decay $O(t^{-\alpha})$ for stable solutions corresponds to characteristic roots satisfying $|\arg(\lambda)| > \alpha\pi/2$; bounded but non-asymptotically stable solutions on the stability boundary occur when characteristic roots satisfy $|\arg(\lambda)| \geq \alpha\pi/2$ and the geometric multiplicity of a characteristic root with $|\arg(\lambda)| = \alpha\pi/2$ is one; the exponential growth rate e^{Mt} in unstable regimes directly follows from his analysis of $|\arg(\lambda)| \leq \alpha\pi/2$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall the representation formula (2.1). There exist $A > 0$ and $B > 0$ such that

$$\begin{aligned}\left| a\int_{-\tau}^{0} \phi(u)\Phi^{\alpha,\alpha}{}_{\tau+u}(t)du \right| &\leq |a| \int_{-\tau}^{0} |\phi(u)\Phi^{\alpha,\alpha}{}_{\tau+u}(t)|\, du \\ &\leq |a| \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)| \int_{-\tau}^{0} |\phi(u)|\, du \\ &= A \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|,\end{aligned}$$

and

$$\left| b\int_{-2\tau}^{0} \phi(u)\Phi^{\alpha,\alpha}{}_{2\tau+u} du \right| \leq B \sup_{t-2\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|.$$

Thus, we have:

$$|x(t)| \leq |\phi(0)\Phi^{\alpha,1}(t)| + A \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)| + B \sup_{t-2\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|.$$

Then, from Proposition 4.2, we conclude that:

(i) If all the roots of (3.1) are located in the left half-plane, then

$$x(t) = |\phi(0)|\, O(t^{-\alpha}) + A\cdot O(t^{-1}) + B\cdot O(t^{-1}) \quad \text{as } t \to \infty.$$

Therefore, $x(t) = O(t^{-\alpha})$ if $\phi(0) \neq 0$ and $x(t) = O(t^{-1})$ if $\phi(0) = 0$ as $t \to \infty$.

(ii) If there exists the root 0 of (3.1) and the other roots of (3.1) are located in the left half-plane, then we obtain

$$x(t) = |\phi(0)|\, O(1) + A \cdot O(t^{\alpha-1}) + B \cdot O(t^{\alpha-1}) \quad \text{as } t \to \infty.$$

Therefore, $x(t) = O(1)$ if $\phi(0) \neq 0$ and $x(t) = O(t^{\alpha-1})$ if $\phi(0) = 0$ as $t \to \infty$.

(iii) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are simple, then $x(t)$ is bounded.

(iv) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are double, then

$$\limsup_{t\to\infty} \left| \frac{x(t)}{t} \right| = \text{const} > 0.$$

(v) If there are some roots s_i of (3.1) in the right half-plane, then

$$\limsup_{t\to\infty} \left| \frac{x(t)}{t^2 e^{Mt}} \right| = \text{const} > 0,$$

where $M = \max_{s_i}\{\text{Re}(s_i)\}$.

Hence, along with Propositions 3.1 and 3.3, Theorem 1.1 holds true. □

5 Conclusion

In this study, we discussed a Caputo-type fractional-order differential equation with two delays

$${}^{C}\text{D}_0^{\alpha} x(t) = ax(t-\tau) + bx(t-2\tau), \quad t \geq 0,$$

where $\alpha \in (0,1)$. Using the Laplace transform, we obtained the representation formula of the solution and the associated characteristic equation. From a detailed root analysis of the characteristic equation, we presented a region enclosed by a line and a curve in the ab-plane to guarantee that all the roots of the characteristic equation are located in the left half-plane for given α and τ. Moreover, we proved that the zero solution is asymptotically stable if and only if the pair (a, b) is an interior point of this region, and it is only stable if the pair (a, b) is on the partial boundary of this region. Because the Caputo fractional derivative coincides with the first-order derivative if $\alpha \to 1^-$, we compared Theorem 1.1 and Theorem C, which describes the asymptotic stability of the corresponding first-order delay differential equation. On the one hand, the expression of the set of (a, b) for asymptotic stability is the same if we set $\alpha = 1$ in the criterion of Theorem 1.1. On the other hand, we can judge from Figure 1.1, which shows the regions of asymptotic stability when α takes different values, that there exists no relationship of one set being a subset of another. This implies that the fractional order does not make it easier or more difficult to stabilize the zero solution than the integer order.

The asymptotic stability of two cases will be the focus of our future studies: (1.1) with $\tau_1 = \tau$, $\tau_2 = 3\tau$, and $0 < \alpha < 1$; and (1.4) with $1 < \alpha < 2$. This work aims to confirm the effect of the fractional-order α.

Statements and declarations

The authors declare that there are no conflicts of interest.

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