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A Note on Projectively Coresolved Gorenstein Flat Complexes and Dimensions

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Abstract: In this article, we first establish a recollement related to projectively coresolved Gorenstein flat (PGF) complexes. Secondly, we define and study PGF dimension of complexes, we denote it $\text{PGF}(X)$ for a complex X . It is shown that the $\text{PGF}(X)$ is equal to the infimum of the set $\{\sup A \mid \text{there exists a diagram of morphisms of complexes } A \leftarrow G \rightarrow X, \text{ such that } G \rightarrow X \text{ is a special PGF precover of } X \text{ and } G \rightarrow A \text{ is a PGF almost isomorphism}\}$.

Key words: projectively coresolved Gorenstein flat (PGF) module; PGF complex; recollement; PGF dimension

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0 Introduction

The famous "flat cover conjecture", that is, all modules have a flat cover over any ring, is addressed by Enochs^[1]. It has now been settled by Bican *et al*^[2]. It is natural to consider the Gorenstein version of "flat cover conjecture". Thus, the existence of Gorenstein flat covers of moduls is concerned by many scholars^[3-5]. In particular, Yang and Liu^[4] proved that all modules have Gorenstein flat covers over a left ring over which the class of Gorenstein flat modules is closed under extensions (GF-closed). Recently, Šaroch and Šťovíček^[6] proved that every ring is always GF-closed in terms of the notion of projectively coresolved Gorenstein flat (PGF) modules. Thus, the Gorenstein version of flat cover conjecture holds as well. Recently, the researcheres extend the notion of PGF modules to the category of complexes, and call it PGF complexes^[7].

The definition of recollements of triangulated categories was first introduced by Beilinson *et al*^[8] to study the triangulated categories of perverse sheaves over singular spaces, and later was used by Scott *et al*^[9] to stratify the derived categories of quasi-hereditary algebras arising from the representation theory of semisimple Lie algebras and algebraic groups.

Homological dimension plays an important role in homological algebra, and has been extended to the setting of complexes by many researchers. Avramov and Foxby^[10] defined the notion of projective dimension^[10] for a complex X ,

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pd(X), as the infimum of the set $\{\sup P \mid P \text{ is a DG-projective resolution of } X\}$. They showed that for any complexes X , $\text{pd}(X) = \inf \{n \in \mathbb{Z} \mid \sup H(P) \leq n, \text{Coker} \delta_{n+1}^P \text{ is projective, where } P \rightarrow X \text{ is a DG - projective resolution}\}$.

Note that if $f: P \rightarrow X$ is a quasi-isomorphism then $\text{Ext}^1(Q, \text{Ker}f) = 0$ for every DG-projective complex Q . Thus, the DG-projective resolution $P \rightarrow X$ is a special DG-projective precover. This means that for any complex X ,

$$\text{pd}(X) = \inf \{n \in \mathbb{Z} \mid \sup H(P) \leq n, \text{Coker} \delta_{n+1}^P \text{ is projective, where } P \rightarrow X \text{ is a special DG - projective precover}\}.$$

Gorenstein projective dimension of complexes, $\text{Gpd}(X)$, was introduced by Veliche and it follows from Ref. [11] that $\text{Gpd}(X) = \inf \{n \in \mathbb{Z} \mid \sup H(P) \leq n, \text{Coker} \delta_{n+1}^P \text{ is Gorenstein projective, where } P \rightarrow X \text{ is a DG - projective resolution}\} = \inf \{n \in \mathbb{Z} \mid \sup H(P) \leq n, \text{Coker} \delta_{n+1}^P \text{ is Gorenstein projective, where } P \rightarrow X \text{ is a special DG - projective precover of } X\}$.

In Ref. [12], the authors introduced and studied the dimension of complexes, which is related to special Gorenstein projective precovers:

$$\text{Gppd}(X) = \inf \{n \in \mathbb{Z} \mid \sup H(G) \leq n, \text{ and } \text{Coker} \delta_{n+1}^G \text{ is Gorenstein projective, where } G \rightarrow X \text{ is a special Gorenstein projective precover of } X\}.$$

If no such G exists, set $\text{Gppd}(X) = \infty$.

Inspired by the above, in this article, we study the recollement and dimension related to PGF complexes.

1 Preliminaries

In this section we recall some necessary notations and definitions. Throughout the article, let R denote an associative ring with an identity and by the term "module" we always mean a left R -module.

A complex X of modules is often displayed as a sequence

$$\cdots \rightarrow X_{n+1} \xrightarrow{\delta_{n+1}^X} X_n \xrightarrow{\delta_n^X} X_{n-1} \rightarrow \cdots$$

with $\delta_n^X \delta_{n+1}^X = 0$ for all $n \in \mathbb{Z}$. The n -th cycle (resp., homology) module is defined as $\text{Ker} \delta_n^X$ (resp., $\text{Ker} \delta_n^X / \text{Im} \delta_{n+1}^X$) and denoted by $Z_n(X)$ (resp., $H_n(X)$). The homology complex $H(X)$ is defined by setting $H(X)_i = H_i(X)$ and $\delta_i^{H(X)} = 0$ for all $i \in \mathbb{Z}$. A complex X is called exact if $H_i(X) = 0$ for all $i \in \mathbb{Z}$. For a complex X , we associate the numbers

$$\sup X = \sup \{i \in \mathbb{Z} \mid X_i \neq 0\} \quad \text{and} \quad \inf X = \inf \{i \in \mathbb{Z} \mid X_i \neq 0\}.$$

The complex X is called bounded above (resp., bounded below) if $\sup X < \infty$ (resp., $\inf X > -\infty$). It is called bounded if it is both bounded above and bounded below.

In the following, we will use $\mathcal{C}(R)$ and $\mathcal{D}(R)$ to denote the category of complexes of modules and the derived category. The symbol \simeq is used to designate quasi-isomorphisms in $\mathcal{C}(R)$. For complexes X and Y , denote by $\text{Hom}(X, Y)$ the Abelian group of morphisms from X to Y in $\mathcal{C}(R)$ and $\text{Ext}^i(X, Y)$ for $i \geq 1$ will denote the groups we get from the right derived functor of $\text{Hom}(-, -)$.

For two complexes X and Y , denote by $\text{Hom}_R(X, Y)$ the complex with $\text{Hom}_R(X, Y)_n = \prod_{i \in \mathbb{Z}} \text{Hom}_R(X_i, Y_{i+n})$ for all $n \in \mathbb{Z}$ and with the boundary operator given by $\delta_n^{\text{Hom}_R(X, Y)}(f) = (\delta_{n+i}^Y f_i - (-1)^{|f|} f_{i-1} \delta_i^X)_{i \in \mathbb{Z}}$ for every $f \in \text{Hom}_R(X, Y)_n$. We denote by $\text{Ext}_R^n(-, -)$ the right derived functors of $\text{Hom}_R(-, -)$ and call it the absolute cohomology functors. Two morphisms α and β in $\text{Hom}(X, Y) = \text{Ker} \delta_0^{\text{Hom}_R(X, Y)}$ are called homotopic, denoted by $\alpha \sim \beta$, if there exists a homomorphism μ of degree 1 such that $\delta_1^{\text{Hom}_R(X, Y)}(\mu) = \alpha - \beta$.

Let X be a complex of right R -modules and Y a complex of left R -modules, denote the usual tensor product of X and Y , where $(X \otimes_R Y)_n = \bigoplus_{i \in \mathbb{Z}} X_i \otimes_R Y_{n-i}$ and $\delta(x \otimes y) = \delta^X(x) \otimes y + (-1)^{|x|} x \otimes \delta^Y(y)$ for $x \in X_i, y \in Y_{n-i}$. We define $X \bar{\otimes} Y$

to be $\frac{(X \otimes_R Y)}{\text{B}(X \otimes_R Y)}$ with the maps

$$\frac{(X \otimes_R Y)_m}{\text{B}_m(X \otimes_R Y)} \rightarrow \frac{(X \otimes_R Y)_{m-1}}{\text{B}_{m-1}(X \otimes_R Y)}, \quad x \otimes y \rightarrow \delta^X(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $\frac{(X \otimes_R Y)_m}{\text{B}_m(X \otimes_R Y)}$. Then we get a complex. Since $X \bar{\otimes} -: \mathcal{C}(R) \rightarrow \mathcal{C}(\mathbb{Z})$ is a right ex-

act functor between Abelian categories with enough projectives, we can construct left derived functors which we denote by $\overline{\text{Tor}}_i(C, -)$.

Let \mathcal{A} be an Abelian category and \mathcal{X} a subcategory of \mathcal{A} . For an object $M \in \mathcal{A}$ write $M \in {}^\perp \mathcal{X}$ if $\text{Ext}_{\mathcal{A}}^1(M, X) = 0$ for each object $X \in \mathcal{X}$. Dually, one can define $M \in \mathcal{X}^\perp$.

Recall that a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of \mathcal{A} is called a cotorsion pair or cotorsion theory provided that $\mathcal{X} = {}^\perp \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^\perp$. A morphism $\phi: X \rightarrow M$ with $X \in \mathcal{X}$ is called an \mathcal{X} -precover of M if for any morphism $f: X' \rightarrow M$ with $X' \in \mathcal{X}$, there is a morphism $g: X' \rightarrow X$ such that $\phi g = f$. An epimorphism $\psi: X \rightarrow M$ with $X \in \mathcal{X}$ is said to be a special \mathcal{X} -precover of M if ψ is an \mathcal{X} -precover of M and $\text{Ker} \psi \in \mathcal{X}^\perp$.

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be complete provided that every object in \mathcal{A} has a special \mathcal{Y} -preenvelope and a special \mathcal{X} -precover. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ cogenerated by a set S of objects, i.e. such that $\mathcal{Y} = S^\perp$, is complete, see Ref. [13]. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be hereditary provided that $\text{Ext}_{\mathcal{A}}^2(X, Y) = 0$ for each object $X \in \mathcal{X}$ and each object $Y \in \mathcal{Y}$. Ref. [3] is a standard reference for cotorsion pairs^[3]. A complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be a projective cotorsion pair^[14] if \mathcal{Y} is thick and $\mathcal{X} \cap \mathcal{Y}$ coincides with the class of projective objects. Recall from Ref. [14] that a class $\mathcal{W} \subseteq \mathcal{A}$ is called thick provided that it is closed under direct summands, extensions, and taking kernels of epimorphisms and cokernels of monomorphisms.

2 A Recollement Related to PGF Complexes

We begin with the definition of PGF modules and PGF complexes. Note that PGF modules not only are Gorenstein flat, but also Gorenstein projective^[6].

Definition 1^[6-7] A module M is said to be PGF if there exists an exact sequence of projective modules

$$\dots \rightarrow Q^1 \rightarrow Q^0 \rightarrow Q^{-1} \rightarrow \dots$$

with $M = \text{Ker}(Q^0 \rightarrow Q^{-1})$ and which remains exact after applying $E \otimes_R -$ for any injective right R -modules E .

A complex X is said to be PGF if there exists an exact sequence of projective complexes

$$\dots \rightarrow P^1 \rightarrow P^0 \rightarrow P^{-1} \rightarrow \dots$$

with $X = \text{Ker}(P^0 \rightarrow P^{-1})$ and which remains exact after applying $I \otimes -$ for any injective complex I of right R -modules.

In the following, we will use **PGF** and **PGF** to denote the category of PGF modules and the category of PGF complexes.

We give some basic results for PGF modules and PGF complexes which are obtained by Refs. [6-7, 14].

Lemma 1 (1) The class of PGF modules is projectively resolving, closed under arbitrary direct sums, under direct summands and transfinite extensions.

(2) **(PGF, PGF[⊥])** is hereditary complete cotorsion pair.

(3) **(PGF, PGF[⊥])** is a projective cotorsion pair which is cogenerated by a set.

(4) If M is a PGF module, then $\text{Tor}_n^R(E, M) = 0$ for any injective right R -modules E and any $n \geq 1$.

(5) A complex G is a PGF complex if and only if G_n is a PGF module for each $n \in \mathbb{Z}$.

Remark 1 Gorenstein flat complexes are exactly the complexes with Gorenstein flat components over right coherent rings, independently^[6, 15]. Šaroch and Šťovíček^[6] showed that the class of Gorenstein flat modules is always closed under extensions over any ring. Yang and Liu^[15] proved that Gorenstein flat complexes are exactly the complexes with Gorenstein flat components whenever the class of Gorenstein flat modules is closed under extensions. Thus, Gorenstein flat complexes are exactly the complexes with Gorenstein flat components.

We obtain the following result which gives equivalent characterizations of that Gorenstein projective modules are Gorenstein flat by Ref. [7].

Lemma 2 The following statements are equivalent:

(1) Every Gorenstein projective modules are Gorenstein flat modules.

(2) Every Gorenstein projective modules are PGF modules.

(3) Every Gorenstein projective complexes are Gorenstein flat complexes.

(4) Every Gorenstein projective complexes are PGF complexes.

Proof (1) \Leftrightarrow (2) follows from Ref. [6].

(1) \Leftrightarrow (3) follows by Ref. [16] and Remark 1.

(2) \Leftrightarrow (4) holds by Ref. [7].

Definition 2^[14] Let \mathcal{D} , \mathcal{D}' and \mathcal{D}'' be triangulated categories. We say that \mathcal{D} is a recollement of \mathcal{D}' and \mathcal{D}'' if there are six triangle functors as in Fig. 1 such that

(1) (F_λ, F, F_ρ) and (G_λ, G, G_ρ) are adjoint triples;

(2) F, G_λ and G_ρ are fully faithful;

(3) For any object $X \in \mathcal{D}$, we have $GX=0$ if and only if $X \cong FX'$ for some $X' \in \mathcal{D}'$.

Recall from Ref. [17] that a triple $\mathcal{H}=(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ of classes of objects in an Abelian category is called a Hovey triple if $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ are complete cotorsion pairs and the class \mathcal{W} is thick. As a consequence of Refs. [6-7, 14], we obtain the following result.

Lemma 3 There exist two projective cotorsion pairs $(\mathcal{PGF}, \mathcal{PGF}^\perp)$ and $(\mathcal{EPGF}, \mathcal{EPGF}^\perp)$ which are cogenerated by sets, where \mathcal{EPGF} denote the class of all exact PGF complexes. Moreover, $(\mathcal{PGF}, \mathcal{PGF}^\perp, \mathcal{A}(R))$ and $(\mathcal{EPGF}, \mathcal{EPGF}^\perp, \mathcal{A}(R))$ are Hovey triples. In particular, every complex has a special PGF precover and a special exact PGF precover.

Using Lemma 3 and Ref. [17], we also obtain two projective model structures on $\mathcal{C}(R)$ in which every object is fibrant, the objects in \mathcal{PGF} (resp., \mathcal{EPGF}) are cofibrant, and the objects in \mathcal{PGF}^\perp (resp., \mathcal{EPGF}^\perp) are trivial. We call these PGF model structures (resp., exact PGF model structures).

We use $K_{\text{ex}}(\text{PGF})$ and $K(\text{PGF})$ to denote the homotopy category of all exact PGF complexes and the homotopy category of all PGF complexes, respectively. Then we obtain the following conclusion.

Theorem 1 Let R be a ring. There is a recollement in Fig. 2 where $E(E)$ and $E(\text{EPGF}^\perp)$ represent special exact preenvelopes and special EPGF^\perp -preenvelopes, respectively, $C(\text{DG})$ and $C(\text{EPGF})$ represent special DG-projective precovers and special exact PGF precovers, respectively, I denotes the inclusion functor.

Proof We can see that $(\mathcal{PGF}, \mathcal{PGF}^\perp)$ and $(\mathcal{EPGF}, \mathcal{EPGF}^\perp)$ are projective cotorsion pairs in $\mathcal{A}(R)$ by Lemma 3. We also notice that $(\text{DGP}, \mathcal{E})$ is a projective cotorsion pair in $\mathcal{C}(R)$, $\mathcal{PGF} \cap \mathcal{E} = \mathcal{EPGF}$ and $\text{DGP} \subseteq \mathcal{PGF}$ by Ref. [7]. As we know, the homotopy category of all DG-projective complexes $K(\text{DGP})$ is isomorphic to $\mathcal{A}(R)$. Thus, the result is obtain by Ref. [14].

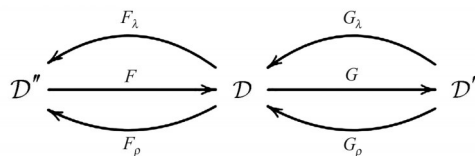


Fig. 1 The definition of recollement

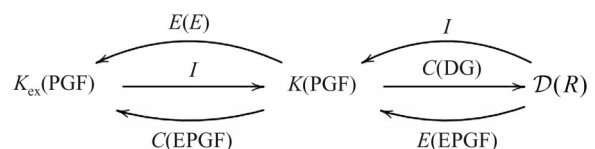


Fig. 2 The recollement of PGF

3 PGF Dimension of Complexes

In this section, we will investigate the dimension with respect to PGF complexes.

Since $(\text{PGF}, \text{PGF}^\perp)$ is a projective cotorsion pair cogenerated by a set, thus by Ref. [14], we conclude that $(\text{dw } \widetilde{\text{PGF}}, \text{dw } \widetilde{\text{PGF}}^\perp)$, in which each degree component of the associated graded object is a PGF module, forms a projective cotorsion pair. By Lemma 1, the complexes in $\text{dw } \widetilde{\text{PGF}}$ are precisely the PGF complexes. Then by Ref. [18], we obtain the following definition.

Definition 3 Let X be a complex. The dimension of X related to special PGF precovers, $\text{PGF}(X)$, is defined as $\text{PGF}(X) = \inf \{ n \in \mathbb{Z} \mid \sup H(G) \leq n, \text{ and } \text{Coker} \delta_{n+1}^G \text{ is a PGF module, where } G \rightarrow X \text{ is a special PGF precover of } X \}$.

If no such G exists, set $\text{PGF}(X) = \infty$.

Definition 4 Let A and G be two PGF complexes. An epimorphism of complexes $f: G \rightarrow A$ is said to be PGF almost isomorphic if $\text{Ker} f$ is exact and bounded below. It is clear that $\text{Ker} f$ is a PGF complex.

Theorem 2 Suppose that G is a PGF complex. Then

$$\begin{aligned} \text{PGF}(G) &= \inf \left\{ n \in \mathbb{Z} \mid \sup H(G) \leq n, \text{Coker} \delta_{n+1}^G \text{ is a PGF module} \right\} \\ &= \inf \left\{ \sup A \mid G \rightarrow A \text{ is a PGF almost isomorphism} \right\}. \end{aligned}$$

Proof Set $s = \inf \left\{ n \in \mathbb{Z} \mid \sup H(G) \leq n, \text{Coker} \delta_{n+1}^G \text{ is a PGF module} \right\}$ and $t = \inf \left\{ \sup A \mid G \rightarrow A \text{ is a PGF almost isomorphism} \right\}$.

Since G is a PGF complex, it can be seen that $\text{PGF}(G) \leq s$. We only need to concentrate on $\text{PGF}(G) \geq s$. If $\text{PGF}(G) = \infty$ then the proof is evident. Suppose that $\text{PGF}(G) = n \in \mathbb{Z}$. Then there exists a special PGF precover $\varphi: F \rightarrow G$ such that $\sup H(F) \leq n$, $\text{Coker} \delta_{n+1}^F$ is a PGF module and $\text{Ext}^1(G, \text{Ker} \varphi) = 0$. Hence the short exact sequence of complexes

$$0 \rightarrow \text{Ker} \varphi \rightarrow F \rightarrow G \rightarrow 0$$

is splits. Then we have $F \cong G \oplus \text{Ker} \varphi$. This implies that $\sup H(G) \leq n$. We can conclude that $s \leq n$. If $\text{PGF}(G) = -\infty$, then for any integer n , we can find a PGF precover $F \rightarrow G$ such that $\sup H(F) \leq n$ and $\text{Coker} \delta_{n+1}^F$ is a PGF module. Similar discussion yields that $s = -\infty$. This leads to the conclusion that $\sup H(G) \leq n$, which is to say $\text{PGF}(G) \geq s$. Thus $\text{PGF}(G) = s$.

We show that $t \leq s$. If $s = \infty$, then the proof is trivial. Assume s is an integer. Then $\sup H(G) \leq s$, and $\text{Coker} \delta_{n+1}^G$ is a PGF module. Consider the commutative diagram (Fig. 3). Define K and A as the following sequences

$$\begin{aligned} K &= \cdots \rightarrow G_{s+2} \rightarrow G_{s+1} \rightarrow \text{Im} \delta_{s+1}^G \rightarrow 0 \cdots, \\ A &= 0 \rightarrow \text{Coker} \delta_{n+1}^G \rightarrow G_{s-1} \rightarrow G_{s-2} \rightarrow \cdots. \end{aligned}$$

It is clearly that K is exact and $\text{Im} \delta_{s+1}^G$ is PGF module. Then the map $G \rightarrow A$ is a PGF almost isomorphic and $\sup A \leq s$. Consequently, $t \leq s$. Suppose $s = -\infty$, hence for any integer n , $\sup H(G) \leq n$ and $\text{Coker} \delta_{n+1}^G$ is a PGF module. Thus G is exact. Using a similar diagram, we can obtain $G \rightarrow A$ is a PGF almost isomorphism such that $\sup A \leq n$. This leads to the conclusion $t = -\infty$. Thus $t \leq s$.

Next, we demonstrate that $s \leq t$. If $t = -\infty$, then the proof is trivial. Assume t is an integer. Then there exists a PGF almost isomorphism $G \rightarrow A$ with $\sup A = t$. Let $E = \text{Ker}(G \rightarrow A)$. Then E is exact and bounded below, which implies that $\text{Coker} \delta_{n+1}^E$ is a PGF module. Consider the commutative diagram (Fig. 4) with exact rows and the exact middle column, then it is evident that the final column of the diagram is exact, which implies that $\text{Coker} \delta_{n+1}^G$ is a PGF module. Additionally, it is evident that $\sup H(G) \leq n$. Consequently, we can conclude that $s \leq t$.

Now, consider the case where $t = -\infty$. For any integer n , there is a PGF almost isomorphism $G \rightarrow A$ such that $\sup A < n$, hence G is exact at $n+1$ and $\text{Coker} \delta_{n+1}^G$ is a PGF module. This means that G is an exact complex, and $\text{Coker} \delta_{n+1}^G$ is a PGF complex for any integer n . This leads to the conclusion that $s = -\infty$.

The following result is an immediate consequence of Definition 3 and Theorem 2.

Theorem 3 Let X be a complex. Then

$$\text{PGF}(X) = \inf \left\{ \sup A \mid \text{there exists a diagram of morphisms of complexes } A \leftarrow G \rightarrow X \text{ such that } G \rightarrow X \text{ is a special PGF precover of } X \text{ and } G \rightarrow A \text{ is a PGF almost isomorphism} \right\}.$$

Take $\mathcal{A} = \text{PGF}$ in Ref. [18], we get the following definition.

Definition 5 Let X be a complex. Then for any complex Y and any integer n , the n -th relative cohomology group $\text{Ext}_{\text{PGF}}^n(X, Y)$ is defined by the equality $\text{Ext}_{\text{PGF}}^n(X, Y) = H^n(\text{Hom}_R(G, Y))$ where $G \rightarrow X$ is a special PGF precover of X .

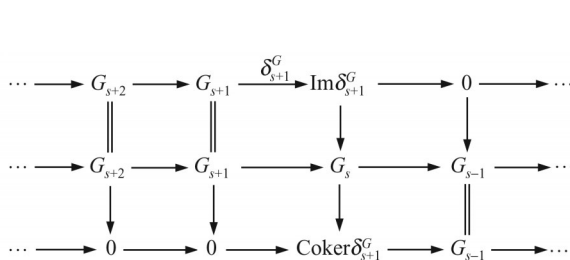


Fig. 3 The commutative diagram about G

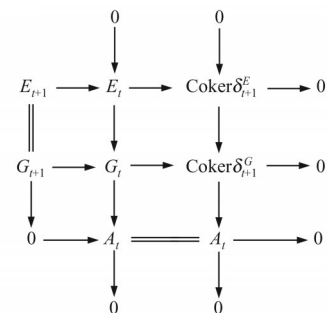


Fig. 4 The commutative diagram about E

As an immediate consequence of Ref. [18], we obtain the following result, which gives an equivalent characterization on PGF dimensions.

Corollary 1 Let X be a complex and n an integer. Then the following conditions are equivalent.

- (1) $\text{PGF}(X) \leq n$;
- (2) $\text{Ext}_{\text{PGF}}^i(X, Y) = 0$ for all $i > n - t$ and any bounded complex Y such that all Y_j and $\text{Ker}(Y_i \rightarrow Y_{i-1})$ are in PGF^\perp , where $t = \sup H(Y)$;
- (3) $\text{Ext}_{\text{PGF}}^i(X, Y) = 0$ for all $i > n$ and any module Y in PGF^\perp ;
- (4) $\sup H(X) \leq n$ and for any special PGF precover $G \rightarrow X$ of X , the module $\text{Coker}(G_{n+1} \rightarrow G_n)$ is in PGF .

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关于 PGF 复形和维数的注记

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摘要: 首先, 建立了相对于 projectively coresolved Gorenstein flat (PGF) 复形的粘合。其次, 定义并研究了 PGF 维数, 对于任意的复形 X , 将其 PGF 维数记作 $\text{PGF}(X)$ 。得到了 $\text{PGF}(X)$ 是 $\{\sup A \mid \text{存在复形的态射图 } A \leftarrow G \rightarrow X, \text{使得 } G \rightarrow X \text{ 是 } X \text{ 的特殊 PGF 预覆盖, 且 } G \rightarrow A \text{ 是 PGF 几乎同构}\}$ 的下确界。

关键词: PGF 模; PGF 复形; 粘合; PGF 维数

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