Vietnam Journal of Mathematics
https://doi.org/10.1007/s10013-026-00808-8

ORIGINAL ARTICLE

Step Systems on Graphs

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Received: 21 March 2025 / Accepted: 25 January 2026

Abstract

Step systems of connected graphs were introduced by Ladislav Nebeský as a means of describing shortest paths solely in terms of local information. A ternary relation T encodes, for each point u, the first step x towards a target v. Eight first-order axioms characterize the ternary relation T that are step systems. Here we show that one of Nebeský's eight axioms is in fact redundant. Moreover, we characterize the step systems of bipartite graphs, partial cubes, and weakly modular graphs in terms of first-order axioms. In each case we show that the corresponding sets of axioms are non-redundant.

Keywords Signpost systems · Independency of axioms · Bipartite graphs · Partial cubes · Weakly modular graphs

Mathematics Subject Classification (2010) Primary 05C12 · Secondary 05C75 · 52A01

1 Introduction

The concept of a *signpost system* on a non-empty set was introduced by Mulder and Nebeský in 2003 in [8]. The idea of a signpost system can be explained as follows. Suppose that we are in an unknown place or a city and we want to find our way from a place u to another place w in the city. At u we follow the signpost to w, by which we first arrive at v. This information is conveniently summarized in the triple (u, v, w). At v we look for the next signpost to w. Signpost systems thus describe a notion of paths in purely local manner.

Ideas very similar to signpost systems occur in distributed computing in the design of so-called routing labeling schemes in a graph, in particular, for routing along shortest paths. Given the source u and the destination w, the routing scheme encodes the destination w and a neighbor v of u on a (shortest) path from u to w. Then the message is routed to v and from the label $L(v)$ of v and the destination w, a neighbor of v is computed. Usually, the size of the labels is intended to be as small as possible, polylogarithmic at the best, [5, 15]. The main difference between signpost systems and routing labeling schemes is that in routing labeling schemes only one neighbor of the current vertex closer to the destination is required, while

This contribution is dedicated to the memory of Andreas W. M. Dress. Although Andreas never wrote about signpost systems or step systems, the topic does connect to his life-long interest in the relationship of discrete structures and metrics – an interest that PFS was infected with when meeting with Andreas as a young PostDoc at the Center for Interdisciplinary Research (ZiF) in 1991.

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Published online: 21 April 2026

in signpost systems and step systems all such neighbors are used. The similarity of signpost systems and step systems with the idea of routing labeling schemes in distributed computing, provides an additional motivation to study the notion of signpost systems and step systems on graphs.

A particularly interesting class of signpost systems is obtained by assuming that v is a neighbor of u along a shortest path from u to w in a connected graph G. Such *step systems*, introduced by Nebeský in 1997 [10], capture in particular the geodesic structure of the underlying graph G. The ternary algebras of signpost systems in general and step systems on a graph in particular were studied further in [9]. In particular, additional conditions on signpost systems that characterize special classes of graphs have been a focus of research. For instance, the step systems of trees and distance hereditary graphs are characterized respectively in [11] and [12]. Also, modular and median graphs are characterized in [8].

In this manuscript we consider undirected simple connected graphs only. In Section 2, we prove the main result, Theorem 3: Among the eight axioms characterizing the step system of a graph, given by Nebeský in [10], the symmetry axiom (E) is redundant, i.e, it follows from the other seven axioms. We then characterize the step systems of bipartite graphs in Section 3, partial cubes in Section 4 and weakly modular graphs in Section 5, respectively.

2 Signpost Systems and Their Underlying Graphs

Let V be a finite nonempty set. A *ternary system* (V, T) on V consists of a set V and a ternary relation $T \subset V^3$. A *signpost system* is a ternary system (V, T) that satisfies the following three axioms:

(A) If $(u, v, x) \in T$, then $(v, u, u) \in T$ for all $u, v, x \in V$.
(B) If $(u, v, x) \in T$, then $(v, u, x) \notin T$ for all $u, v, x \in V$.
(H) If $u \neq v$, then there exists an $x \in V$ such that $(u, x, v) \in T$ for all $u, v \in V$.

The triples in T are called *signposts*. The *underlying graph* G of T has vertex set V and edge set $E = E(G)$ such that, for all $u, v \in V$, holds

$$uv \in E(G) \iff (u, v, v) \in T.$$

For later reference we note that using (A) twice yields the equivalence $(u, v, v) \in T$ if and only if $(v, u, u) \in T$ for $u \neq v$, and (B) implies that $(u, u, v) \notin T$.

Let $G = (V, E)$ be a finite, simple, connected graph. By $V(G)$ and $E(G)$, we mean that G is a graph with vertex set V and edge set E. For any $u, v \in V$, we denote the geodesic distance between u and v by $d(u, v)$. It is the length of a shortest u, v-path or u, v-geodesic. G is called a *bipartite graph*, if $V(G)$ can be partitioned into two disjoint sets V_1 and V_2 such that $E(G)$ consists of vertices u, v with either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$. For a subgraph H of G (i.e, $V(H) \subset V(G)$ and $E(H) \subset E(G)$), we call H an *induced subgraph* of G, if for $u, v \in V(H)$, $uv \in E(G)$, then $uv \in E(H)$. H is an *isometric subgraph* of G, if the distance between any two vertices u, v in H is the same as the distance $d(u, v)$ in G. If the subgraph H is a cycle in G and if H is isometric in G, then H is called an *isometric cycle* of G. For any two vertices u, v of G, the geodesic interval or simply the *interval* denoted as $I(u, v)$ is a well known tool in metric graph theory; $I(u, v)$ is defined as $I(u, v) = \{x \in V(G) | d(u, x) + d(x, v) = d(u, v)\}$. A subset W of $V(G)$ is said to be a *convex* subset of $V(G)$, if $I(u, v) \subset W$, for every $u, v \in W$. A subgraph H induced by a convex set in G is called a *convex subgraph* of G. Clearly, every convex subgraph of G is an isometric subgraph of G, but the converse need not be true. A convex subset W of $V(G)$ is called a *half space* of G, if W and its set complement (i.e, $V(G) \setminus W$) are convex subsets of $V(G)$. For an edge uv in G, the set W_{uv} consisting of vertices closer to u than to v is defined as $W_{uv} = \{x \in V(G) | d(u,x) < d(v,x)\}$. The graphs in which all W_{uv}'s are half spaces are special graphs, which we discuss in Section 4.

Definition 1 The step system of G is the ternary system T defined as follows:

$$(u,v,w) \in T \iff d(u,v) = 1 \text{ and } d(v,w) = d(u,w) - 1.$$

We call $(u,v,w) \in T$ a *step* in G.

Note that $(u,v,w) \in T$ implies that u is necessarily distinct from v and w, but v and w may be the same vertex. At first sight, signpost and step systems seem quite similar. Of course, this should be the case because they both describe the same thing. However, a closer look reveals important differences between signpost system and step system with a given underlying graph G. When studying the properties of both systems, in particular, when one wants to characterize the betweenness relation defined by a signpost system and a step system using axioms on T, the axioms and proofs for both systems show essential differences.

Theorem 1 [10] *Let V be a finite non-empty set and $T \subseteq V^3$. Let G be the underlying graph of T and suppose that G is connected. Then T is the set of all steps in G if and only if T is a signpost system, i.e., satisfies* (A), (B)*, and* (H)*, as well as the following five properties for all $u, v, x, y \in V$:*

(C) *If $(u,v,x) \in T$ and $(x,y,v) \in T$ then $(x,y,u) \in T$;*
(D) *If $(u,v,x) \in T$ and $(x,y,v) \in T$ then $(u,v,y) \in T$;*
(E) *If $(u,v,x) \in T$ and $(u,y,v) \in T$ then $y = v$;*
(F) *If $(u,v,x) \in T$, $(v,u,y) \in T$, and $(x,y,y) \in T$ then $(x,y,u) \in T$;*
(G) *If $(u,v,x) \in T$ and $(x,y,y) \in T$ then $(x,y,u) \in T$ or $(y,x,v) \in T$ or $(u,v,y) \in T$.*

We note that [10], apparently mistakenly, states (G) as *either* of the three alternatives. However, the corresponding axiom (Y4) in [13] does not use this restriction. Setting $v = x$ and $y = u$ yields the precondition $(u,v,v) \in T$ and $(v,u,u) \in T$, and concludes "either $(v,u,u) \in T$ or $(u,v,v) \in T$ or $(u,v,u) \in T$". Moreover, the step systems of graphs clearly can satisfy two of the three alternatives simultaneously. We therefore use (G) here in the form stated as (Y4) in [13].

The following relaxation of signpost systems will be useful in the proofs below, even though it may not be of particular interest in its own right.

Definition 2 A ternary relation $T \subseteq V^3$ is a *pseudo-step system* if it satisfies (A), (B), (C), (D), (F), and (G).

Lemma 2 *Pseudo-step systems are hereditary, that is, if T is pseudo-step system on V and $W \subseteq V$, then $T_W := \{(u,v,w) \in W^3 | (u,v,w) \in T\}$ is again a pseudo-step system.*

Proof One easily checks that each of the axioms appearing in the definition of pseudo-step systems imply the existence or non-existence of triples preconditioned only on the existence of triples formed by the same vertices. Therefore, if one of the axioms is statisfied on V, it remains true on any subset $W \subseteq V$. □

We note that pseudo-step systems satisfying (E) are also hereditary. However, the assertion is not true for signpost systems, because axiom (H) can be violated by taking subsets. To see that this is true in particular also for step systems, consider a connected graph G with $|V| \geq 4$ vertices, where x_1 has degree 1 and x_2 is the unique neighbor of x_1. Let T be the step system of G. Clearly, for all $y \neq x_1$, we have $(x_1, z, y) \in T$ if and only if $z = x_2$. Now consider the restriction T' of T to the elements $W := \{x_1, x_3, \ldots, x_n\}$. Since $(x_1, z, y) \notin T'$ for all $z, y \in W$, the ternary system T' does not satisfy (H). We note that T' is pseudo-step system by virtue of Lemma 2.

We next show that axiom (E) is redundant:

Theorem 3 *Let V be a non-empty finite set and suppose $T \subseteq V^3$ satisfies axioms* (A), (B), (C), (D), (F), (G) *and* (H). *Then T satisfies* (E).

Proof The strategy of the proof is as follows: First we show that a pseudo-step system that violates (E) must contain a specific pair of triples up to renaming. In the second step we show that the presence of this pair of triples contradicts (H) for $3 \leq |V| \leq 5$. In the third step we argue that the case $|V| \geq 5$ can be reduced to the case $|V| = 5$, whence a pseudo-step system that violates (E) never satisfies (H). Both the second and third step relies on the heredity of pseudo-step systems, i.e., Lemma 2: we repeatedly use the fact that any implication of the axioms (A) through (G) derived for T' on $V' \subseteq V$ remains valid for T on V, where T' is the restriction of T to V'. In particular, we therefore can reuse the presence and absence of triples inferred for $|V| = 3$, 4, and 5 also in all larger sets of elements.

Claim 1 Supposes T satisfies (A) and (B) but violates (E). Then $|V| \geq 3$ and there are three distinct elements $u, v, x \in V$ such that $(u, v, v) \in T$ and $(u, x, v) \in T$.

Proof If T violates (E) there are triples $(u, v, x) \in T$ and $(u, y, v) \in T$ with $v \neq y$. By (A) we obtain $(v, u, u) \in T$ and $(u, v, v) \in T$. If $u = v$, then $(v, v, v) \in T$, which contradicts (B). If $u = y$, then $(u, u, v) \in T$, again contradicting (B), and if $u = x$ we have $(u, v, u) \in T$, then by (B), $(v, u, u) \notin T$, a contradiction. Thus u, v, and y are pairwise distinct. Since $(u, v, v) \in T$, $(u, y, v) \in T$ and $y \neq v$, we also get a violation of (E) as claimed. □

Now we consider a pseudo-step system that violates (E). As an immediate consequence of Claim 1 we may reorder the elements such that $(v_1, v_2, v_2) \in T$ and $(v_1, v_3, v_2) \in T$.

Claim 2 Let T be a pseudo-step system on $V = \{v_1, v_2, v_3\}$ such that $(v_1, v_2, v_2) \in T$ and $(v_1, v_3, v_2) \in T$. Then $(v_2, x, v_3) \notin T$ and $(v_3, x, v_2) \notin T$ for all $x \in \{v_1, v_2, v_3\}$; i.e., T violates (H).

Proof By (A) we have $(v_3, v_1, v_1) \in T$. Together with the assumption, (F) implies $(v_1, v_2, v_3) \in T$, and hence by (B) we have $(v_2, v_1, v_3) \notin T$. On the other hand, (A) implies $(v_2, v_1, v_1) \in T$. Together with the contrapositive of (C) this yields $(v_3, v_1, v_2) \notin T$. Now suppose $(v_2, v_3, v_3) \in T$ and thus by (A) also $(v_3, v_2, v_2) \in T$. Together with $(v_1, v_3, v_2) \in T$, (C) implies $(v_2, v_3, v_1) \in T$ and thus (B) yield $(v_3, v_2, v_1) \notin T$. However, $(v_1, v_2, v_3) \in T$ and $(v_3, v_2, v_2) \in T$ yields $(v_3, v_2, v_1) \in T$ by (C), a contradiction. Thus $(v_2, v_3, v_3) \notin T$ and $(v_3, v_2, v_2) \notin T$. Finally, $(v_3, v_3, v_2) \notin T$ and $(v_2, v_2, v_3) \notin T$. □

Claim 3 Let T be a pseudo-step system on $V = \{v_1, v_2, v_3, v_4\}$ with $(v_1, v_2, v_2) \in T$ and $(v_1, v_3, v_2) \in T$. Then $(v_3, z, v_2) \notin T$ or $(v_2, z, v_3) \notin T$ for all $z \in V$, i.e., T violates (H).

Proof We already know from Claim 2 that the statement holds for $z \in \{v_1, v_2, v_3\}$. Consider $z = v_4$ and assume, for contradiction, that $(v_2, v_4, v_3) \in T$ and $(v_3, v_4, v_2) \in T$. Then, by (A), we also have $(v_2, v_4, v_4) \in T$ and $(v_4, v_2, v_2) \in T$. By (C), $(v_1, v_3, v_2) \in T$

and $(v_2, v_4, v_3) \in T$ implies $(v_2, v_4, v_1) \in T$ and (D) yields $(v_1, v_3, v_4) \in T$. Likewise by (C), $(v_1, v_2, v_3) \in T$, and $(v_3, v_4, v_2) \in T$ implies $(v_3, v_4, v_1) \in T$. Using (C) again, $(v_1, v_3, v_4) \in T$ and $(v_4, v_3, v_3) \in T$ implies $(v_4, v_3, v_1) \in T$, and thus $(v_3, v_4, v_1) \notin T$, a contradiction. □

Claim 4 Let T be a pseudo-step system on $V = \{v_1, v_2, v_3, v_4, v_5\}$ with $(v_1, v_2, v_2) \in T$ and $(v_1, v_3, v_2) \in T$. Then, at least one of the following statements is true for all $z \in V$: $(v_3, z, v_2) \notin T$, $(v_2, z, v_3) \notin T$, $(v_4, z, v_2) \notin T$, or $(v_4, z, v_3) \notin T$. That is, T violates (H).

Proof Assume, for contradiction that (H) holds, and thus in particular, we have $(v_3, z, v_2) \in T$ and $(v_2, z', v_3) \in T$ for some $z, z' \in V$. The proofs of Claims 2 and 3 show that $z, z' \in \{v_4, v_5\}$ and only the three cases (1) $(v_3, v_5, v_2) \in T$ and $(v_2, v_5, v_3) \in T$, (2) $(v_3, v_4, v_2) \in T$ and $(v_2, v_5, v_3) \in T$, or (3) $(v_3, v_5, v_2) \in T$ and $(v_2, v_4, v_3) \in T$ are possible. These are further reduced by observing that renaming $v_4 \leftrightarrow v_5$ reduces case (1) to Claim 3, and thus $(v_3, z, v_2) \notin T$ or $(v_2, z, v_3) \notin T$.

Renaming $v_3 \leftrightarrow v_2$ reduces case (2) to case (3). Thus, we may assume w.l.o.g., that (v_3, v_4, v_2) and $(v_2, v_5, v_3) \in T$ and hence T contains the following set of triplets: $\{(v_1, v_2, v_2), (v_1, v_3, v_2), (v_2, v_1, v_1), (v_1, v_3, v_3), (v_1, v_2, v_3), (v_1, v_2, v_4), (v_3, v_1, v_1), (v_3, v_1, v_4), (v_3, v_4, v_2), (v_3, v_4, v_1), (v_3, v_4, v_4), (v_4, v_3, v_3), (v_2, v_5, v_3), (v_2, v_5, v_5), (v_5, v_2, v_2)\}$.

By (C) and (D), $(v_1, v_3, v_2) \in T$ and $(v_2, v_5, v_3) \in T$ implies $(v_2, v_5, v_1) \in T$ and $(v_1, v_3, v_5) \in T$. Thus applying (F) to $(v_5, v_2, v_2), (v_2, v_5, v_1)$ and (v_2, v_1, v_1) implies $(v_2, v_1, v_5) \in T$. (v_4, z, v_2) with all possible choices of z:

- $z = v_1$: If $(v_4, v_1, v_2) \in T$, then by (C), $(v_4, v_1, v_2) \in T$ and $(v_2, v_1, v_1) \in T$ implies $(v_2, v_1, v_4) \in T$. However, we have $(v_1, v_2, v_4) \in T$, a contradiction by (B).
- $z = v_2$: If $(v_4, v_2, v_2) \in T$, then by (A), $(v_2, v_4, v_4) \in T$ and then by (C), (v_3, v_4, v_2), $(v_2, v_4, v_4) \in T$ implies $(v_2, v_4, v_3) \in T$. Now by (D), $(v_1, v_3, v_4) \in T$. However, we have $(v_3, v_1, v_4) \in T$, a contradiction by (B).
- $z = v_3, v_4$: $(v_4, v_3, v_2) \in T$ contradicts $(v_3, v_4, v_2) \in T$ by (B); $(v_4, v_4, v_2) \in T$ is not possible.
- $z = v_5$: If $(v_4, v_5, v_2) \in T$ then (A) implies $(v_4, v_5, v_5) \in T$ and $(v_5, v_4, v_4) \in T$. Since $(v_2, v_1, v_5) \in T$, (C) yields $(v_2, v_1, v_4) \in T$. However, we have $(v_1, v_2, v_4) \in T$, a contradiction by (B).

Since every choice of z leads to a contradiction, T cannot contain a triple of the form (v_4, z, v_2). Hence T violates (H). □

Claim 5 There is no pseudo-step system on $|V| > 5$ that satisfies (H) and violates (E).

Proof By assumption, all pairs of distinct vertices $v_j, v_k \in V$ there are $z, z' \in V$ such that $(v_j, z, v_k) \in T$ and $(v_k, z', v_j) \in T$ by (H). Moreover, by Claim 1, violation of (E) implies that there is a triple $(v_i, v_k, v_j) \in T$ such that $(v_i, v_j, v_j) \in T$. If $z = z'$, we may relabel the vertices such that $v_i \leftrightarrow v_1, v_j \leftrightarrow v_2, v_k \leftrightarrow v_3$. In this case, we know from Claim 2 that $z = z' \in \{v_1, v_2, v_3\}$ yields a contradiction. Alternatively, relabeling $z = z' \leftrightarrow v_4$ yields a contradiction by Claim 3. If $z \neq z'$ we relabel such that $v_i \leftrightarrow v_1, v_j \leftrightarrow v_2, v_k \leftrightarrow v_3, z \leftrightarrow v_4$ and $z' \leftrightarrow v_5$, and thus we obtain the situation in Claim 4, for which we have already shown that it leads to a contradiction. □

Taken together, a pseudo-step system that satisfies (H) cannot violate (E) for any non-empty set V. Thus, any pseudo-step system that satisfies (H), also satisfies (E). □

We note in passing that (H) and (E) are not equivalent for pseudo-step systems, since pseudo-step systems satisfying (E) are hereditary, while step systems are not.

Now Theorem 1 can be restated in a simplified version as follows:

Corollary 4 *Let V be a finite non-empty set and $T \subseteq V^3$. Let G be the underlying graph of T and suppose that G is connected. Then T is the set of all steps in G if and only if T fulfils the axioms* (A), (B), (C), (D), (F), (G), *and* (H).

Finally, we note that a further simplification is not possible.

Proposition 5 *None of the seven axioms* (A), (B), (C), (D), (F), (G), *and* (H) *is implied by the remaining six axioms.*

Proof In Appendix A.1 we construct examples of ternary relations that satisfy six of the axioms and violate the seventh. □

3 Step System of a Bipartite Graph

In [11], the step systems of trees are characterized as the signpost systems satisfying the following two additional properties:

(i) If $u \neq v$ then there exists at most one $x \in V$ such that $(u, x, v) \in T$ for all $u, v \in V$;
(BP) $(u, v, v) \in T \Rightarrow (u, v, x)$ or $(v, u, x) \in T$, for all $u, v, x \in V$.

In the following we show that step systems satisfying (BP) are exactly the step systems of bipartite graphs.

Proposition 6 *Let $T \subseteq V^3$ be a signpost system satisfying axioms* (F) *and* (BP). *Then T satisfies Nebeský's axioms of step systems.*

Proof Let T be a signpost system satisfying (F) and (BP).

Claim (A), (B), (H), (F) and (BP) implies (G).

Proof of Claim Let $(u, v, x), (x, y, y) \in T$ and assume for contradiction $(x, y, u) \notin T$ and $(y, x, v) \notin T$ and $(u, v, y) \notin T$. By axiom (BP) we have $(y, x, u) \in T$ and $(x, y, v) \in T$ and $(v, u, y) \in T$. Now (F) implies $(v, u, x) \in T$, a contradiction, and thus T satisfies (G). □

Claim (A), (B), (H), (F) and (BP) implies (C).

Proof of Claim Let $(u, v, x), (x, y, v) \in T$ and assume for contradiction that $(x, y, u) \notin T$. Therefore $(y, x, u) \in T$ by (BP). Now $(x, y, v), (y, x, u), (v, u, u) \in T$ implies $(v, u, x) \in T$ by (F), a contradiction, since $(u, v, x) \in T$. Hence T satisfies (C) axiom. □

Claim (A), (B), (H), (C) and (BP) implies (F).

Proof of Claim Let $(u, v, x), (v, u, y), (x, y, y) \in T$ and assume, for contradiction $(x, y, u) \notin T$. Therefore $(y, x, u) \in T$ by (BP) axiom. Now $(y, x, u), (u, v, x) \in T$ implies $(u, v, y) \in T$ by (C), a contradiction, since $(v, u, y) \in T$. Hence T satisfies (F). □

Claim (A), (B), (H), (C), (F) and (BP) implies (D).

Proof of Claim Let $(u, v, x), (x, y, v) \in T$ and assume, for contradiction, that $(u, v, y) \notin T$. Since $(u, v, v) \in T$ by (A) and $(u, v, y) \notin T$ and, by (BP), we have $(v, u, y) \in T$. Now $(x, y, v), (v, u, y) \in T$ implies $(v, u, x) \in T$ by (C), a contradiction since $(u, v, x) \in T$. Hence T satisfies (D). $\square$

Taken together, therefore, T satisfies all axioms of the step system of a connected graph G. $\square$

It is well known that a graph is bipartite if and only if it contains no odd cycles. This statement can be restricted to several more restricted classes of odd cycles, e.g. to induced odd cycles. The following statement also appears to be known. We give a short proof to keep this contribution self-consistent, since we could not find a reference.

Proposition 7 *A graph G is bipartite if and only if every isometric cycle is even.*

Proof Suppose by way of contradiction that G contains an odd cycle and let C be a shortest such odd cycle. Since C is not an isometric cycle, C contains two vertices x, y such that $d(x, y)$ is smaller than the length of the two (x, y)-paths P_1, P_2 of C. We can suppose that x and y are closest such vertices of C. Let P be a shortest (x, y)-path. Then $|P| < \min\{|P1|, |P2|\}$ and $P \cap P_1 = P \cap P_2 = \{x, y\}$. Then $P \cup P_1$ and $P \cup P_2$ are cycles shorter than C. Since one of these cycles is odd, we obtain a contradiction with the choice of C. $\square$

Theorem 8 *Let $T \subseteq V^3$ be a signpost system with underlying graph G. If G is connected, then T is the step system of G and G is bipartite if and only if T satisfies axioms* (F) *and* (BP).

Proof By Proposition 6 and Theorem 1, it follows that T is the step system of G. Now assume that G is connected. Assume, for a contradiction, that G is not bipartite. Therefore, by Proposition 7 there exists an isometric odd cycle C of length, say $2k + 1$ in G. Let u, v be two adjacent vertices of C and w be the vertex of C opposite to the edge uv. Then $d(u, w) = d(v, w) = k$, thus $(u, v, w), (v, u, w) \notin T$. Since $(u, v, v) \in T$, we get a contradiction with axiom (BP). Conversely, let G be a connected bipartite graph and let T be the step system of G. Clearly, T satisfies the axiom (F). To prove that T satisfies the axiom (BP), pick any three vertices u, v, x such that $(u, v, v) \in T$. This implies that u and v are adjacent. Since G is bipartite and u adjacent to v, $d(u, x) \neq d(v, x)$, say $d(v, x) < d(u, x)$. But then $(u, v, x) \in T$, as required. $\square$

Proposition 9 *None of the five axioms* (A), (B), (F), (H) *and* (BP) *is implied by the remaining four axioms.*

Proof In Appendix A.2 we construct ternary relations that satisfy four of the five axioms and violate the fifth. $\square$

Interestingly, the signpost systems of bipartite graphs cannot be extended by including additional triples.

Definition 3 A signpost system T is *maximal*, if for any triplet $t \notin T$, $T \cup \{t\}$ is not a signpost system.

Theorem 10 *Let G be a connected graph with vertex set V and let T be the step system of G. Then G is bipartite if and only if T is a maximal signpost system on G.*

Proof Let G be a bipartite graph and T be the step system of G. Therefore T satisfies the (BP) axiom by Theorem 8. Suppose T is not maximal. Then there exist $(u, v, w) \in V^3 \setminus T$ such that $T \cup \{(u, v, w)\}$ is a signpost on V. Therefore $(v, u, w) \notin T$, since otherwise axiom (B) would be violated. Now both (u, v, w) and $(v, u, w) \notin T$. This contradicts the fact that T satisfies the (BP) axiom and hence T is maximal.

Conversely, let T be a maximal signpost system with underlying graph G and assume, for contradiction, that G is not bipartite, and thus there is an odd isometric cycle $uvv_1v_2 \dots v_{2n}w$ $u_{2n}, u_{2n-1} \dots u_1u$ in G, and hence $(u, v, w) \notin T$ and $(v, u, w) \notin T$. Therefore $T \cup \{(u, v, w)\}$ is a signpost on V, a contradicting the maximality of T. □

Similarly, we may ask which signpost systems are minimal in the same sense.

Proposition 11 *Let T be a signpost system such that the underlying graph is a tree. Then, for any $(u, v, w) \in T$, the relation $T \setminus \{(u, v, w)\}$ is not a signpost system.*

Proof Let $T \subset V^3$ be the step system of a tree G. Then for $u \neq v$, there exists at most one vertex $x \in V$ such that $(u, x, v) \in T$. On the other hand, by (H), there is at least one $x \in V$ such that $(u, x, v) \in T$. Taken together, there is exactly one such $x \in V$ for every $u \neq v$. Moreover $(u, x, u) \notin T$. Hence removal of any triplet from T leads to a violation of (H). □

4 Step Systems of Partial Cubes

Partial cubes were introduced by Graham and Pollak in [6] as isometric subgraphs of hypercubes. Djoković in [4] characterized partial cubes in terms of convexity of half spaces. A graph G is a partial cube if and only if G is bipartite and for any edge uv, the sets W_{uv} and W_{vu} are convex and hence half spaces [4].

This characterization can be translated to the language of step systems. To this end we make use of the following notation [7] for "paths" in a signpost system T: we write $(u, x_0, x_1, \dots, x_n, v) \in T$ if $(u, x_0, v) \in T$, $(x_0, x_1, v) \in T$, ..., $(x_{n-1}, x_n, v) \in T$. Thus, $(u, x_0, x_1, \dots, x_n, v) \in T$ implies that $u, x_0, x_1, \dots, x_n, v$ is a u, v-shortest path in the underlying graph G. Therefore, if T is the step system of G, then W is a convex subset in $V(G)$ if and only if $x_0, x_1, \dots, x_n \in W$ for all paths $(u, x_0, x_1, \dots, x_n, v) \in T$. Moreover, the half spaces of a graph G can be expressed in terms of the step system on T as $W_{uv} = \{x \in V : (v, u, x) \in T\}$.

Mulder and Nebesky in [7] introduced the following smoothness axiom as a sufficient condition for connectedness of the underlying graph of signpost system T:

(S) $(u, v, x) \in T$, $(u, v, y) \in T$ and $(x, z, y) \in T$ implies $(u, v, z) \in T$.

As shown in [7], (S) implies (D).

Lemma 12 *Let T be a signpost system satisfying the axioms (S) and (BP). Then T satisfies axiom (F).*

Proof Suppose $(u, v, x) \in T$, $(v, u, y) \in T$, and $(x, y, y) \in T$. Assume, for contradiction, that $(x, y, u) \notin T$. Then $(y, x, u) \in T$ by axiom (BP). Now we have $(v, u, y) \in T$ and $(y, x, u) \in T$. Since (S) implies (D), we obtain $(v, u, x) \in T$, which by (B) contradicts $(u, v, x) \in T$. □

Lemma 13 *Let $T \subseteq V^3$ be a signpost system with a connected underlying graph G satisfying axioms* (F) *and* (BP). *Then T satisfies axiom* (S) *if and only if, for any edge uv in G, the sets W_{uv} and W_{vu} are half spaces of G.*

Proof By Theorem 8, T is the step system of G and G is bipartite. First, suppose W_{uv} is convex for every edge uv in G. Consider triplets $(u, v, w) \in T$, $(u, v, y) \in T$, and $(w, x, y) \in T$. We have $w, y \in W_{vu}$. Moreover, $x \in I(w, y)$ and convexity of W_{vu} implies $x \in W_{vu}$ and hence $(u, v, x) \in T$. Hence T satisfies (S). Conversely, if W_{vu} is not convex, then W_{vu} contains two non-adjacent vertices x and y, such that $I(x, y) \cap W_{vu} = \{x, y\}$. Hence there is a neighbor z of x in $I(x, y)$ such that $z \notin W_{vu}$. Thus $(u, v, z) \notin T$. However, $x, y \in W_{vu}$ implies $(u, v, x) \in T$, and $(u, v, y) \in T$. This contradicts (S). □

Theorem 14 *Let $T \subseteq V^3$ be a signpost system with underlying graph G. Then T is the step system of G and G is a partial cube if and only if T satisfies axioms* (S) *and* (BP).

Proof If G is a partial cube, then it is connected. Conversely the underlying graph of signpost system satisfying (S) is connected [14]. Hence, we may assume that G is connected and T is a signpost system with a connected underlying graph. The statement now follows from Theorem 8, Lemma 12 and Lemma 13. □

Proposition 15 *None of the five axioms* (A), (B), (H), (BP) *and* (S) *is implied by the remaining four axioms.*

Proof In Appendix A.3 we construct ternary relations that satisfy four of the five axioms and violate the fifth. □

5 Step System of Weakly Modular Graphs

Weakly modular graphs have been introduced by Chepoi [3], see also [1]. A local-to-global theory of weakly modular graphs and their subclasses are developed by Chalopin *et al.* [2]. A graph is *weakly modular* if it satisfies the following two conditions:

Triangle condition For any three vertices u, v, w with $1 = d(v, w) < d(v, u) = d(u, w)$, there exists a common neighbor z of v and w such that $d(u, z) = d(u, v) - 1$.

Quadrangle condition For any four vertices u, v, w, y with $d(v, y) = d(w, y) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, y) - 1$, there exists a common neighbor z of v and w such that $d(u, z) = d(u, v) - 1$.

These conditions are readily translated into corresponding axioms for signpost systems:

(TC) For any three vertices $u, v, x \in V$, $(u, v, x) \notin T$, $(v, u, x) \notin T$ and $(u, v, v) \in T$ implies that there exists $z \in V$ such that $(u, z, x) \in T$ and $(v, z, x) \in T$.

(QC) For any four vertices $u, v, x, y \in V$ with $(u, x, v) \in T$, and $(u, y, v) \in T$, and $(x, y, y) \notin T$ there exists z such that $(x, z, v) \in T$ and $(y, z, v) \in T$.

Theorem 16 *Let $T \subset V^3$ be a step system of a graph G satisfying axioms* (TC) *and* (QC). *Then G is a weakly modular graph.*

Proof Let u, v and x be three vertices of G with $uv \in E(G)$ and vertex x equidistant from u and v. Since T is the step system of G we have $(u, v, x) \notin T$ and $(v, u, x) \notin T$. But $(u, v, v) \in T$. By axiom (TC) there exists $z \in V$ such that $(u, z, x) \in T$ and $(v, z, x) \in T$. Now (A) implies $(u, z, z) \in T$ and $(v, z, z) \in T$. Thus uzv forms a triangle.

Again, let $u, v, x, y \in V(G)$ with $d(v, x) + 1 = d(v, y) + 1 = d(u, v)$ and ux and uy are edges and xy is not an edge. Then $(u, x, x) \in T$ and $(u, y, y) \in T$ and $(x, y, y) \notin T$. Since

$d(v, x) = d(u, v) - 1$ and ux is an edge, we see that $ux \dots v$ is a shortest u, v-path. This implies $(u, x, v) \in T$. Similarly, we obtain $(u, y, v) \in T$. By (QC) axiom, therefore, there exists $z \in V$ such that $(x, z, v) \in T$ and $(y, z, v) \in T$, and thus $uxzy$ is a quadrangle. Thus G satisfies both the triangle and the quadrangle condition, and hence G is weakly modular. □

The following corollary follows from Corollary 4.

Corollary 17 *Let V be a finite non-empty set and $T \subseteq V^3$ with underlying graph G. Then G is a weakly modular graph and T is the step system of G if and only if T fulfils the axioms* (A)*,* (B)*,* (C)*,* (D)*,* (F)*,* (G)*,* (H)*,* (TC)*, and* (QC).

Finally, we show that axioms in Corollary 17 are independent of each other.

Proposition 18 *None of the nine axioms* (A)*,* (B)*,* (C)*,* (D)*,* (F)*,* (G)*,* (H)*,* (TC)*, and* (QC) *is implied by the remaining eight axioms.*

Proof In Appendix A.4 we construct ternary relations that satisfy eight of the nine axioms and violate the ninth. □

A Independence of Axioms

The examples used to show that certain sets of axioms are independent of each other follow a common construction principle. For each of the axioms (α), we construct a graph H_α and a corresponding ternary system T_α on its vertex set such that T_α satisfies all relevant axioms except (α) and T_α is not the step system of the graph H_{α_i}. The graphs H_α are taken from the collection of small graphs shown in Fig. 1.

A.1 Proof of Proposition 5

We construct examples of ternary relations that satisfy six of the seven axioms (A), (B), (C), (D), (F), (G), and (H) while violating the seventh.

Concerning the independence of (A) Let $V = \{v_1, v_2, v_3\}$. Let $T_A \subset V^3$ be defined as $T_A = \{(v_1, v_2, v_3), (v_2, v_3, v_1), (v_3, v_1, v_2), (v_1, v_2, v_2), (v_2, v_3, v_3), (v_3, v_1, v_1), (v_2, v_3, v_2), (v_3, v_1, v_3), (v_1, v_2, v_1)\}$. The underlying graph H_A of T_A is a K_3 with vertices v_1, v_2, v_3, see Fig. 1(c). Now T_A satisfies axioms (B), (C), (D), (F), (G) and (H). But T_A does not satisfy axiom (A), since $(v_1, v_2, v_2) \in T_A$ but $(v_2, v_1, v_1) \notin T_A$. Clearly T_A is not the step system of the underlying graph $H_A = K_3$ of T_A.

Concerning the independence of (B) Let $V = \{v_1, v_2\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_2, v_1, v_1)\}$. The underlying graph of T is a P_2, see Fig. 1(a) and T is the step system of P_2. Let $T_B = T \cup \{(v_2, v_1, v_2), (v_1, v_2, v_1)\}$. Clearly T_B satisfies axioms (A), (C), (D), (F), (G), and (H). But T_B does not satisfy axiom (B), since $(v_2, v_1, v_2) \in T_B$ and $(v_1, v_2, v_2) \in T_B$. The underlying graph H_B of T_B is still P_2, but T_B is not the step system of P_2.

Concerning the independence of (C) Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_2, v_3), (v_1, v_2, v_4), (v_1, v_2, v_5), (v_1, v_6, v_6), (v_1, v_6, v_5),$

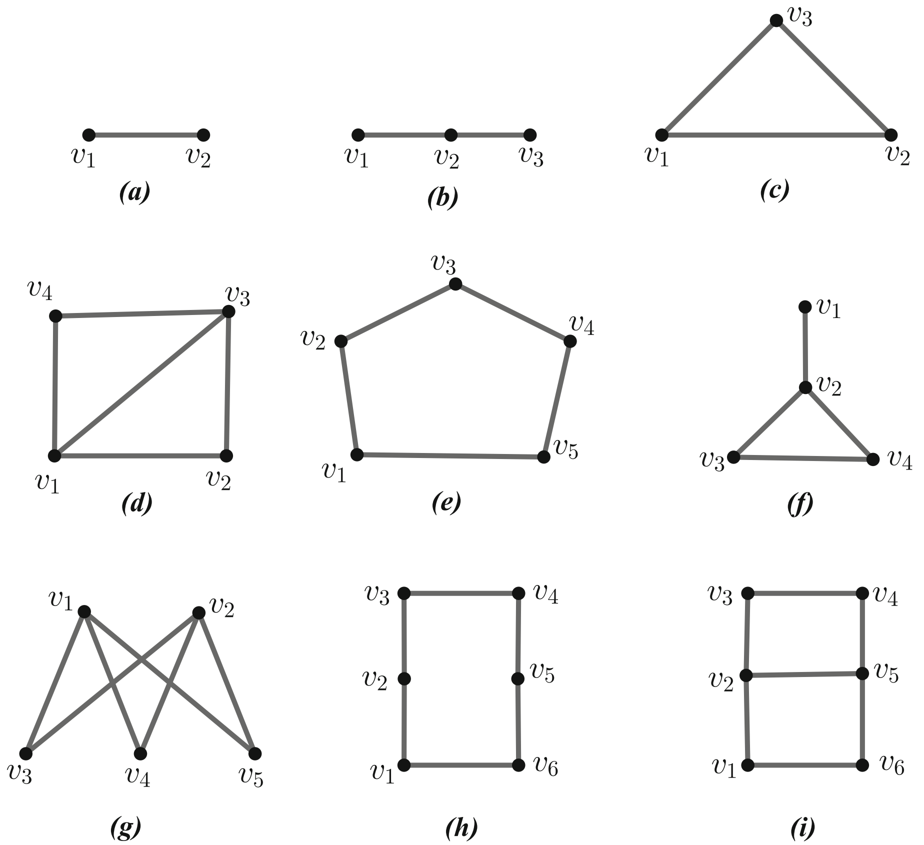

Fig. 1 Graphs appearing in the proofs of Propositions 5, 9, 15, 18: (a) P_2, (b) P_3, (c) K_3, (d) $K_4 - e$, (e) C_5, (f) paw graph, (g) $K_{2,3}$, (h) C_6, (i) domino graph

(v_1, v_6, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_5, v_5), (v_2, v_1, v_6), (v_2, v_3, v_4), (v_2, v_5, v_4), (v_2, v_5, v_6), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_2, v_5), (v_3, v_2, v_6), (v_3, v_4, v_5), (v_3, v_4, v_6), (v_4, v_3, v_3), (v_4, v_5, v_5), (v_4, v_3, v_1), (v_4, v_3, v_2), (v_4, v_5, v_1), (v_4, v_5, v_2), (v_4, v_5, v_6), (v_5, v_2, v_2), (v_5, v_4, v_4), (v_5, v_6, v_6), (v_5, v_2, v_1), (v_5, v_2, v_3), (v_5, v_4, v_3), (v_5, v_6, v_1), (v_6, v_1, v_1), (v_6, v_5, v_5), (v_6, v_1, v_2), (v_6, v_1, v_3), (v_6, v_5, v_2), (v_6, v_5, v_3), $(v_6, v_5, v_4)\}$. The underlying graph of T is a domino as shown in Fig. 1(i), and T is the step system of the domino. Let $T_C = T \setminus \{(v_1, v_2, v_4)\}$. Clearly, T_C satisfies axioms (A), (B), (D), (F), (G), and (H). But T_C does not satisfy axiom (C), since $(v_4, v_3, v_1), (v_1, v_2, v_3) \in T_C$, but $(v_1, v_2, v_4) \notin T_C$. The underlying graph H_C of T_C is still a domino, but T_C is not the step system of the domino.

Concerning the independence of (D) Let $V = \{v_1, v_2, v_3, v_4\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_2, v_3), (v_1, v_2, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_4, v_4), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_4, v_2, v_2), (v_4, v_3, v_3), (v_4, v_2, v_1)\}$. The underlying graph of T is a paw graph, see Fig. 1(f) and T is the step system of the paw. The ternary relation $T_D = T \cup \{(v_3, v_4, v_1)\}$ satisfies axioms (A), (B), (C), (F), (G), and (H). But T_D does not satisfy axiom (D), because $(v_3, v_4, v_1) \in T_D$ and $(v_1, v_2, v_4) \in T_D$, but $(v_3, v_4, v_2) \notin T_D$. The underlying graph H_D of T_D is still a paw graph, but T_D is not the step system of the paw.

Concerning the independence of (F) Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_5, v_5), (v_1, v_2, v_3), (v_1, v_5, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_1, v_5), (v_2, v_3, v_4), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_4, v_5), (v_4, v_3, v_3), (v_4, v_5, v_5), (v_4, v_3, v_2), (v_4, v_5, v_1), (v_5, v_1, v_1), (v_5, v_4, v_4), (v_5, v_1, v_2), (v_5, v_4, v_3)\}$. Clearly, the underlying graph of T is a C_5 with vertices as shown in Fig. 1(e) and T is the step system of C_5. Let $T_F = T \cup \{(v_1, v_2, v_4), (v_4, v_3, v_1)\}$. Clearly T_F satisfies axioms (A), (B), (C), (D), (G) and (H). But T_F does not satisfy axiom (F), since $(v_2, v_1, v_5), (v_1, v_2, v_4), (v_5, v_4, v_4) \in T_F$ but $(v_5, v_4, v_2) \notin T_F$. The underlying graph H_F of T_F is still a C_5 as shown in Fig. 1(e), but T_F is not the step system of C_5.

Concerning the independence of (G) Let $V = \{v_1, v_2, v_3, v_4\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_2, v_3), (v_1, v_4, v_4), (v_1, v_4, v_3), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_4, v_4), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_4, v_1), (v_4, v_1, v_1), (v_4, v_2, v_2), (v_4, v_3, v_3)\}$. Clearly, the underlying graph of T is a $K_4 - x$, see Fig. 1(d) and T is the step system of $K_4 - x$. Let $T_G = T - \{(v_1, v_4, v_3), (v_3, v_4, v_1)\}$. Clearly T_G satisfies axioms (A), (B), (C), (D), (F), and (H). But T_G does not satisfy axiom (G), since $(v_3, v_2, v_1), (v_1, v_4, v_4) \in T_G$, but $(v_1, v_4, v_3) \notin T_G$, $(v_4, v_1, v_2) \notin T_G$ and $(v_3, v_2, v_4) \notin T_G$. The underlying graph H_G of T_G is still a $K_4 - x$ with vertices as in Fig. 1(d), but T_G is not the step system of $K_4 - x$.

Concerning the independence of (H) Let $V = \{v_1, v_2, v_3\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_2, v_3), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_3, v_2, v_2), (v_3, v_2, v_1)$. The underlying graph of T is a P_3 (see Fig. 1(b)), and T is the step system of P_3. Let $T_H = T \setminus \{(v_1, v_2, v_3), (v_3, v_2, v_1)\}$. Clearly T_H satisfies axioms (A), (B), (C), (D), (F) and (G). But T_H does not satisfy axiom (H), since for $v_1 \neq v_3$ there exists no z such that $(v_1, z, v_3) \in T_H$. The underlying graph H_H of T_H is still a P_3 with vertices as in Fig. 1(b), but T_H is not the step system of P_3.

A.2 Proof of Proposition 9

For proving the independence of seven of Nebeský's eight axioms in Appendix A.1 we used K_3 to prove the independence of (A), and P_2 to prove the independence of (B). Here also we can use the same examples to prove the independence of (A) and (B).

Concerning the independence of (H) Let $V = \{v_1, v_2, v_3\}$ and $T_H \subset V^3$ be defined as $T_H = \{(v_1, v_2, v_2), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_3, v_2, v_2), (v_2, v_1, v_3), (v_2, v_3, v_1)\}$. The underlying graph of T_H is a P_3, see Fig. 1(b). T_H satisfies axioms (A), (B), (F) and (BP). However, T_H violates (H) since for $v_1 \neq v_3$ there exists no z such that $(v_1, z, v_3) \in T_H$. Here the underlying graph P_3 of T_H is a bipartite but T_H is not the step system of P_3.

Concerning the independence of (F) Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_5, v_5), (v_1, v_2, v_3), (v_1, v_5, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_1, v_5), (v_2, v_3, v_4), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_4, v_5), (v_4, v_3, v_3), (v_4, v_5, v_5), (v_4, v_3, v_2), (v_4, v_5, v_1), (v_5, v_1, v_1), (v_5, v_4, v_4), (v_5, v_1, v_2), (v_5, v_4, v_3)\}$. Clearly the underlying graph of T is a C_5 with vertices as shown in Fig. 1(e) and T is the step system of C_5. Now consider $T_F = T \cup \{(v_1, v_2, v_4), (v_4, v_3, v_1), (v_1, v_5, v_3), (v_2, v_3, v_5), (v_4, v_5, v_2)\}$. Now T_F satisfies axioms (A), (B), (H) and (BP), but not axiom (F); since $(v_2, v_1, v_5), (v_1, v_2, v_4), (v_5, v_4, v_4) \in T_F$ but $(v_5, v_4, v_2) \notin T_F$. The underlying graph of T_F is still a C_5 but T_F is not the step system of the C_5 with vertices as shown in Fig. 1 (e) and C_5 is not bipartite.

Concerning the independence of (BP) Let $V = \{v_1, v_2, v_3\}$ and $T_{BP} \subset V^3$ be defined as $T_{BP} = \{(v_1, v_2, v_2), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_3, v_2, v_2), (v_1, v_3, v_3), (v_3, v_2, v_1)\}$. Clearly

T_{BP} satisfies axioms (A), (B), (F), and (H). But T_{BP} does not satisfy axiom (BP), since $(v_1, v_2, v_2) \in T$, but (v_1, v_2, v_3) and $(v_2, v_1, v_3) \notin T$. The underlying graph of T_{BP} is a C_3 as shown in Fig. 1(c). But C_3 is not a bipartite graph and, moreover, T_{BP} is not the step system of C_3.

A.3 Proof of Proposition 15

For the independence of axioms (A), (B), (H) and (BP), the same examples as in Appendix A.2 suffice since their step systems satisfy the smoothness axiom (S). In order to show that (S) is not implied by (A), (B), (H) and (BP), consider the ternary relation T_S on $V = \{v_1, v_2, v_3, v_4, v_5\}$ defined by $T_S = \{(v_1, v_3, v_3), (v_1, v_4, v_4), (v_1, v_5, v_5), (v_1, v_3, v_2), (v_1, v_4, v_2), (v_1, v_5, v_2), (v_2, v_3, v_3), (v_2, v_4, v_4), (v_2, v_5, v_5), (v_2, v_3, v_1), (v_2, v_4, v_1), (v_2, v_5, v_1), (v_3, v_1, v_1), (v_3, v_2, v_2), (v_3, v_1, v_4), (v_3, v_1, v_5), (v_3, v_2, v_4), (v_3, v_2, v_5), (v_4, v_1, v_1), (v_4, v_2, v_2), (v_4, v_1, v_3), (v_4, v_1, v_5), (v_4, v_2, v_3), (v_4, v_2, v_5), (v_5, v_1, v_1), (v_5, v_2, v_2), (v_5, v_1, v_3), (v_5, v_1, v_4), (v_5, v_2, v_3), (v_5, v_2, v_4)\}$. The underlying graph of T_S is a $K_{2,3}$ and T_S is the step system of $K_{2,3}$. One easily checks that T_S satisfies (A), (B), (H) and (BP). However, $(v_3, v_1, v_4) \in T_S$, but violates (S), since $(v_3, v_1, v_5) \in T_S$, $(v_4, v_2, v_5) \in T_S$, and $(v_3, v_1, v_2) \notin T_S$.

A.4 Proof of Proposition 18

Concerning the independence of (A) Let $V = \{v_1, v_2, v_3\}$. Consider T_A as in the proof of the corresponding claim in Appendix A.1. Recall that the underlying graph is a C_3, see Fig. 1(c). For any edge the remaining vertex is equidistant from the end vertices of the edge, and hence takes the role of z in axiom (TC). Moreover, since C_3 has only three vertices, axiom (QC) is satisfied trivially. Thus T_A satisfies all the axioms for a weakly modular graph except (A).

Concerning the independence of (B) Let $V = \{v_1, v_2\}$ and consider $T \subset V^3$ and T_B as in the proof of the corresponding claim in Appendix A.1. Recall that the underlying graph of T_B is a P_2, see Fig. 1(a), and T_B is the step system of P_2. It satisfies the seven axioms of a step system with the exception of (B). Since V contains only two vertices, both (TC) and (QC) axioms are satisfied trivially.

Concerning the independence of (C) Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and consider $T \subset V^3$ and T_C as in the proof of the corresponding claim in Appendix A.1. The underlying graph of T_C is the domino graph, Fig. 1(i), and T_C is the step system of the domino. Clearly T_C, violates axiom (C) and satisfies all the seven axioms of a step system, also T_C satisfies axioms (TC) and (QC).

Concerning the independence of (D) The example used for the corresponding statement in Appendix A.1 also pertains to the case of weakly modular graphs since T_D also satisfies axioms (TC) and (QC).

Concerning the independence of (F) Let $V = \{v_1, v_2, v_3\}$. Let $T \subset V^3$ be defined as $T = \{(v_1, v_2, v_2), (v_1, v_3, v_3), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_3, v_1, v_1), (v_3, v_2, v_2)\}$. Consider T_F defined as $T_F = T \cup \{(v_2, v_1, v_3), (v_3, v_1, v_2)\}$. The underlying graph of T_F is a K_3 with vertices v_1, v_2, v_3, see Fig. 1(c). The modified relation T_F satisfies all seven axioms for a step system with the exception of (F), since $(v_1, v_2, v_2), (v_2, v_1, v_3), (v_2, v_3, v_3) \in T_F$, but $(v_2, v_3, v_1) \notin T_F$. Moreover, T_F satisfies (TC) and (QC) trivially.

Concerning the independence of (G) Let $V = \{v_1, v_2, v_3, v_4\}$ and consider $T \subset V^3$ and T_G as in the proof of the corresponding claim in Proposition 5. The underlying graph of T_G is a $K_4 - x$, see Fig. 1(d), but T_G is not the step system of $K_4 - x$. The relation T_G clearly satisfies (TC) and (QC) trivially, but T_G does not satisfy axiom (G).

Concerning the independence of (H) Let $V = \{v_1, v_2, v_3\}$ and T_H be defined as $T_H = \{(v_1, v_2, v_2), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_3, v_2, v_2)\}$. The underlying graph of T_H is a P_3, see Fig. 1(b). T_H satisfies all axioms of a step system, except the axiom (H) because there exists no z such that $(v_1, z, v_3) \in T$. Since there exists no vertex equidistant from the end vertices of any of the two edges in the P_3 graph, axiom (TC) is satisfied trivially. Since V contains only three vertices (QC) axiom is also satisfied trivially. Thus T_H satisfies all the axioms of the step system of a weakly modular graph except axiom (H). However, T_H is not the step system of the P_3 graph.

Concerning the independence of (TC) Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and T_{TC} be defined as $T_{TC} = \{(v_1, v_2, v_2), (v_1, v_5, v_5), (v_1, v_2, v_3), (v_1, v_5, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_1, v_5), (v_2, v_3, v_4), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_4, v_5), (v_4, v_3, v_3), (v_4, v_5, v_5), (v_4, v_3, v_2), (v_4, v_5, v_1), (v_5, v_1, v_1), (v_5, v_4, v_4), (v_5, v_1, v_2), (v_5, v_4, v_3)\}$. The underlying graph of T_{TC} is a C_5 with vertices as shown in Fig. 1(e) and T_{TC} is the step system of the C_5. Therefore, T_{TC} satisfies all the axioms of a step system. T_{TC} satisfies the axiom (QC) trivially, since there exists no $(u, x, v), (u, y, v) \in T_{TC}$ for any $u, v \in V$. But the (TC) axiom fails here, since for $(v_1, v_5, v_5) \in T_{TC}$, $(v_1, v_5, v_3), (v_5, v_1, v_3) \notin T_{TC}$ there exists no z such that $(v_1, z, v_3), (v_5, z, v_3) \in T_{TC}$. So C_5 is not weakly modular.

Concerning the independence of (QC) Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and Consider $T \subset V^3$ defined by $T_{QC} = \{(v_1, v_2, v_2), (v_1, v_6, v_6), (v_1, v_2, v_3), (v_1, v_2, v_4), (v_1, v_6, v_5), (v_1, v_6, v_4), (v_2, v_1, v_1), (v_2, v_3, v_3), (v_2, v_3, v_5), (v_2, v_3, v_4), (v_2, v_1, v_6), (v_2, v_1, v_5), (v_3, v_2, v_2), (v_3, v_4, v_4), (v_3, v_2, v_1), (v_3, v_2, v_6), (v_3, v_4, v_5), (v_3, v_4, v_6), (v_4, v_3, v_3), (v_4, v_5, v_5), (v_4, v_3, v_2), (v_4, v_3, v_1), (v_4, v_5, v_6), (v_4, v_5, v_1), (v_5, v_6, v_6), (v_5, v_4, v_4), (v_5, v_4, v_2), (v_5, v_4, v_3), (v_5, v_6, v_1), (v_5, v_6, v_2), (v_6, v_1, v_1), (v_6, v_5, v_5), (v_6, v_1, v_2), (v_6, v_1, v_3), (v_6, v_5, v_4), (v_6, v_5, v_3)\}$. The underlying graph of T_{QC} is a C_6 with vertices as shown in Fig. 1(h). Here T_{QC} satisfies all the axioms of the step system of a weakly modular graph except axiom (QC), since $(v_1, v_6, v_4) \in T_{QC}$, $(v_1, v_2, v_4) \in T_{QC}$ and $(v_6, v_2, v_2) \notin T_{QC}$, but there does not exist an $z \in V$ such that $(v_6, z, v_4) \in T_{QC}$ and $(v_2, z, v_4) \in T_{QC}$ and C_6 is not weakly modular. It follows that axiom (QC) is independent.

Acknowledgements We thank an anonymous review for very helpful comments.

Author Contributions MC and PFS designed the study. All authors contributed the mathematical results and the drafting of the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL. This work was supported in part by the DST, Govt. of India (Grant No. DST/INT/DAAD/P-03/2023), the DAAD, Germany (Grant No. 57683501), PFS acknowledges the financial support by the Federal Ministry of Research, Technology and Space of Germany (BMFTR) through DAAD project 57616814 (SECAI, School of Embedded Composite AI), and jointly with the Sächsische Staatsministerium für Wissenschaft, Kultur und Tourismus in the programme Center of Excellence for AI-research *Center for Scalable Data Analytics and Artificial Intelligence Dresden/Leipzig*, project identification number: SCADS24B.

Availability of data and materials There are no data associated with this work.

Declarations

Competing interests The authors declare that they have no competing interests, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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