FACTORIZATION IN HAAR SYSTEM HARDY SPACES

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ABSTRACT. A Haar system Hardy space is the completion of the linear span of the Haar system $(h_I)_I$, either under a rearrangement-invariant norm $\|\cdot\|$ or under the associated square function norm

$$\Big\|\sum_I a_I h_I\Big\|_* = \Big\|\Big(\sum_I a_I^2 h_I^2\Big)^{1/2}\Big\|.$$

Apart from L^p, $1 \leq p < \infty$, the class of these spaces includes all separable rearrangement-invariant function spaces on $[0,1]$ and also the dyadic Hardy space H^1. Using a unified and systematic approach, we prove that a Haar system Hardy space Y with $Y \neq C(\Delta)$ ($C(\Delta)$ denotes the continuous functions on the Cantor set) has the following properties, which are closely related to the primariness of Y: For every bounded linear operator T on Y, the identity I_Y factors either through T or through $I_Y - T$, and if T has large diagonal with respect to the Haar system, then the identity factors through T. In particular, we obtain that

$$\mathcal{M}_Y = \{T \in \mathcal{B}(Y) : I_Y \neq ATB \text{ for all } A, B \in \mathcal{B}(Y)\}$$

is the unique maximal ideal of the algebra $\mathcal{B}(Y)$ of bounded linear operators on Y. Moreover, we prove similar factorization results for the spaces $\ell^p(Y)$, $1 \leq p \leq \infty$, and use them to show that they are primary.

CONTENTS

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1. Introduction

In 1970, J. Lindenstrauss gave a talk on decompositions of Banach spaces (see [31] for the abstract), which spurred a research program consisting of the following two main research directions:

- ▷ Constructing infinite-dimensional Banach spaces E which are indecomposable, i.e., for any decomposition of E into two complemented subspaces, one of them is finite-dimensional.
- ▷ Identifying the Banach spaces E which are primary, i.e., whenever, $E = F \oplus G$, then either F or G is isomorphic to E.

The foundations for creating indecomposable Banach spaces were laid by the works of B. S. Tsirelson [8] (see also [11, 7]) and Th. Schlumprecht [48]. But it was not until 1993 that the first indecomposable Banach space X_{GM} was constructed in the seminal work [16] by W. T. Gowers and B. Maurey—in fact, they even showed that X_{GM} is hereditarily indecomposable. This fruitful line of research led to solutions of long-standing problems, including the unconditional basic sequence problem and Banach's hyperplane problem [16] (see also [15, 36]) as well as the scalar-plus-compact problem solved by S. A. Argyros and R. G. Haydon [3].

The study of primary Banach spaces, on the other hand, goes back to A. Pełczyński who proved in his 1960 work [46] that the classical sequence spaces c_0 and ℓ^p, $1 \leq p < \infty$, are prime, i.e., every infinite-dimensional complemented subspace is isomorphic to the whole space (clearly, prime spaces are primary).

We will now give a terse historical overview of developments most relevant to the present work. The Lebesgue spaces L^p, $1 \leq p < \infty$, were shown to be primary in 1975 by P. Enflo and B. Maurey [34] (the proof for L^p, $1 < p < \infty$, is due to P. Enflo, and it was extended to L^1 by B. Maurey, see [1]). Subsequently, D. Alspach, P. Enflo and E. Odell [1] gave an alternative proof for the primariness of L^p, $1 < p < \infty$, and extended this result to separable rearrangement-invariant (r.i.) function spaces on $[0, 1]$ with non-trivial Boyd indices (see [32] or [17] for full proofs). In particular, this constitutes an alternative proof for L^p, $1 < p < \infty$. Shortly thereafter, P. Enflo and T. W. Starbird [10] obtained the primariness of L^1 via E-operators. Later, the dyadic Hardy space H^1 and its dual BMO were proved to be primary by P. F. X. Müller (see [40] and [39], respectively). M. Capon proved the primariness of the two-parameter spaces $L^p(L^q)$, $1 < p, q < \infty$ and $L^p(E)$, where E denotes a Banach space with a symmetric Schauder basis (see [5] and [6]). Capon's and Alspach-Enflo-Odell's methods were then successfully adapted by P. F. X. Müller in [41] to show that the two-parameter dyadic Hardy space $H^1(\delta^2)$ is primary. The primariness of its dual $\mathrm{BMO}(\delta^2)$ was later obtained in [30] by utilizing Bourgain's localization method.

The method of P. Enflo and B. Maurey [34] was to construct a pointwise multiplier $M_g : L^p \to L^p$, $f \mapsto g \cdot f$, which approximates a given operator $T : L^p \to L^p$ on a large subspace of L^p:

$$\mathbb{E}^{\mathcal{B}}(\chi_B \cdot (Tf)) \approx M_g f, \qquad f \in L^p(B, \mathcal{B}),$$

where $\mathcal{B}$ is a carefully constructed σ-algebra and $|B| \geq 1/2$. In a second step, they then stabilize the function g on another large subspace:

$$\mathbb{E}^{\mathcal{C}}(\chi_C \cdot (Tf)) \approx c \cdot f, \qquad f \in L^p(C, \mathcal{C}),$$

where $\mathcal{C} \subset \mathcal{B}$ and $C \subset B$, $|C| > 0$.

A variant of the approach taken by D. Alspach, P. Enflo and E. Odell [1] for L^p, $1 < p < \infty$, is to construct a block basis $(\tilde{h}_I)_I$ of the Haar system $(h_I)_I$ which almost diagonalizes a given operator $T : L^p \to L^p$. More precisely, $(\tilde{h}_I)_I$ is equivalent to $(h_I)_I$

and spans a subspace of L^p which is complemented by a projection P, and we have

$$PT\tilde{h}_I \approx d_I\tilde{h}_I, \qquad I \in \mathcal{D},$$

where $\mathcal{D}$ denotes the set of dyadic intervals and $(d_I)_I$ is a suitable family of scalars. Subsequently, either the *Haar multiplier* D given by $Dh_I = d_Ih_I$ or the operator $I_{L^p} - D$ can, by virtue of the unconditionality of the Haar system in L^p, $1 < p < \infty$, be inverted on a large subspace of L^p. Finally, primariness follows from Pełczyński's decomposition method [46] (see also [54, II.B.24]).

The approach developed by D. Alspach, P. Enflo and E. Odell [1] turned out to be the more flexible one, and it was further refined by P. F. X. Müller [37, 38, 39, 40] (see also [42, 43]) to be suitable for the dyadic Hardy space H^1 and its dual BMO. However, since the Haar system fails to be unconditional in L^1, the method described above cannot be applied to L^1 in the same manner. Only recently, the first named author, P. Motakis, P. F. X. Müller and Th. Schlumprecht [27] successfully extended this technique to the space L^1: First, they introduced *strategically reproducible bases* as part of a general framework which allows one to reduce factorization problems for general bounded linear operators to the case of *diagonal operators* (i.e., in L^1, to the case of Haar multipliers). Then they proved that the identity on L^1 factors through every bounded Haar multiplier whose entries are bounded away from 0, utilizing a result by E. M. Semenov and S. N. Uksusov [49] (see also [51]), which characterizes the bounded Haar multipliers on L^1 (for extensions to the vector-valued case, see [14, 29, 52, 53]).

In this paper, we will prove factorization results for *Haar system Hardy spaces*, a large class of Banach spaces in which the Haar system is a Schauder basis. They are constructed as the completion of the linear span of the Haar system either under a rearrangement-invariant norm $\|\cdot\|$ or under the associated square function norm $\|\cdot\|_*$ given by

$$\Big\|\sum_I a_Ih_I\Big\|_* = \Big\|\Big(\sum_I a_I^2h_I^2\Big)^{1/2}\Big\|.$$

This class encompasses all separable r.i. spaces on $[0, 1)$ and their square function versions, including the classical Lebesgue spaces L^p as well as the dyadic Hardy spaces H^p, $1 \le p < \infty$. We develop a unified approach, which can deal with all these spaces simultaneously; in particular, we obtain a unified proof of the classical results that L^p, $1 \le p < \infty$ and H^1 are primary. We then extend our factorization results to infinite direct sums of Haar system Hardy spaces.

We will now give a general outline of our approach, in which we systematically reduce a given operator T on a Haar system Hardy space Y to a constant multiple of the identity operator: First, we reduce T to a bounded Haar multiplier via the framework of strategically reproducible bases and strategically supporting systems. If the Haar system $(h_I)_I$ is unconditional, then every bounded sequence of scalars $(d_I)_I$ determines a bounded Haar multiplier by $Dh_I = d_Ih_I$, $I \in \mathcal{D}$. On the contrary, the result of E. M. Semenov and S. N. Uksusov [49] shows that there are far fewer bounded Haar multipliers on L^1. Thus, a unified approach must necessarily be able to deal with any Haar multiplier without exploiting that there are only few bounded Haar multipliers. Indeed, our approach reduces any bounded Haar multiplier to one with very small variation, i.e., to a stable Haar multiplier. This reduction is achieved by using randomized block bases of the Haar system. By a perturbation argument, we then arrive at a scalar multiple cI_Y of the identity operator, completing our reduction procedure. One notable aspect of our method is that we have a priori knowledge about the constant c as soon as the Haar multiplier D is determined.

Before proceeding, we will introduce some terminology. Let E be a Banach space and let $S, T\colon E \to E$ be bounded linear operators. We say that S *factors through* T *(with* *constant* $C \geq 0$*)* if there exist bounded linear operators $A, B\colon E \to E$ with $S = ATB$ (and $\|A\|\|B\| \leq C$). The first important property is the *primary factorization property.*

Definition 1.1. We say that a Banach space E has the *primary factorization property* if for every bounded linear operator $T\colon E \to E$, the identity I_E either factors through T or through $I_E - T$.

If a Banach space E has the primary factorization property and satisfies Pełczyński's *accordion property*, i.e., $E \sim \ell^p(E)$ for some $1 \leq p \leq \infty$, then E is primary. This follows from Pełczyński's decomposition method and the following observation: If P is a bounded linear projection on E, then the primary factorization property of E implies that either $P(E)$ or $(I_E - P)(E)$ contains a complemented subspace that is isomorphic to E.

We will now describe how the primary factorization property is related to the theory of operator ideals. Let E be a Banach space and let $\mathcal{B}(E)$ denote the algebra of bounded linear operators on E. Then we define

$$\mathcal{M}_E = \{T \in \mathcal{B}(E) : I_E \neq ATB \text{ for all } A, B \in \mathcal{B}(E)\}.$$

The set $\mathcal{M}_E$ is an ideal of $\mathcal{B}(E)$ if and only if it is closed under addition. It was observed by D. Dosev and W. B. Johnson [9] that if $\mathcal{M}_E$ is an ideal of $\mathcal{B}(E)$, then it is automatically the *unique maximal ideal* of $\mathcal{B}(E)$. On the other hand, the assertion that $\mathcal{M}_E$ is closed under addition is equivalent to the primary factorization property of E (see [9, Proposition 5.1]). In [20], T. Kania and the first named author introduced *strategically supporting* systems in dual pairs of Banach spaces to describe sufficient conditions for $\mathcal{M}_E$ to be the unique maximal ideal of $\mathcal{B}(E)$. Typically, this concept is used to find a large complemented subspace where a given operator has additional properties, e.g., large diagonal: A bounded linear operator T on a Banach space E with a Schauder basis $(e_j)_{j=1}^{\infty}$ and biorthogonal functionals $(e_j^*)_{j=1}^{\infty}$ has *large diagonal* if $\inf_{j\in\mathbb{N}} |\langle e_j^*, Te_j\rangle| > 0$. The study of operators with large diagonal goes back to A. D. Andrew [2], and they were later explicitly investigated in [4, 21, 23, 22, 25, 24, 26]. The first named author, P. Motakis, P. F. X. Müller and Th. Schlumprecht then developed these approaches into a systematic framework in [27, 28, 29]. A part of this framework is the *factorization property*, which was analyzed using strategically reproducible Schauder bases.

Definition 1.2. Let E be a Banach space with a Schauder basis $(e_j)_{j=1}^{\infty}$ and biorthogonal functionals $(e_j^*)_{j=1}^{\infty}$. We say that $(e_j)_{j=1}^{\infty}$ has the *factorization property* if the identity I_E factors through every bounded linear operator $T\colon E \to E$ which has large diagonal with respect to $(e_j)_{j=1}^{\infty}$.

For further development of this framework and some new applications to stopping time Banach spaces, we refer to [20]. More recently, Kh. V. Navoyan [44] showed that under mild assumptions, the Haar basis of a *Haar system space* X has the factorization property, provided that it is unconditional in X. A *Haar system space* is the completion of the linear span of the Haar system under a rearrangement-invariant norm; these spaces were introduced in [29]. Before presenting our main results in Section 3, we will establish necessary notation and terminology.

2. Notation and basic definitions

We denote by $\mathcal{D}$ the collection of all dyadic intervals in $[0, 1)$, namely

$$\mathcal{D} = \Big\{ \Big[\frac{i-1}{2^j}, \frac{i}{2^j}\Big) : j \in \mathbb{N}_0,\ 1 \leq i \leq 2^j \Big\}.$$

In addition, for each $n \in \mathbb{N}_0$, we define

$$\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\} \qquad \text{and} \qquad \mathcal{D}_{\leq n} = \bigcup_{k=0}^{n} \mathcal{D}_k, \quad \mathcal{D}_{<n} = \bigcup_{k=0}^{n-1} \mathcal{D}_k.$$

For each dyadic interval $I \in \mathcal{D}$, let I^+ denote the left half of I and I^- its right half, i.e., I^+ is the largest dyadic interval $J \in \mathcal{D}$ with $J \subsetneq I$ and $\inf J = \inf I$, and we have $I^- = I \setminus I^+$. If we use the symbol $\pm$ multiple times in an equation, we mean either always $+$ or always $-$. Sometimes, instead of I^+ or I^-, we will write I^ε, where $\varepsilon \in \{\pm 1\}$. Conversely, for $I \in \mathcal{D} \setminus \{[0,1)\}$, we denote by $\pi(I)$ the dyadic predecessor of I, i.e., the unique dyadic interval $J \in \mathcal{D}$ with $I = J^+$ or $I = J^-$. Finally, for any subcollection $\mathcal{B} \subset \mathcal{D}$, we put $\mathcal{B}^* = \bigcup_{I \in \mathcal{B}} I$.

Next, we define the bijective function $\iota \colon \mathcal{D} \to \mathbb{N}$ by

$$\Big[\frac{i}{2^j}, \frac{i+1}{2^j}\Big) \overset{\iota}{\mapsto} 2^j + i.$$

The function ι defines a linear order on $\mathcal{D}$, and we will frequently consider sequences indexed by dyadic intervals, identifying $\mathcal{D}$ with $\mathbb{N}$. The Haar system $(h_I)_{I \in \mathcal{D}}$ is defined by

$$h_I = \chi_{I^+} - \chi_{I^-}, \qquad I \in \mathcal{D},$$

where χ_A denotes the characteristic function of a subset $A \subset [0,1)$. We additionally define $h_\varnothing = \chi_{[0,1)}$ and put $\mathcal{D}^+ = \mathcal{D} \cup \{\varnothing\}$. We also put $\iota(\varnothing) = 0$. We will usually write $I \leq J$ if $\iota(I) \leq \iota(J)$, $I, J \in \mathcal{D}^+$. Henceforth, whenever we write $\sum_{I \in \mathcal{D}^+}$, we will always mean that the sum is taken with this linear order ι. Recall that the Haar system $(h_I)_{I \in \mathcal{D}^+}$, in the linear order defined by ι, is a monotone Schauder basis of L^p, $1 \leq p < \infty$ (and unconditional if $1 < p < \infty$). For $x = \sum_{I \in \mathcal{D}^+} a_I h_I \in L^1$, we define the *Haar support* of x to be the set of all $I \in \mathcal{D}^+$ with $a_I \neq 0$. More generally, if $(e_j)_{j=1}^\infty$ is a Schauder basis of a Banach space E, then for $x = \sum_{j=1}^\infty a_j e_j \in E$, we define $\operatorname{supp} x = \{j \in \mathbb{N} : a_j \neq 0\}$.

We only consider real Banach spaces. If $(x_n)_{n=1}^\infty$ is a sequence in a Banach space E, then we denote its closed linear span in E by $[x_n]_{n \in \mathbb{N}}$. If E and F are Banach spaces and $C \geq 1$, we say that E and F are C-isomorphic, i.e., $E \overset{C}{\sim} F$, if there exists an isomorphism T from E onto F with $\|T\|\|T^{-1}\| \leq C$. Moreover, if $C \geq 0$, we say that a closed subspace F of E is C-complemented in E if there exists a linear projection $P \colon E \to E$ with $P(E) = F$ and $\|P\| \leq C$. Finally, we say that two measurable functions $x, y \colon [0,1) \to \mathbb{R}$ are *equimeasurable* if their absolute values $|x|$ and $|y|$ have the same distribution.

2.1. **Haar system Hardy spaces.**

The class of *Haar system Hardy spaces* will be defined as an extension of the class of *Haar system spaces*, which was introduced in [29] as follows.

Definition 2.1. A *Haar system space* X is the completion of $H := \operatorname{span}\{h_I : I \in \mathcal{D}^+\} = \operatorname{span}\{\chi_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ that satisfies the following properties:

(i) If x, y are in H and $|x|$, $|y|$ have the same distribution, then $\|x\|_X = \|y\|_X$.
(ii) $\|\chi_{[0,1)}\|_X = 1$.

We denote the class of Haar system spaces by $\mathcal{H}(\delta)$. Moreover, given $X \in \mathcal{H}(\delta)$, we define the closed subspace X_0 of X as the closure of $H_0 := \operatorname{span}\{h_I : I \in \mathcal{D}\}$ in X. We denote the class of these subspaces by $\mathcal{H}_0(\delta)$.

Note that if a norm on H satisfies Property (i), then we can always scale it so that it satisfies Property (ii). One can show that the Haar system $(h_I)_{I \in \mathcal{D}^+}$, in the linear order defined by ι, is a monotone Schauder basis of any Haar system space X (see Proposition 4.1).

Besides the spaces L^p, $1 \le p < \infty$, and the closure of H in L^∞, the class $\mathcal{H}(\delta)$ includes all rearrangement-invariant function spaces on $[0,1)$ (e.g., Orlicz function spaces) in which the span of the Haar system $(h_I)_{I\in\mathcal{D}^+}$ is dense. According to [32, Proposition 2.c.1], this is true for all separable rearrangement-invariant function spaces.

By $(r_n)_{n=0}^\infty$, we denote the sequence of standard Rademacher functions, i.e.,

$$r_n = \sum_{I\in\mathcal{D}_n} h_I, \qquad n \in \mathbb{N}_0.$$

We will now introduce the class of *Haar system Hardy spaces*. To this end, we first define the set $\mathcal{R}$ as

$$\mathcal{R} = \{(r_{\iota(I)})_{I\in\mathcal{D}^+}, (r_0)_{I\in\mathcal{D}^+}\}.$$

Hence, if $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+} \in \mathcal{R}$, then $\mathbf{r}$ is either an independent sequence of ±1-valued random variables (indexed by dyadic intervals) or a constant sequence. Starting with a Haar system space X and a sequence $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+} \in \mathcal{R}$, we obtain a Haar system Hardy space by taking the completion of H under a new norm:

Definition 2.2. Given $X \in \mathcal{H}(\delta)$ and $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+} \in \mathcal{R}$, we define the *(one-parameter) Haar system Hardy space* $X(\mathbf{r})$ as the completion of $H = \operatorname{span}\{h_I : I \in \mathcal{D}^+\}$ under the norm $\|\cdot\|_{X(\mathbf{r})}$ given by

$$\Big\|\sum_{I\in\mathcal{D}^+} a_I h_I\Big\|_{X(\mathbf{r})} = \Big\| s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}^+} r_I(u) a_I h_I(s)\Big| \,\mathrm{d}u\Big\|_X.$$

We will denote the class of one-parameter Haar system Hardy spaces by $\mathcal{HH}(\delta)$. Moreover, given $X(\mathbf{r}) \in \mathcal{HH}(\delta)$, we define the closed subspace $X_0(\mathbf{r}) = [h_I]_{I\in\mathcal{D}} \subset X(\mathbf{r})$. For notational convenience, we will also refer to the subspaces $X_0(\mathbf{r})$ as Haar system Hardy spaces. We denote the class of these subspaces by $\mathcal{HH}_0(\delta)$.

Clearly, if $\mathbf{r} = (r_I)_{I\in\mathcal{D}^+}$ is an independent sequence, then $(h_I)_{I\in\mathcal{D}^+}$ is a 1-unconditional Schauder basis of $X(\mathbf{r})$. If, on the other hand, $r_I = r_0$ for all $I \in \mathcal{D}^+$, then we have $\|\cdot\|_{X(\mathbf{r})} = \|\cdot\|_X$ and thus $X(\mathbf{r}) = X$. We already know that in this case, $(h_I)_{I\in\mathcal{D}^+}$ is a monotone Schauder basis of $X(\mathbf{r})$ (but it need not be unconditional). Finally, note that if x is a finite linear combination of disjointly supported Haar functions, then we always have $\|x\|_{X(\mathbf{r})} = \|x\|_X$.

Remark 2.3. Let $X \in \mathcal{H}(\delta)$, suppose that $\mathbf{r} \in \mathcal{R}$ is independent, and let $(a_I)_{I\in\mathcal{D}}$ be a scalar sequence with $a_I \neq 0$ for at most finitely many I. Then, using Khintchine's inequality and the fact that $|x| \le |y|$ pointwise implies $\|x\|_X \le \|y\|_X$ for all $x, y \in H$ (see Proposition 4.1 (iv)), we obtain

$$\Big\|\sum_{I\in\mathcal{D}^+} a_I h_I\Big\|_{X(\mathbf{r})} \sim \Big\|\Big(\sum_{I\in\mathcal{D}^+} |a_I|^2 h_I(s)^2\Big)^{1/2}\Big\|_X. \tag{2.1}$$

Thus, for $X = L^1$ and $\mathbf{r} \in \mathcal{R}$ independent, we see that $X(\mathbf{r})$ is isomorphic to the dyadic Hardy space H^1, and for $X = L^p$, $1 < p < \infty$, we have $X(\mathbf{r}) \sim H^p \sim L^p$, where H^p denotes the dyadic Hardy space with parameter p. In fact, $\|\cdot\|_X \sim \|\cdot\|_{X(\mathbf{r})}$ holds whenever X is a separable r.i. function space with non-trivial Boyd indices (according to the remark following [32, Proposition 2.d.8]). In all these cases, the identity operator provides an isomorphism. Finally, if X is the closure of H in L^∞ and $\mathbf{r} \in \mathcal{R}$ is independent, then by (2.1), the space $X_0(\mathbf{r})$ is isomorphic to the closure of H_0 in the non-separable space SL^∞ (see [19, 23]).

2.2. **Additional definitions.** We begin this subsection by summarizing and extending the definitions related to factorization of operators that were given in Section 1. First, we introduce the following additional factorization modes.

Definition 2.4. Let E denote a Banach space. Let $S, T\colon E \to E$ denote bounded linear operators, and let $C, \eta \geq 0$.

(i) We say that S *factors through* T *with constant* C *and error* η if there exist linear operators $A, B\colon E \to E$ with $\|A\|\|B\| \leq C$ such that $\|S - ATB\| \leq \eta$.
(ii) If (i) holds and we additionally have $AB = I_E$, then we say that S *projectionally* factors through T with constant C and error η.
(iii) We say that S (projectionally) factors through T *with constant* C^+ and error η if for every $\gamma > 0$, the operator S (projectionally) factors through T with constant $C + \gamma$ and error η.

If we omit the phrase "with error η" in (i), (ii) or (iii), then we take that to mean that the error is 0.

Next, we make some elementary observations that will be useful later.

Remark 2.5. Let $R, S, T\colon E \to E$ denote bounded linear operators and suppose that R (projectionally) factors through S with constant C_1 and error η_1, and that S (projectionally) factors through T with constant C_2 and error η_2. In [29, Proposition 2.3], it was observed that R (projectionally) factors through T with constant C_1C_2 and error $\eta_1 + C_1\eta_2$.

Remark 2.6. Let $S, T\colon E \to E$ denote bounded linear operators and suppose that S is an isomorphism. If S factors through T with constant $C \geq 0$ and error $\eta \geq 0$ and $\eta\|S^{-1}\| < 1$, then S factors through T with constant $\frac{C}{1-\eta\|S^{-1}\|}$ (and error 0). Indeed, let $A, B\colon E \to E$ be bounded linear operators with $\|S - ATB\| \leq \eta$ and $\|A\|\|B\| \leq C$. Then we have

$$\|I_E - S^{-1}ATB\| \leq \eta\|S^{-1}\| < 1,$$

so $Q := (S^{-1}ATB)^{-1}$ exists and satisfies $\|Q\| \leq 1/(1 - \eta\|S^{-1}\|)$. The statement follows since we have $S = ATBQ$.

Remark 2.7. Let $S, T\colon E \to E$ denote bounded linear operators and suppose that S projectionally factors through T with constant $C \geq 1$ and error $\eta > 0$. Then $I_E - S$ projectionally factors through $I_E - T$ with constant $C \geq 1$ and error $\eta > 0$.

The next definition includes quantitative and uniform versions of the (primary) factorization property as well as some more variations of these concepts.

Definition 2.8. Let E denote a Banach space with a Schauder basis $(e_j)_{j=1}^\infty$ and biorthogonal functionals $(e_j^*)_{j=1}^\infty$, and let $C \geq 0$.

(i) Let $\delta > 0$, and let $T\colon E \to E$ be a bounded linear operator. We say that
  - ▷ T is *diagonal (with respect to* $(e_j)_{j=1}^\infty$*)* if $\langle e_k^*, Te_j\rangle = 0$ for all $k \neq j$. Diagonal operators with respect to the Haar system are called *Haar multipliers*.
  - ▷ T has δ*-large diagonal (with respect to* $(e_j)_{j=1}^\infty$*)* if $|\langle e_j^*, Te_j\rangle| \geq \delta$ for all $j \in \mathbb{N}$.
  - ▷ T has δ-large *positive* diagonal if $\langle e_j^*, Te_j\rangle \geq \delta$ for all $j \in \mathbb{N}$.
  - ▷ T has δ-large *negative* diagonal if $\langle e_j^*, Te_j\rangle \leq -\delta$ for all $j \in \mathbb{N}$.

(ii) We say that E has the C*-primary (diagonal) factorization property (with respect to* $(e_j)_{j=1}^\infty$*)* if for any bounded linear operator $T\colon E \to E$ (which is diagonal with respect to $(e_j)_{j=1}^\infty$), the identity I_E either factors through T or through $I_E - T$ with constant C^+.

(iii) Let $K\colon (0,\infty) \to (0,\infty)$. We say that $(e_j)_{j=1}^\infty$ has the $K(\delta)$-*(diagonal) factorization property* if for every bounded linear (diagonal) operator $T\colon E \to E$ with δ-large diagonal with respect to $(e_j)_{j=1}^\infty$, the identity I_E factors through T with constant $K(\delta)^+$.

(iv) If (iii) holds with "δ-large diagonal" replaced by "δ-large *positive* diagonal", then we say that $(e_j)_{j=1}^\infty$ has the $K(\delta)$-*positive* (diagonal) factorization property.

Remark 2.9. Let $\lambda \geq 1$ and suppose that for each $k \in \mathbb{N}$, E_k is a Banach space with a Schauder basis $(e_{k,j})_{j=1}^\infty$ whose basis constant is bounded by λ. Let $1 \leq p \leq \infty$. We identify each space E_k with the subspace of $\ell^p((E_k)_{k=1}^\infty)$ consisting of those sequences for which all coordinates, except the kth one, are equal to zero.

Now let $(\tilde{e}_m)_{m=1}^\infty$ be an enumeration of $(e_{k,j})_{k,j=1}^\infty$ with the property that whenever we have $e_{k,i} = \tilde{e}_l$ and $e_{k,j} = \tilde{e}_m$ for some $k, i, j, l, m \in \mathbb{N}$, then the inequality $i < j$ implies $l < m$. Then for every $1 \leq p < \infty$, $(\tilde{e}_m)_{m=1}^\infty$ is a Schauder basis of $\ell^p((E_k)_{k=1}^\infty)$ with basis constant at most λ. The associated biorthogonal functionals are given by $\tilde{e}_m^* = e_{k,j}^*$ (for $\tilde{e}_m = e_{k,j}$), where $e_{k,j}^*$ acts on the kth component of $\ell^p((E_k)_{k=1}^\infty)$.

Clearly, for $p = \infty$, $(\tilde{e}_m)_{m=1}^\infty$ is not a Schauder basis of $\ell^p((E_k)_{k=1}^\infty)$, but we can still define the notion of a large diagonal: Let $\delta > 0$. A bounded linear operator $T\colon \ell^\infty((E_k)_{k=1}^\infty) \to \ell^\infty((E_k)_{k=1}^\infty)$ has δ-large diagonal with respect to $(\tilde{e}_m)_{m=1}^\infty$ if $|\langle \tilde{e}_m^*, T\tilde{e}_m\rangle| \geq \delta$ holds for all $m \in \mathbb{N}$. The $K(\delta)$-factorization property of $(\tilde{e}_m)_{m=1}^\infty$ in $\ell^\infty((E_k)_{k=1}^\infty)$ is then defined like above.

In order to prove the (primary) factorization property for ℓ^∞-sums of Haar system Hardy spaces, we need to assume our spaces form a sequence that is *uniformly asymptotically curved* with respect to the array consisting of their Haar bases. This property was introduced in [28].

Definition 2.10. Let E be a Banach space with a Schauder basis $(e_j)_{j=1}^\infty$. Moreover, let $(E_k)_{k=1}^\infty$ be a sequence of Banach spaces, and for each $k \in \mathbb{N}$, let $(e_{k,j})_{j=1}^\infty$ denote a Schauder basis of E_k. By an *array*, we mean an indexed family $(x_{k,j})_{k,j=1}^\infty$ with $x_{k,j} \in E_k$ for all $k, j \in \mathbb{N}$.

- ▷ We say that E is *asymptotically curved with respect to* $(e_j)_{j=1}^\infty$ if for every bounded block basis $(x_j)_{j=1}^\infty$ of $(e_j)_{j=1}^\infty$, we have
$$\lim_{n\to\infty} \frac{1}{n}\Big\| \sum_{j=1}^n x_j \Big\|_E = 0.$$
- ▷ We say that the sequence $(E_k)_{k=1}^\infty$ *uniformly asymptotically curved with respect to the array* $(e_{k,j})_{k,j=1}^\infty$ if the following holds: For every bounded array $(x_{k,j})_{k,j=1}^\infty$ with the property that $(x_{k,j})_{j=1}^\infty$ is a block basis of $(e_{k,j})_{j=1}^\infty$ for all $k \in \mathbb{N}$, we have
$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{n}\Big\| \sum_{j=1}^n x_{k,j} \Big\|_{E_k} = 0.$$

We refer to Section Section 6 for a discussion of these concepts.

3. Main results

In order to avoid having to deal with the constant function $h_\varnothing$ separately in the proofs, we will state our results for the subspaces $X_0(\mathbf{r})$ instead of $X(\mathbf{r})$. We will explain in Remark 3.3 how to obtain the corresponding versions of Theorem 3.1 and Theorem 3.2 for the spaces $X(\mathbf{r})$. In the following, we always assume that $Y, Y_k \in \mathcal{HH}_0(\delta)$, $k \in \mathbb{N}$. The Haar basis of Y is denoted by $(h_I)_{I\in\mathcal{D}}$, and for every $k \in \mathbb{N}$, the Haar basis of Y_k is denoted by $(h_{k,I})_{I\in\mathcal{D}}$. Given $1 \leq p \leq \infty$, we identify each space Y_k with the subspace of $\ell^p((Y_k)_{k=1}^\infty)$ consisting of sequences supported in the kth component, and we enumerate $(h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}}$ according to Remark 2.9, thus obtaining a monotone Schauder basis of $\ell^p((Y_k)_{k=1}^\infty)$ for $1 \leq p < \infty$.

Theorem 3.1. *Suppose that the sequence of Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in Y, and let E denote one of the following Banach spaces:*

- *(i) $E = Y$*
- *(ii) $E = \ell^p(Y)$ for some $1 \leq p < \infty$*
- *(iii) $E = \ell^\infty(Y)$ if Y is asymptotically curved with respect to $(h_I)_{I\in\mathcal{D}}$.*

Then E has the 4-primary factorization property, and hence, $\mathcal{M}_E$ is the unique maximal ideal of $\mathcal{B}(E)$. In particular, the spaces in (ii) and (iii) are primary.

We will prove Theorem 3.1 in Section 11. Next, we state the main results which involve the factorization property.

Theorem 3.2. *Suppose that the sequence of Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in Y and in each Y_k, $k \in \mathbb{N}$, and let E and $(e_m)_{m=1}^\infty$ denote one of the following pairs of Banach spaces and sequences:*

- *(i) $E = Y$ and $(e_m)_{m=1}^\infty = (h_I)_{I\in\mathcal{D}}$*
- *(ii) $E = \ell^p((Y_k)_{k=1}^\infty)$ for some $1 \leq p < \infty$ and $(e_m)_{m=1}^\infty = (h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}}$*
- *(iii) $E = \ell^\infty((Y_k)_{k=1}^\infty)$ and $(e_m)_{m=1}^\infty = (h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}}$ if $(Y_k)_{k=1}^\infty$ is uniformly asymptotically curved with respect to $(h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}}$.*

Then $(e_m)_{m=1}^\infty$ has the $4/\delta$-factorization property in E.

The proof of Theorem 3.2 can be found in Section 11. We will now discuss the hypotheses of Theorem 3.1 and Theorem 3.2. In Section 6, we will provide sufficient conditions for (uniform) asymptotic curvedness in Haar system Hardy spaces (see Proposition 6.6, Remark 6.7 and Remark 6.8).

The requirement that the (standard) sequence of Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in a Haar system Hardy space $X(\mathbf{r}) \in \mathcal{HH}(\delta)$ or, equivalently, in $Y = X_0(\mathbf{r})$ is not a strong limitation: If $\mathbf{r}$ is independent, then this condition is always satisfied because using (2.1), we see that the sequence $(r_n)_{n=0}^\infty$ in $X(\mathbf{r})$ is equivalent to the unit vector basis of ℓ^2. On the other hand, if $\mathbf{r}$ is constant, recall that $X(\mathbf{r}) = X$. Then the proof of [32, Proposition 2.c.10], which is a result by V. A. Rodin and E. M. Semenov [47], shows that the following conditions are equivalent:

- (i) The sequence of standard Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in X.
- (ii) The sequence $(r_n)_{n=0}^\infty$ in X is not equivalent to the unit vector basis of ℓ^1.
- (iii) The norm $\|\cdot\|_X$ is not equivalent to the L^∞-norm on H (i.e., $X \neq C(\Delta)$).
- (iv) We have $\lim_{n\to\infty} \|\chi_{[0,2^{-n})}\|_X = 0$.

The implication (ii)$\Longrightarrow$(i) also follows from Rosenthal's ℓ^1 Theorem (see [29, Remark 2.15]).

Remark 3.3. Later on, we will show that for every $X(\mathbf{r}) \in \mathcal{HH}(\delta)$, the closed subspace $Y = X_0(\mathbf{r})$ is isomorphic to $X(\mathbf{r})$ and that there exists an isomorphism $S\colon X(\mathbf{r}) \to X_0(\mathbf{r})$ with $\|S\| \leq 9$ and $\|S^{-1}\| \leq 18$ which maps $(h_I)_{I\in\mathcal{D}^+}$ bijectively onto a permutation of $(h_I)_{I\in\mathcal{D}}$ (see Proposition 4.10 and Remark 4.11). Since the isomorphism S is a rearrangement of the Haar system, it preserves large diagonals of bounded linear operators, i.e., if $\delta > 0$ and $T\colon X(\mathbf{r}) \to X(\mathbf{r})$ has δ-large diagonal with respect to $(h_I)_{I\in\mathcal{D}^+}$, then $STS^{-1} \in \mathcal{B}(Y)$ has δ-large diagonal with respect to $(h_I)_{I\in\mathcal{D}}$. We thus see that the results of Theorem 3.1 and Theorem 3.2 concerning the primary factorization property and the factorization property respectively, carry over from Y and $(h_I)_{I\in\mathcal{D}}$ to $X(\mathbf{r})$ and $(h_I)_{I\in\mathcal{D}^+}$, albeit at the price of increasing the factorization constant. The same is true for the results in Theorem 3.2 (ii) and (iii).

We will now introduce the *characteristic set* of a bounded Haar multiplier, which contains a priori information on the factorization appearing in Theorem 3.6.

Definition 3.4. Given a bounded Haar multiplier $D\colon Y \to Y$, we define the *characteristic set* $\Lambda(D)$ by

$$\Lambda(D) := \{c \in \mathbb{R} : c \text{ is a cluster point of } (\langle r_n, Dr_n\rangle)_{n=0}^{\infty}\}.$$

Remark 3.5. Since for each $n \in \mathbb{N}$, the Rademacher function r_n has norm 1 both in Y and in Y^* (see Corollary 7.2), the boundedness of D implies that $\Lambda(D)$ is non-empty and bounded: We have $|c| \leq \|D\|$ for all $c \in \Lambda(D)$. Moreover, note that $\Lambda(I_Y - D) = \{1 - c : c \in \Lambda(D)\}$. Finally, if D has δ-large positive diagonal for some $\delta > 0$, i.e., $\langle h_I, Dh_I\rangle \geq \delta|I|$ for all $I \in \mathcal{D}$, then we have $\inf \Lambda(D) \geq \delta$.

Theorem 3.6. *Let $D\colon Y \to Y$ be a bounded Haar multiplier, and let $c \in \Lambda(D)$. Then the following assertions are true:*

(i) For every $\eta > 0$, cI_Y projectionally factors through D with constant 1 and error η.
(ii) If $c \neq 0$, then the identity I_Y factors through D with constant $(1/|c|)^+$.
(iii) The identity I_Y either factors through D or through $I_Y - D$ with constant 2^+; more precisely, with constant $\min(1/|c|, 1/|1-c|)^+$, where we define $1/0 = \infty$.

For the proof of Theorem 3.6, see Section 8. Note that Theorem 3.6 (i) does not simply state that *some* multiple of the identity cI_Y projectionally factors through D (with some constant and error), but it also provides a priori knowledge about the constant c: We may choose c to be any element of the characteristic set $\Lambda(D)$, and we also have some knowledge about what this set can look like (see Remark 3.5).

4. Properties of Haar system Hardy spaces

Before discussing the properties of Haar system Hardy spaces, we recall the following basic results on Haar system spaces.

Proposition 4.1. *Let $X \in \mathcal{H}(\delta)$. Then following assertions are true.*

(i) For every $f \in H = \operatorname{span}\{\chi_I : I \in \mathcal{D}\}$, we have $\|f\|_{L^1} \leq \|f\|_X \leq \|f\|_{L^\infty}$. Therefore, X can be naturally identified with a space of measurable scalar valued functions on $[0,1)$ and $\overline{H}^{\|\cdot\|_{L^\infty}} \subset X \subset L^1$.
(ii) The Haar system $(h_I)_{I\in\mathcal{D}^+}$, in the usual linear order, is a monotone Schauder basis of X.
(iii) H naturally coincides with a subspace of X^*, and its closure $\overline{H}$ in X^* is also a Haar system space.
(iv) For all $f, g \in H$ with $|f| \leq |g|$, we have $\|f\|_X \leq \|g\|_X$.

Proof. We refer to [29, Proposition 2.13] for a proof of (i)–(iii); assertion (iv) follows from the observation that for each $n \in \mathbb{N}$, the family $(\chi_I : I \in \mathcal{D}_n)$ is 1-unconditional in X. □

Remark 4.2. In Proposition 4.1 (iii), we identify each $g \in H$ with the bounded linear functional $x_g^* \in X^*$ defined as the continuous extension of $f \mapsto \int_0^1 fg$, $f \in H$.

Next, we show that if X is a Haar system space, then conditional expectations with respect to certain finite σ-algebras are contractions on X.

Lemma 4.3. *Let $X \in \mathcal{H}(\delta)$, and let $\mathcal{F}$ denote a σ-algebra generated by a partition $(A_i : 1 \leq i \leq m)$ of $[0,1)$, where each set A_i is a finite union of dyadic intervals. Then*

$$\|\mathbb{E}^{\mathcal{F}} x\|_X \leq \|x\|_X, \qquad x \in H.$$

Proof. Let $M \in \mathbb{N}$ and suppose that $x = \sum_{I\in\mathcal{D}_M} a_I\chi_I$. Pick $N > M$ and sets of pairwise disjoint dyadic intervals $\mathcal{A}_i = \{K_{i,k} : 1 \le k \le n_i\} \subset \mathcal{D}_N$ such that $\bigcup_{k=1}^{n_i} K_{i,k} = A_i$, $1 \le i \le m$. Put

$$R_N = \{\rho\colon \mathcal{D}_N \to \mathcal{D}_N : \rho \text{ is bijective and } \rho(\mathcal{A}_i) = \mathcal{A}_i \text{ for all } 1 \le i \le m\}$$

and let $\operatorname{ave}_{\rho\in R_N}$ denote the average over all ρ in R_N. Observe that since the intervals $K_{i,k}$, $1 \le i \le m$, $1 \le k \le n_i$ are pairwise disjoint, using Definition 2.1 (i), we obtain

$$\begin{aligned}\|x\|_X &= \Big\|\sum_{i=1}^m\sum_{k=1}^{n_i}\sum_{\substack{I\in\mathcal{D}_M\\ I\supset K_{i,k}}} a_I\chi_{K_{i,k}}\Big\|_X = \operatorname*{ave}_{\rho\in R_N}\Big\|\sum_{i=1}^m\sum_{k=1}^{n_i}\sum_{\substack{I\in\mathcal{D}_M\\ I\supset K_{i,k}}} a_I\chi_{\rho(K_{i,k})}\Big\|_X\\ &\ge \Big\|\sum_{i=1}^m\sum_{k=1}^{n_i}\sum_{\substack{I\in\mathcal{D}_M\\ I\supset K_{i,k}}} a_I \operatorname*{ave}_{\rho\in R_N}\chi_{\rho(K_{i,k})}\Big\|_X.\end{aligned}$$

Note that for fixed $1 \le i \le m$, $1 \le k \le n_i$ we have

$$\operatorname*{ave}_{\rho\in R_N}\chi_{\rho(K_{i,k})} = \frac{1}{|R_N|}\sum_{\rho\in R_N}\chi_{\rho(K_{i,k})} = \sum_{K\in\mathcal{A}_i}\frac{1}{|R_N|}\sum_{\substack{\rho\in R_N\\ \rho(K_{i,k})=K}}\chi_K = \sum_{K\in\mathcal{A}_i}\frac{1}{|\mathcal{A}_i|}\chi_K = \frac{|K_{i,k}|}{|A_i|}\chi_{A_i}.$$

Inserting this identity in the above inequality yields

$$\begin{aligned}\|x\|_X &\ge \Big\|\sum_{i=1}^m\sum_{I\in\mathcal{D}_M} a_I\sum_{\substack{1\le k\le n_i\\ K_{i,k}\subset I}}\frac{|K_{i,k}|}{|A_i|}\chi_{A_i}\Big\|_X = \Big\|\sum_{i=1}^m\sum_{I\in\mathcal{D}_M} a_I\frac{|A_i\cap I|}{|A_i|}\chi_{A_i}\Big\|_X\\ &= \Big\|\sum_{I\in\mathcal{D}_M} a_I\mathbb{E}^{\mathcal{F}}\chi_I\Big\|_X = \|\mathbb{E}^{\mathcal{F}}x\|_X,\end{aligned}$$

as claimed. □

If $X(\mathbf{r}) \in \mathcal{HH}(\delta)$ is a Haar system Hardy space, then just like in the case of Haar system spaces, we can identify $H = \operatorname{span}\{h_I : I \in \mathcal{D}^+\}$ with a subspace of the dual space of $X(\mathbf{r})$. In the next lemma, we compute the norm of a Haar function h_I, viewed as an element of $X(\mathbf{r})^*$.

Lemma 4.4. *Let $X(\mathbf{r}) \in \mathcal{HH}(\delta)$. Then for every $I \in \mathcal{D}$, we have $\|h_I\|_{X(\mathbf{r})}\|h_I\|_{X(\mathbf{r})^*} = |I|$ and $\|h_I\|_{X_0(\mathbf{r})}\|h_I\|_{X_0(\mathbf{r})^*} = |I|$.*

Proof. We only prove the second equality since the proof of the first one is analogous. Fix $I \in \mathcal{D}$ and let $x = \sum_{J\in\mathcal{D}} a_Jh_J \in H_0$. Now observe that

$$\|a_Ih_I\|_{X(\mathbf{r})} \le \frac12\Big\|a_Ih_I + \sum_{\substack{J\in\mathcal{D}\\ J<I}} a_Jh_J\Big\|_{X(\mathbf{r})} + \frac12\Big\|a_Ih_I - \sum_{\substack{J\in\mathcal{D}\\ J<I}} a_Jh_J\Big\|_{X(\mathbf{r})}.$$

The two summands on the right-hand side are equal because the two functions inside the norms are equimeasurable. Thus, we have

$$\|a_Ih_I\|_{X(\mathbf{r})} \le \Big\|\sum_{\substack{J\in\mathcal{D}\\ J\le I}} a_Jh_J\Big\|_{X(\mathbf{r})} \le \|x\|_{X(\mathbf{r})}.$$

Consequently, we obtain that $\|h_I\|_{X_0(\mathbf{r})^*} \le |I|/\|h_I\|_{X(\mathbf{r})}$. For the other inequality, note that

$$\|h_I\|_{X_0(\mathbf{r})^*} \ge \Big\langle h_I, \frac{h_I}{\|h_I\|_{X(\mathbf{r})}}\Big\rangle = \frac{|I|}{\|h_I\|_{X(\mathbf{r})}}.$$ □

The following lemma provides an upper bound on the norm of a Haar multiplier, in the spirit of the theorem on Haar multipliers on L^1 by E. M. Semenov and S. N. Uksusov [49] (see also [51] and [29]).

Lemma 4.5. *Let $X_0(\mathbf{r}) \in \mathcal{HH}_0(\delta)$ and suppose that $(d_I)_{I\in\mathcal{D}}$ is a scalar sequence that satisfies*

$$|||(d_I)_{I\in\mathcal{D}}||| := |d_{[0,1)}| + 2 \sum_{I\in\mathcal{D}\setminus\{[0,1)\}} |d_I - d_{\pi(I)}| < \infty.$$

Then the Haar multiplier $D\colon X_0(\mathbf{r}) \to X_0(\mathbf{r})$, defined as the linear extension of $Dh_I = d_I h_I$, $I \in \mathcal{D}$, is bounded:

$$\|D\| \le |||(d_I)_{I\in\mathcal{D}}|||. \tag{4.1}$$

Moreover, if the Rademacher sequence $\mathbf{r}$ is independent, then we have

$$\|D\| \le \sup_{I\in\mathcal{D}} |d_I|. \tag{4.2}$$

Remark 4.6. In contrast to [49], our upper bound (4.1) does not only involve the largest variation of $(d_I)_{I\in\mathcal{D}}$ along branches of the dyadic tree, but the sum of *all* differences between entries d_I and their predecessors $d_{\pi(I)}$. This larger upper bound is sufficient for our purposes.

Proof of Lemma 4.5. Firstly, under the condition that the sequence $\mathbf{r}$ is independent, estimate (4.2) follows immediately from the 1-unconditionality of the Haar system in $X_0(\mathbf{r})$. Secondly, considering that

$$|d_I| \le |d_{[0,1)}| + |d_I - d_{[0,1)}| \le |d_{[0,1)}| + \sum_{J:I\subset J\subsetneq[0,1)} |d_J - d_{\pi(J)}|, \qquad I \in \mathcal{D},$$

we only have to show (4.1) in the case where $\mathbf{r}$ is a constant sequence, i.e., $X_0(\mathbf{r}) = X_0$. To this end, let $x = \sum_{I\in\mathcal{D}} a_I h_I \in H_0$ and observe that

$$\begin{aligned}\|Dx - d_{[0,1)}x\|_X &= \Big\|\sum_{I\in\mathcal{D}}(d_I - d_{[0,1)})a_I h_I\Big\|_X = \Big\|\sum_{I\in\mathcal{D}}\ \sum_{J:I\subset J\subsetneq[0,1)} (d_J - d_{\pi(J)})a_I h_I\Big\|_X\\ &\le \sum_{J\subsetneq[0,1)} |d_J - d_{\pi(J)}| \cdot \Big\|\sum_{I:I\subset J} a_I h_I\Big\|_X.\end{aligned}$$

Now, for $n \in \mathbb{N}_0$, let $\mathbb{E}_n$ denote the conditional expectation with respect to the σ-algebra generated by $\mathcal{D}_n$. Then, using Lemma 4.3 and Proposition 4.1 (iv), we obtain

$$\begin{aligned}\|Dx - d_{[0,1)}x\|_X &= \sum_{n=1}^{\infty}\sum_{J\in\mathcal{D}_n} |d_J - d_{\pi(J)}| \cdot \|\chi_J \cdot (I_X - \mathbb{E}_n)x\|_X\\ &\le 2\sum_{J\subsetneq[0,1)} |d_J - d_{\pi(J)}| \cdot \|x\|_X,\end{aligned}$$

which shows (4.1). □

Lemma 4.7. *Let $X_0(\mathbf{r}) \in \mathcal{HH}_0(\delta)$, and let $(d_I)_{I\in\mathcal{D}} \in \{0,1\}^{\mathcal{D}}$ be a sequence with the property that for all $I \in \mathcal{D}$ with $d_I = 0$, we have $d_{I^+} = d_{I^-} = 0$. Then the operator $D\colon X_0(\mathbf{r}) \to X_0(\mathbf{r})$, defined as the continuous linear extension of $Dh_I = d_I h_I$, $I \in \mathcal{D}$, is a contraction, i.e.,*

$$\|Dx\|_{X(\mathbf{r})} \le \|x\|_{X(\mathbf{r})}, \qquad x \in X_0(\mathbf{r}).$$

Proof. If the sequence $\mathbf{r}$ is independent, the result follows immediately from the 1-unconditionality of the Haar system in $X_0(\mathbf{r})$. So we only have to prove the lemma in the case where $\mathbf{r}$ is a constant sequence, i.e., $X_0(\mathbf{r}) = X_0$. Moreover, we may assume that $d_{[0,1)} = 1$.

First, given $I \in \mathcal{D}$, we define $M(I) = \bigcap\{J \in \mathcal{D} : J \supset I,\ d_J = 1\} \in \mathcal{D}$. Let $x \in H_0$ be given by $x = \sum_{I\in\mathcal{D}_{\leq n}} a_I h_I$ for some natural number n. Put $\mathcal{M} = \{M(I) : I \in \mathcal{D}_n\}$ and observe that $\mathcal{M}$ is a partition of $[0,1)$ and that for every $J \in \mathcal{M}$, the function Dx is constant on J^+ and on J^- since $d_I = 0$ for all $I \in \mathcal{D}_{\leq n}$ with $I \subsetneq J$. Now let $\mathcal{F}$ be the σ-algebra on $[0,1)$ generated by the intervals $J^\pm$, $J \in \mathcal{M}$, and note that since

$$x - Dx = \sum_{J\in\mathcal{M}} \sum_{\substack{I\in\mathcal{D}_{\leq n}\\ I\subsetneq J}} a_I h_I,$$

it follows that $Dx = \mathbb{E}^{\mathcal{F}} x$. Thus, by Lemma 4.3, we have $\|Dx\|_X = \|\mathbb{E}^{\mathcal{F}} x\|_X \leq \|x\|_X$. □

Remark 4.8. Note that in the preceding lemma, Dx may be interpreted as a stopped martingale with respect to the dyadic filtration, where the stopping time at $s \in [0,1)$ is determined by the index I at which the sequence $(d_I : I \in \mathcal{D},\ s \in I)$ switches from 1 to 0.

Next, we prove that for every Haar system Hardy space $X(\mathbf{r}) \in \mathcal{HH}(\delta)$, the closed subspace $X_0(\mathbf{r})$ is isomorphic to $X(\mathbf{r})$.

Lemma 4.9. *Let $X_0(\mathbf{r}) \in \mathcal{HH}_0(\delta)$ and put*

$$E = [h_I : I \in \mathcal{D},\ \inf I = 0],$$
$$E_0 = [h_I : I \in \mathcal{D},\ \inf I = 0,\ \sup I \leq \tfrac{1}{2}].$$

Then the operator $W\colon E \to E_0$ defined by

$$\sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} a_I h_I \mapsto \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} a_I h_{I^+}$$

satisfies the estimates

$$\frac{1}{2}\|x\|_{X(\mathbf{r})} \leq \|Wx\|_{X(\mathbf{r})} \leq \|x\|_{X(\mathbf{r})}, \qquad x \in X_0(\mathbf{r}).$$

Proof. Let $x = \sum_{I\in\mathcal{D}} a_I h_I \in E$ be a finite linear combination of the Haar system and observe that the two functions

$$\begin{aligned} &s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_I(s) \Big| \,\mathrm{d}u,\\ &s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s) + \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s-1/2) \Big| \,\mathrm{d}u \end{aligned} \tag{4.3}$$

are equimeasurable. Moreover, we note that the two terms of the second function in (4.3) are disjointly supported. Hence, by Definition 2.1 and Proposition 4.1, we obtain

$$\|x\|_{X(\mathbf{r})} = \Big\| s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s) + \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s-1/2) \Big| \,\mathrm{d}u \Big\|_X \geq \|Wx\|_{X(\mathbf{r})}.$$

For the other inequality, note that the functions

$$s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s) \Big| \,\mathrm{d}u \qquad \text{and} \qquad s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u) a_I h_{I^+}(s-1/2) \Big| \,\mathrm{d}u$$

are equimeasurable, and hence, by Definition 2.1, we obtain

$$\|Wx\|_{X(\mathbf{r})} \geq \frac{1}{2}\Big\| s \mapsto \int_0^1 \Big| \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u)a_I h_{I^+}(s) + \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} r_I(u)a_I h_{I^+}(s-1/2)\Big| \,\mathrm{d}u\Big\|_X$$
$$= \frac{1}{2}\|x\|_{X(\mathbf{r})}. \qquad \square$$

Proposition 4.10. *Let $X(\mathbf{r}) \in \mathcal{HH}(\delta)$. Then the spaces $X_0(\mathbf{r})$ and $X(\mathbf{r})$ are* 162*-isomorphic to each other.*

Proof. We are going to prove that $X_0(\mathbf{r})$ contains a complemented subspace that is isomorphic to its hyperplanes. By Lemma 4.7, the projection $P\colon X_0(\mathbf{r}) \to X_0(\mathbf{r})$, given by

$$\sum_{I\in\mathcal{D}} a_I h_I \mapsto \sum_{\substack{I\in\mathcal{D}\\ \inf I=0}} a_I h_I,$$

is a well-defined contraction. Put

$$E = P(X_0(\mathbf{r})) = [h_I : I \in \mathcal{D},\ \inf I = 0],$$
$$E_0 = [h_I : I \in \mathcal{D},\ \inf I = 0,\ \sup I \leq \tfrac{1}{2}].$$

Then E is a 1-complemented subspace of $X_0(\mathbf{r})$, and hence, $X_0(\mathbf{r})$ is 3-isomorphic to $E\oplus F$ for some F. We know from Lemma 4.9 that E is 2-isomorphic to its hyperplane E_0. Moreover, it is clear that $X(\mathbf{r})$ is 3-isomorphic to $X_0(\mathbf{r}) \oplus \mathbb{R}$ and E is 3-isomorphic to $E_0 \oplus \mathbb{R}$. Thus, we obtain

$$X(\mathbf{r}) \overset{3}{\sim} X_0(\mathbf{r}) \oplus \mathbb{R} \overset{3}{\sim} E \oplus F \oplus \mathbb{R} \overset{2}{\sim} E_0 \oplus F \oplus \mathbb{R} \overset{3}{\sim} E \oplus F \overset{3}{\sim} X_0(\mathbf{r}), \tag{4.4}$$

which completes the proof. $\square$

Remark 4.11. It follows from the proof of Proposition 4.10 that an isomorphism $S\colon X(\mathbf{r}) \to X_0(\mathbf{r})$ is given by the continuous linear extension of

$$Sh_I = \begin{cases} h_{[0,1)}, & I = \varnothing,\\ h_{I^+}, & I \in \mathcal{D},\ \inf I = 0,\\ h_I, & I \in \mathcal{D},\ \inf I \neq 0. \end{cases}$$

In fact, a more detailed analysis of (4.4) shows that we always have $\|S\| \leq 9$ and $\|S^{-1}\| \leq 18$.

5. Faithful Haar systems

We will now discuss *faithful Haar systems*, a term which was coined in [29]. A faithful Haar system is a system of functions on $[0, 1)$ which are blocks of the Haar system and share many structural properties with the original Haar system. These and more generalized systems were used extensively throughout the last decades. In particular, we would like to highlight the classical works of Gamlen-Gaudet [13], Enflo-Maurey [34], Alspach-Enflo-Odell [1] and Maurey [35]. In order to ensure that the orthogonal projection onto such a generalized system is bounded on BMO, P. W. Jones [18] found conditions which are nowadays referred to as *Jones' compatibility conditions* (see also [42, p. 105]). For variants of Jones' conditions, see e.g. [41, 39, 23].

We will now introduce $(\varkappa_I)_{I\in\mathcal{D}}$*-faithful Haar systems*, which, loosely speaking, allow for small gaps (in contrast to faithful Haar systems).

Definition 5.1. Let $\mathcal{B}_I$ be a finite subcollection of $\mathcal{D}$ for each $I \in \mathcal{D}$, and let $(\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm1\}^{\mathcal{D}}$ be a sequence of signs. Put $\tilde{h}_I = \sum_{K\in\mathcal{B}_I} \varepsilon_K h_K$, $I \in \mathcal{D}$. Moreover, let $(\varkappa_I)_{I\in\mathcal{D}}$ be a sequence of positive numbers with $0 < \varkappa_I \leq 1$ for all $I \in \mathcal{D}$. We say that $(\tilde{h}_I)_{I\in\mathcal{D}}$ is a $(\varkappa_I)_{I\in\mathcal{D}}$-*faithful Haar system* if the following conditions are satisfied:

(i) Each collection $\mathcal{B}_I$, $I \in \mathcal{D}$, consists of pairwise disjoint dyadic intervals, and we have $\mathcal{B}_I \cap \mathcal{B}_J = \emptyset$ for all $I \neq J \in \mathcal{D}$.
(ii) For every $I \in \mathcal{D}$, we have $\mathcal{B}^*_{I\pm} \subset \{\tilde{h}_I = \pm1\}$ and $|\mathcal{B}^*_{I\pm}| \geq \varkappa_I \cdot \frac{1}{2}|\mathcal{B}^*_I|$.

If $\mathcal{B}^*_{[0,1)} = [0,1)$ and $\varkappa_I = 1$ for all $I \in \mathcal{D}$, then we simply say that $(\tilde{h}_I)_{I\in\mathcal{D}}$ is *faithful*, and in this case, we will usually denote the system by $(\hat{h}_I)_{I\in\mathcal{D}}$. If a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system $(\tilde{h}_I)_{I\in\mathcal{D}}$ additionally satisfies $\mathcal{B}_I \subset \mathcal{D}_{n_I}$, $I \in \mathcal{D}$, for a strictly increasing sequence $(n_I)_{I\in\mathcal{D}}$ of non-negative integers, then we say that it is *relative to the frequencies* $(n_I)_{I\in\mathcal{D}}$.

Remark 5.2. We will summarize elementary yet important properties of faithful Haar systems $(\hat{h}_I)_{I\in\mathcal{D}}$. Our first observation is that

$$|\mathcal{B}^*_I| = |I| \quad \text{and} \quad \mathcal{B}^*_{I\pm} = \{\hat{h}_I = \pm1\}, \qquad I \in \mathcal{D}.$$

Any faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ and the standard Haar system $(h_I)_{I\in\mathcal{D}}$ are distributionally equivalent, i.e., if $(a_I)_{I\in\mathcal{D}}$ is a sequence of scalars with $a_I \neq 0$ for at most finitely many $I \in \mathcal{D}$, then the functions $\sum_{I\in\mathcal{D}} a_I \hat{h}_I$ and $\sum_{I\in\mathcal{D}} a_I h_I$ have the same distribution. Moreover, for each $n \in \mathbb{N}_0$, the sets $\mathcal{B}^*_I$, $I \in \mathcal{D}_n$, form a partition of $[0,1)$, and we have the following equation relating the local and global properties of the system $(\hat{h}_I)_{I\in\mathcal{D}}$:

$$\frac{|K \cap \mathcal{B}^*_J|}{|\mathcal{B}^*_J|} = \frac{|K|}{|I|}, \qquad K \in \mathcal{B}_I,\ J \subset I \in \mathcal{D}. \tag{5.1}$$

A general $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system may violate equation (5.1). In some versions of Jones' compatibility conditions, this equation is replaced by an inequality.

Next, we introduce some convenient notation for collections of dyadic intervals.

Notation 5.3. Let $\mathcal{A} \subset \mathcal{D}$.

(i) We set

$$\mathcal{G}_0(\mathcal{A}) = \{I \in \mathcal{A} : I \text{ is maximal with respect to inclusion}\}.$$

(ii) For $n \in \mathbb{N}$, we recursively define the collections

$$\mathcal{G}_n(\mathcal{A}) = \mathcal{G}_0\Big(\mathcal{A} \setminus \bigcup_{k=0}^{n-1} \mathcal{G}_k(\mathcal{A})\Big).$$

(iii) We say that $\mathcal{A}$ *has finite generations* if $\mathcal{G}_n(\mathcal{A})$ is finite for every $n \in \mathbb{N}_0$.
(iv) We put $\limsup(\mathcal{A}) = \bigcap_{n=0}^{\infty} \mathcal{G}^*_n(\mathcal{A})$, where $\mathcal{G}^*_n(\mathcal{A}) := \mathcal{G}_n(\mathcal{A})^*$ for all $n \in \mathbb{N}_0$.

Note that for every $n \in \mathbb{N}_0$, the elements of $\mathcal{G}_n(\mathcal{A})$ are pairwise disjoint. Moreover, note that if $n \geq 1$ and $I \in \mathcal{G}_n(\mathcal{A})$, then there exists a unique dyadic interval $J \in \mathcal{G}_{n-1}(\mathcal{A})$ such that $I \subset J^+$ or $I \subset J^-$. Hence, we have $\mathcal{G}^*_n(\mathcal{A}) \subset \mathcal{G}^*_{n-1}(\mathcal{A})$ for all $n \geq 1$. Figure 1 shows the collections $\mathcal{G}_0(\mathcal{A}), \mathcal{G}_1(\mathcal{A})$ and $\mathcal{G}_2(\mathcal{A})$ for a specific choice of $\mathcal{A} \subset \mathcal{D}$.

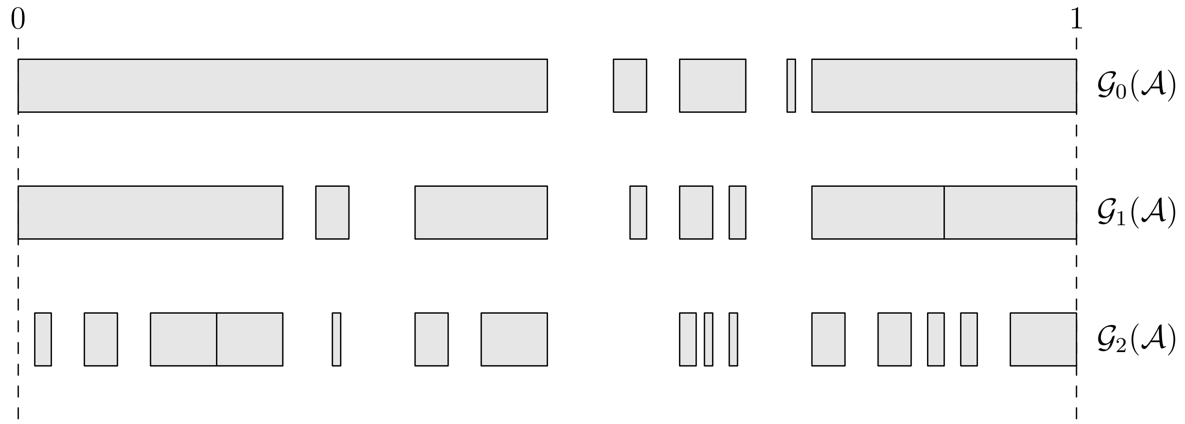

FIGURE 1. The collections $\mathcal{G}_n(\mathcal{A})$

Remark 5.4. Suppose that $\hat{\mathcal{A}} \subset \mathcal{D}$ has finite generations and that $\mathcal{G}_n^*(\hat{\mathcal{A}}) = [0, 1)$ for all $n \in \mathbb{N}_0$. Moreover, let $(\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ be a sequence of signs. Then we can construct a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ by putting

$$\hat{h}_I = \sum_{K\in\hat{\mathcal{B}}_I} \varepsilon_K h_K,$$

where $\hat{\mathcal{B}}_{[0,1)} = \mathcal{G}_0(\hat{\mathcal{A}})$ and

$$\hat{\mathcal{B}}_{I^{\pm}} = \big\{K \in \mathcal{G}_{n+1}(\hat{\mathcal{A}}) : K \subset \{\hat{h}_I = \pm 1\}\big\}, \qquad I \in \mathcal{D}_n,\ n \in \mathbb{N}_0.$$

Next, we show that every $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system can be extended to a faithful Haar system by adding additional Haar functions which “fill the gaps”. This is illustrated in Figure 2. Moreover, we prove that there exists a Haar multiplier with norm 1 which maps the new system to the original one (see Lemma 5.6, below).

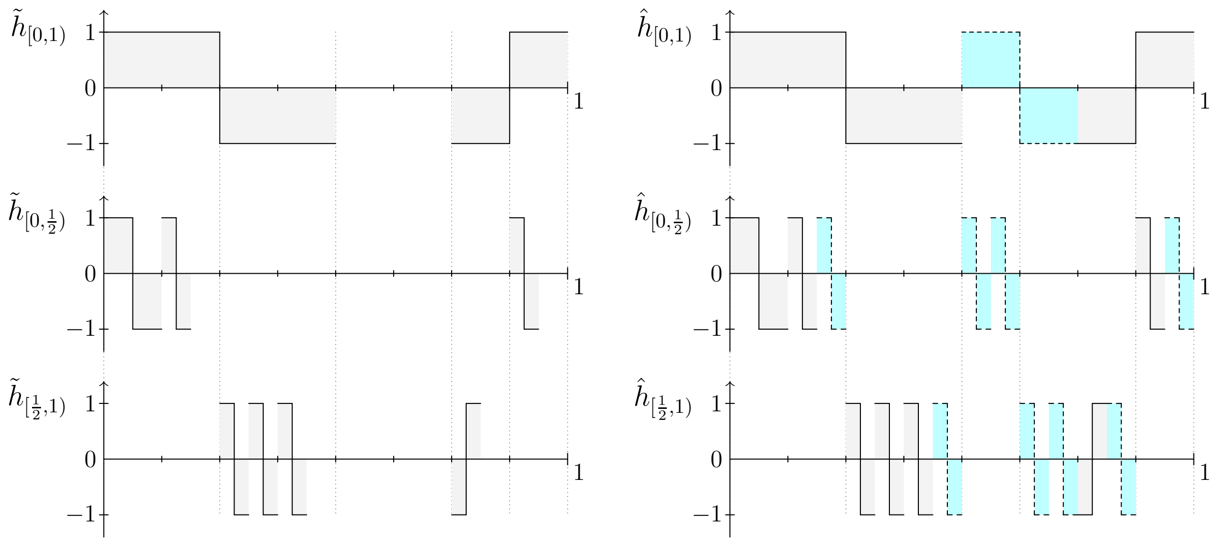

FIGURE 2. The first three functions of a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system $(\tilde{h}_I)_{I\in\mathcal{D}}$ which is extended to a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ by adding the dashed Haar functions with the light blue shading

Lemma 5.5. *Let $Y \in \mathcal{HH}_0(\delta)$ and suppose that $\mathcal{A} \subset \mathcal{D}$ has finite generations. Then there exists another collection $\hat{\mathcal{A}} \subset \mathcal{D}$ which has finite generations such that $\mathcal{G}_n^*(\hat{\mathcal{A}}) = [0, 1)$*

and $\mathcal{G}_n(\mathcal{A}) \subset \mathcal{G}_n(\hat{\mathcal{A}})$ *for all* $n \in \mathbb{N}_0$*. Moreover, there exists a bounded Haar multiplier* $R\colon Y \to Y$ *with* $\|R\| \le 1$ *such that for every* $n \in \mathbb{N}_0$ *and* $K \in \mathcal{G}_n(\hat{\mathcal{A}})$*, we have*

$$Rh_K = \begin{cases} h_K, & K \in \mathcal{G}_n(\mathcal{A}), \\ 0, & K \in \mathcal{G}_n(\hat{\mathcal{A}}) \setminus \mathcal{G}_n(\mathcal{A}). \end{cases} \tag{5.2}$$

Proof. Since $\mathcal{G}_0^*(\mathcal{A})$ is a finite union of dyadic intervals, there exists $n_0 \in \mathbb{N}_0$ such that the complement $[0,1)\setminus\mathcal{G}_0^*(\mathcal{A})$ is a disjoint union of finitely many intervals from $\mathcal{D}_{n_0}$. By adding these intervals to $\mathcal{G}_0(\mathcal{A})$, we obtain a finite collection $\hat{\mathcal{G}}_0 \subset \mathcal{D}$ with $\hat{\mathcal{G}}_0^* = [0,1)$. Next, since $\mathcal{G}_1^*(\mathcal{A})$ is a finite union of dyadic intervals, we can find $n_1 \in \mathbb{N}_0$ such that $[0,1) \setminus \mathcal{G}_1^*(\mathcal{A})$ is a disjoint union of finitely many intervals in $\mathcal{D}_{n_1}$. By adding these intervals to $\mathcal{G}_1(\mathcal{A})$, we obtain a collection $\hat{\mathcal{G}}_1$ with $\hat{\mathcal{G}}_1^* = [0,1)$, and by choosing n_1 sufficiently large, we can ensure that for each $K \in \hat{\mathcal{G}}_1$, there exists $L \in \hat{\mathcal{G}}_0$ such that $K \subset L^+$ or L^-. By continuing this process, we obtain a sequence of collections $(\hat{\mathcal{G}}_n)_{n\in\mathbb{N}_0}$. Then the collection $\hat{\mathcal{A}} := \bigcup_{n=0}^{\infty} \hat{\mathcal{G}}_n$ has the desired properties since $\mathcal{G}_n(\hat{\mathcal{A}}) = \hat{\mathcal{G}}_n$ for all $n \in \mathbb{N}_0$.

Now we define the sequence $(\rho_I)_{I\in\mathcal{D}}$ as

$$\rho_I = \begin{cases} 1, & \text{if } I \supset K \text{ for some } K \in \mathcal{A}, \\ 0, & \text{else.} \end{cases}$$

Observe that if $\rho_I = 1$ for some $I \in \mathcal{D}$, then it follows that $\rho_J = 1$ for all $J \supset I$. Hence, this sequence satisfies the conditions of Lemma 4.7. The corresponding Haar multiplier $R\colon Y \to Y$ defined by $Rh_I = \rho_I h_I$, $I \in \mathcal{D}$, satisfies $\|R\| \le 1$ and $Rh_K = h_K$ for all $K \in \mathcal{A}$. Moreover, if $L \in \mathcal{G}_n(\hat{\mathcal{A}}) \setminus \mathcal{G}_n(\mathcal{A})$ for some $n \in \mathbb{N}_0$, then we have $Rh_L = 0$ because on the one hand, we cannot have $L \supset K$ for any $K \in \mathcal{G}_m(\mathcal{A})$ with $m < n$, and on the other hand, $K \in \bigcup_{m=n}^{\infty} \mathcal{G}_m(\mathcal{A})$ implies that $K \subset \mathcal{G}_n^*(\mathcal{A})$, and so K is disjoint from L. □

Lemma 5.6. *Let* $Y \in \mathcal{HH}_0(\delta)$ *and suppose that* $(\tilde{h}_I)_{I\in\mathcal{D}}$ *is a* $(\varkappa_I)_{I\in\mathcal{D}}$*-faithful Haar system for some sequence* $(\varkappa_I)_{I\in\mathcal{D}}$ *of scalars in* $(0,1]$*. Then there exists a faithful Haar system* $(\hat{h}_I)_{I\in\mathcal{D}}$ *and a Haar multiplier* $R\colon Y \to Y$ *with* $\|R\| \le 1$ *such that* $R\hat{h}_I = \tilde{h}_I$ *for all* $I \in \mathcal{D}$*.*

Proof. Write $\tilde{h}_I = \sum_{K\in\mathcal{B}_I} \varepsilon_K h_K$, where $\mathcal{B}_I \subset \mathcal{D}$, $I \in \mathcal{D}$, and $(\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$. Then the collection $\mathcal{A} := \bigcup_{I\in\mathcal{D}} \mathcal{B}_I$ has finite generations. By Lemma 5.5, there exists another collection $\hat{\mathcal{A}} \subset \mathcal{D}$ with finite generations such that $\mathcal{G}_n^*(\hat{\mathcal{A}}) = [0,1)$ and $\mathcal{G}_n(\mathcal{A}) \subset \mathcal{G}_n(\hat{\mathcal{A}})$ for all $n \in \mathbb{N}_0$. Moreover, there exists a bounded Haar multiplier $R\colon Y \to Y$ with $\|R\| \le 1$ such that equation (5.2) is satisfied for all $n \in \mathbb{N}_0$ and $K \in \mathcal{G}_n(\hat{\mathcal{A}})$. Now let the faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ and the associated collections $(\hat{\mathcal{B}}_I)_{I\in\mathcal{D}}$ be defined as in Remark 5.4 using the collection $\hat{\mathcal{A}}$ and the signs $(\varepsilon_K)_{K\in\mathcal{D}}$. Clearly, we have $\hat{\mathcal{B}}_I \cap \mathcal{G}_n(\mathcal{A}) = \mathcal{B}_I$ for all $I \in \mathcal{D}_n$, $n \in \mathbb{N}_0$. Thus, it follows from equation (5.2) that $R\hat{h}_I = \tilde{h}_I$ for all $I \in \mathcal{D}$. □

6. Asymptotically curved Banach spaces

In this section, we discuss asymptotically curved Banach spaces and uniformly asymptotically curved sequences of Banach spaces (see Definition 2.10). We will need the following additional concepts.

Definition 6.1. Let $(x_j)_{j=1}^{\infty}$ denote a sequence in a Banach space E and let $1 < \tau < \infty$. We say that $(x_j)_{j=1}^{\infty}$ satisfies an *upper* τ*-estimate (in* E*) (with constant* $C > 0$*)* if

$$\Big\|\sum_{j=1}^{n} x_j\Big\|_E \le C\Big(\sum_{j=1}^{n} \|x_j\|_E^{\tau}\Big)^{1/\tau}, \qquad n \in \mathbb{N}.$$

We say that $(x_j)_{j=1}^\infty$ satisfies an *upper* ∞*-estimate (in* E*) (with constant* $C > 0$*)* if

$$\Big\|\sum_{j=1}^n x_j\Big\|_E \leq C \max_{1\leq j\leq n} \|x_j\|_E, \qquad n \in \mathbb{N}.$$

Next, we recall the notion of (Rademacher) type for a Banach space (see, e.g., [32, Definition 1.e.12]).

Definition 6.2. Let E be a Banach space, and let $1 < \tau \leq 2$. We say that E *is of (Rademacher) type* τ if there exists a constant $C > 0$ such that for every finite sequence of vectors $(x_j)_{j=1}^n$ in E, we have

$$\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|_E \,\mathrm{d}t \leq C\Big(\sum_{j=1}^n \|x_j\|_E^\tau\Big)^{1/\tau}.$$

If this holds, we say that E is of (Rademacher) type τ *with constant* C.

The proof of the following lemma is both elementary and straightforward, and therefore omitted.

Lemma 6.3. *Let* $(e_j)_{j=1}^\infty$ *denote a Schauder basis of a Banach space* E *and let* $C > 0$*. Then the following statements are true:*

- *(i) Suppose that* $(e_j)_{j=1}^\infty$ *is* C*-unconditional and* E *has Rademacher type* τ *with constant* C *for some* $1 < \tau \leq 2$*, then every block basis* $(x_j)_{j=1}^\infty$ *of* $(e_j)_{j=1}^\infty$ *satisfies an upper* τ*-estimate with constant* C^2*.*
- *(ii) Suppose that every bounded block basis* $(x_j)_{j=1}^\infty$ *of* $(e_j)_{j=1}^\infty$ *satisfies an upper* τ*-estimate for some* $1 < \tau \leq \infty$*, then* E *is asymptotically curved (with respect to* $(e_j)_{j=1}^\infty$*).*

The following uniform version of Lemma 6.3 (ii) is taken from [28].

Lemma 6.4. *For each* $k \in \mathbb{N}$*, let* $(e_{k,j})_{j=1}^\infty$ *denote a Schauder basis of a Banach space* E_k*. Moreover, let* $1 < \tau \leq \infty$ *and* $C > 0$*, and suppose that for each* $k \in \mathbb{N}$*, every bounded block basis of* $(e_{k,j})_{j=1}^\infty$ *satisfies an upper* τ*-estimate in* E_k *with constant* C*. Then* $(E_k)_{k=1}^\infty$ *is uniformly asymptotically curved with respect to the array* $(e_{k,j})_{k,j=1}^\infty$*.*

Another way to obtain a uniformly asymptotically curved sequence of Banach spaces is by repeatedly taking the same asymptotically curved space, thus forming a constant sequence. This is proved in the following lemma.

Lemma 6.5. *Let* E *be a Banach space with a Schauder basis* $(e_j)_{j=1}^\infty$ *and suppose that* E *is asymptotically curved with respect to* $(e_j)_{j=1}^\infty$*. Put* $e_{k,j} = e_j$ *for all* $k \in \mathbb{N}$*. Then* $(E, E, \dots)$ *is uniformly asymptotically curved with respect to* $(e_{k,j})_{k,j=1}^\infty$*.*

Proof. Let $(x_{k,j})_{k,j=1}^\infty$ be an array such that for every $k \in \mathbb{N}$, $(x_{k,j})_{j=1}^\infty$ is a block basis of $(e_j)_{j=1}^\infty$, and such that for some $C > 0$, we have

$$\|x_{k,j}\|_E \leq C, \qquad k, j \in \mathbb{N}.$$

We have to show that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{n}\Big\|\sum_{j=1}^n x_{k,j}\Big\|_E = 0.$$

Assume for a contradiction that there exist $\varepsilon > 0$ and sequences $(n_i)_{i=1}^\infty$ and $(k_i)_{i=1}^\infty$ of natural numbers such that $(n_i)_{i=1}^\infty$ is strictly increasing and

$$\frac{1}{n_i}\Big\|\sum_{j=1}^{n_i} x_{k_i,j}\Big\|_E \geq \varepsilon, \qquad i \in \mathbb{N}. \tag{6.1}$$

Put $l_i = \max\operatorname{supp} x_{k_i,n_i}$ for all $i \in \mathbb{N}$. By passing to a subsequence, we may assume that $n_i \geq (1+4C/\varepsilon)l_{i-1}$ for all $i \geq 2$. We will now construct a bounded block basis $(y_j)_{j=1}^\infty$ of $(e_j)_{j=1}^\infty$ and a strictly increasing sequence $(N_i)_{i=1}^\infty$ of natural numbers such that

$$\frac{1}{N_i}\Big\|\sum_{j=1}^{N_i} y_j\Big\|_E \geq \frac{\varepsilon}{2}, \qquad i \in \mathbb{N}, \tag{6.2}$$

thus contradicting the hypothesis that E is asymptotically curved with respect to $(e_j)_{j=1}^\infty$.

Put $N_1 = n_1$ and

$$y_1 = x_{k_1,1}, \dots, y_{n_1} = x_{k_1,n_1}.$$

By (6.1), inequality (6.2) holds for $i = 1$. Now let $i \geq 2$ and assume that we have already chosen $N_1, \dots, N_{i-1}$ and picked $y_1, \dots, y_{N_{i-1}} \in \{x_{k,j} : k, j \in \mathbb{N}\}$ such that $(y_1, \dots, y_{N_{i-1}})$ is a finite block basis of $(e_j)_{j=1}^\infty$ and such that $y_{N_{i-1}} = x_{k_{i-1},n_{i-1}}$. Consider the vectors $x_{k_i,1}, \dots, x_{k_i,n_i}$. If we skip the first l_{i-1} of these vectors, then the supports of the remaining vectors are clearly subsets of $\{l_{i-1}+1, l_{i-1}+2, \dots\}$. Thus, if we define $N_i = N_{i-1} + n_i - l_{i-1} > N_{i-1}$ and

$$y_{N_{i-1}+1} = x_{k_i,l_{i-1}+1}, \dots, y_{N_i} = x_{k_i,n_i},$$

then $(y_1, \dots, y_{N_i})$ is a finite block basis of $(e_j)_{j=1}^\infty$. Observe that $N_{i-1} \leq \max\operatorname{supp} y_{N_{i-1}} = l_{i-1}$ and hence $N_i \leq n_i$. Using these observations, exploiting that $\|y_j\|_E \leq C$ for all j, and using (6.1), we obtain

$$\begin{aligned}\Big\|\sum_{j=1}^{N_i} y_j\Big\|_E &\geq \Big\|\sum_{j=N_{i-1}+1}^{N_i} y_j\Big\|_E - CN_{i-1} = \Big\|\sum_{j=l_{i-1}+1}^{n_i} x_{k_i,j}\Big\|_E - CN_{i-1}\\ &\geq \Big\|\sum_{j=1}^{n_i} x_{k_i,j}\Big\|_E - Cl_{i-1} - CN_{i-1} \geq \varepsilon n_i - 2\cdot Cl_{i-1}\\ &\geq \frac{\varepsilon}{2}n_i + \frac{\varepsilon}{2}\cdot\frac{4C}{\varepsilon}l_{i-1} - 2Cl_{i-1} \geq \frac{\varepsilon}{2}N_i,\end{aligned}$$

which proves (6.2). □

Next, we provide some conditions under which Haar system Hardy spaces and sequences of such spaces are (uniformly) asymptotically curved. Clearly, it does not make a difference whether we consider Haar system Hardy spaces spaces $X(\mathbf{r}) \in \mathcal{HH}(\delta)$ equipped with the Haar basis $(h_I)_{I\in\mathcal{D}^+}$ or their closed subspaces $X_0(\mathbf{r})$ equipped with the Haar basis $(h_I)_{I\in\mathcal{D}}$.

Proposition 6.6. *Let $1 \leq p < \infty$, let $\mathbf{r} \in \mathcal{R}$, and put $X = L^p$ if $1 \leq p < \infty$ and $X = [h_I]_{I\in\mathcal{D}^+} \subset L^\infty$ if $p = \infty$. Then the space $X(\mathbf{r})$ is asymptotically curved with respect to the Haar system $(h_I)_{I\in\mathcal{D}^+}$ if and only if*

$$1 < p < \infty \qquad \text{or} \qquad p = \infty \text{ and } \mathbf{r} \text{ is independent.}$$

Proof. Considering a sequence of disjointly supported functions with norm 1 which are blocks of the Haar system shows that for $X_1 = L^1$ and $\mathbf{r} \in \mathcal{R}$, the space $X_1(\mathbf{r}) \in \mathcal{HH}(\delta)$ is not asymptotically curved with respect to the Haar basis. Moreover, since any independent sequence of Rademacher functions in L^∞ is equivalent to the unit vector basis of ℓ^1, we find that $X_2 = [h_I]_{I\in\mathcal{D}^+} \subset L^\infty$ is also not asymptotically curved.

However, if $\mathbf{r}$ is an independent sequence, then $X_2(\mathbf{r})$ is in fact asymptotically curved with respect to the Haar basis according to Lemma 6.3 (ii), as every bounded block basis of $(h_I)_{I\in\mathcal{D}^+}$ satisfies an upper 2-estimate (cf. [23, Lemma 4.2]). To see this, let $(x_j)_{j=1}^\infty$

denote a bounded block basis of $(h_I)_{I\in\mathcal{D}^+}$ and observe that for all $n \in \mathbb{N}$

$$\Big\|\sum_{j=1}^n x_j\Big\|_{X_2(\mathbf{r})} = \sup_t \int_0^1 \Big|\sum_{j=1}^n \sum_{K\in\mathcal{D}^+} r_K(u)\frac{\langle h_K, x_j\rangle}{|K|}h_K(t)\Big|\,\mathrm{d}u$$

$$\le \sup_t \Big(\sum_{j=1}^n \sum_{K\in\mathcal{D}^+}\Big(\frac{\langle h_K, x_j\rangle}{|K|}h_K(t)\Big)^2\Big)^{1/2}$$

$$\le \Big(\sum_{j=1}^n \sup_t \sum_{K\in\mathcal{D}^+}\Big(\frac{\langle h_K, x_j\rangle}{|K|}h_K(t)\Big)^2\Big)^{1/2}.$$

We conclude this argument by Khintchine's inequality, which yields

$$\sup_t \Big(\sum_{K\in\mathcal{D}^+}\Big(\frac{\langle h_K, x_I\rangle}{|K|}h_K(t)\Big)^2\Big)^{1/2} \le C\|x_I\|_{X_2(\mathbf{r})}$$

for some absolute constant $C > 0$.

Finally, let $1 < p < \infty$, $\mathbf{r} \in \mathcal{R}$, and put $X_3 = L^p$. Recall that by Remark 2.3, the identity operator is an isomorphism between $X_3(\mathbf{r})$ and X_3. Since L^p has Rademacher type $\min(2, p)$, we record that by Lemma 6.3, $X_3(\mathbf{r})$ is asymptotically curved as well. □

Remark 6.7. If $X(\mathbf{r}) \in \mathcal{HH}(\delta)$ is an arbitrary Haar system Hardy space, then one can use Lemma 6.3 to check if $X(\mathbf{r})$ is asymptotically curved, either by verifying the condition in (ii) directly or by applying (i) if it is known that $X(\mathbf{r})$ has Rademacher type $\tau > 1$ and that the Haar system is an unconditional basis of $X(\mathbf{r})$. Recall that the Haar basis is always 1-unconditional in $X(\mathbf{r})$ if $\mathbf{r}$ is an independent sequence of Rademacher functions. Moreover, according to [32, Theorem 2.c.6], the Haar system is unconditional in a separable r.i. function space X on $[0, 1)$ if and only if X has non-trivial Boyd indices.

Remark 6.8. Finally, consider a sequence of Haar system Hardy spaces $(Z_k)_{k=1}^\infty$, and for each $k \in \mathbb{N}$, let $(h_{k,I})_{I\in\mathcal{D}^+}$ denote the Haar basis of Z_k. Suppose that either of the following conditions is satisfied:

- ▷ There exist $1 < \tau \le 2$ and a uniform constant $C > 0$ such that for every k, the Haar system is C-unconditional in Z_k and the space Z_k has Rademacher type τ with constant C.
- ▷ We have $Z_k = Z_1$ for all $k \in \mathbb{N}$, and Z_1 is asymptotically curved with respect to the Haar system.

Then $(Z_k)_{k=1}^\infty$ is uniformly asymptotically curved with respect to $(h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}^+}$. This follows from Lemma 6.3 together with Lemma 6.4 in the first case and from Lemma 6.5 in the second case.

7. Embeddings and projections on Haar system Hardy spaces

In this section, we will define the fundamental operators A and B associated with a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system, and we will prove that they are bounded if the numbers $\varkappa_I$ are sufficiently small. In doing so, we will lay the foundation for proving our main results.

Proposition 7.1. *Let $Y \in \mathcal{HH}_0(\delta)$. Let $(\hat{h}_I)_{I\in\mathcal{D}}$ be a faithful Haar system and define the operators $\hat{A}, \hat{B} \colon Y \to Y$ by*

$$\hat{B}x = \sum_{I\in\mathcal{D}} \frac{\langle h_I, x\rangle}{|I|}\hat{h}_I \qquad \textit{and} \qquad \hat{A}x = \sum_{I\in\mathcal{D}} \frac{\langle \hat{h}_I, x\rangle}{|I|}h_I. \tag{7.1}$$

Then we have $\hat{A}\hat{B} = I_Y$ and $\|\hat{A}\| = \|\hat{B}\| = 1$.

Before proving Proposition 7.1, we state the following direct consequence.

Corollary 7.2. *Let $\mathcal{B} \subset \mathcal{D}$ be a finite collection of pairwise disjoint dyadic intervals, and let $(\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ be a sequence of signs. Then we have*

$$\Big\|\sum_{K\in\mathcal{B}} \varepsilon_K h_K\Big\|_{Y^*} \leq 1.$$

Proof of Corollary 7.2. The proof follows either by an elementary direct computation or, alternatively, by exploiting the estimate for $\|\hat{A}\|$ in Proposition 7.1 and using Lemma 4.7 (see also Lemma 5.6). □

Proof of Proposition 7.1. Write $Y = X_0(\mathbf{r})$ for suitable $X \in \mathcal{H}(\delta)$ and $\mathbf{r} \in \mathcal{R}$. Suppose that our faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ is given by

$$\hat{h}_I = \sum_{K\in\mathcal{B}_I} \varepsilon_K h_K, \qquad I \in \mathcal{D},$$

where $\varepsilon = (\varepsilon_K)_{K\in\mathcal{D}}$ is a sequence of signs and $\mathcal{B}_I$ is a finite subset of $\mathcal{D}$ for each $I \in \mathcal{D}$. In order to prove $\|\hat{B}\| = 1$, we fix $x = \sum_{I\in\mathcal{D}} a_I h_I \in H_0$, where $a_I \neq 0$ for at most finitely many $I \in \mathcal{D}$. Then we have

$$\hat{B}x = \sum_{I\in\mathcal{D}} a_I \hat{h}_I = \sum_{I\in\mathcal{D}} \sum_{K\in\mathcal{B}_I} a_I \varepsilon_K h_K$$

and hence

$$\|\hat{B}x\|_{X(\mathbf{r})} = \Big\|s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}} \sum_{K\in\mathcal{B}_I} r_K(u) a_I \varepsilon_K h_K(s)\Big|\, \mathrm{d}u\Big\|_X. \tag{7.2}$$

Now for fixed $s \in [0,1)$, note that by the faithfulness of $(\hat{h}_I)_{I\in\mathcal{D}}$, for every $I \in \mathcal{D}$ there exists at most one interval $K \in \mathcal{B}_I$ with $s \in h_K$. Thus, we may replace $r_K(u)$ by $r_I(u)$ in (7.2), obtaining

$$\begin{aligned}
\|\hat{B}x\|_{X(\mathbf{r})} &= \Big\|s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}} r_I(u) a_I \sum_{K\in\mathcal{B}_I} \varepsilon_K h_K(s)\Big|\, \mathrm{d}u\Big\|_X \\
&= \Big\|s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}} r_I(u) a_I \hat{h}_I(s)\Big|\, \mathrm{d}u\Big\|_X \\
&= \Big\|s \mapsto \int_0^1 \Big|\sum_{I\in\mathcal{D}} r_I(u) a_I h_I(s)\Big|\, \mathrm{d}u\Big\|_X = \|x\|_{X(\mathbf{r})}.
\end{aligned}$$

Next, we prove that $\|\hat{A}\| = 1$. Let $x \in H_0$ be defined as above and let $N \in \mathbb{N}$ be sufficiently large such that for all $K \in \bigcup_{I\in\mathcal{D}\setminus\mathcal{D}_{<N}} \mathcal{B}_I$, we have $a_K = 0$ (hence, we have $\langle \hat{h}_I, x\rangle = 0$ for all $I \in \mathcal{D} \setminus \mathcal{D}_{<N}$). Let $\mathcal{F}$ denote the σ-algebra generated by the sets $\mathcal{B}_I^*$, $I \in \mathcal{D}_N$. We will show that for every $K \in \mathcal{D}$, we have

$$\mathbb{E}^{\mathcal{F}} h_K = \begin{cases} 0, & K \in \mathcal{D} \setminus \bigcup_{I\in\mathcal{D}_{<N}} \mathcal{B}_I, \\ \varepsilon_K \frac{|K|}{|I|} \hat{h}_I, & K \in \mathcal{B}_I,\ I \in \mathcal{D}_{<N}. \end{cases} \tag{7.3}$$

For every $I \in \mathcal{D}_N$, we have the following expansion, which is completely analogous to its well-known counterpart for the standard Haar system:

$$\chi_{\mathcal{B}_I^*} = 2^{-N}\chi_{[0,1)} + \sum_{\substack{J\in\mathcal{D}\\ J\supsetneq I}} \frac{2^{-N}}{|J|} h_J(I) \hat{h}_J,$$

where $h_J(I)$ is the value that h_J assumes on I. Thus, for all $K \in \mathcal{D} \setminus \bigcup_{J\in\mathcal{D}_{<N}} \mathcal{B}_J$, we have $\mathbb{E}^{\mathcal{F}} h_K = 0$. On the other hand, if $I \in \mathcal{D}_N$ and $K \in \mathcal{B}_{J_0}$ for some $J_0 \in \mathcal{D}_{<N}$ with $J_0 \supsetneq I$, then we have

$$\langle \chi_{\mathcal{B}_I^*}, h_K\rangle = \sum_{\substack{J\in\mathcal{D}\\ J\supsetneq I}} \frac{|I|}{|J|} h_J(I)\langle \hat{h}_J, h_K\rangle = \frac{|I|}{|J_0|} h_{J_0}(I)\varepsilon_K |K|.$$

This implies that

$$\mathbb{E}^{\mathcal{F}} h_K = \sum_{\substack{I\in\mathcal{D}_N\\ I\subset J_0}} \frac{1}{|I|}\langle \chi_{\mathcal{B}_I^*}, h_K\rangle \chi_{\mathcal{B}_I^*} = \varepsilon_K \frac{|K|}{|J_0|} \sum_{\substack{I\in\mathcal{D}_N\\ I\subset J_0}} h_{J_0}(I)\chi_{\mathcal{B}_I^*} = \varepsilon_K \frac{|K|}{|J_0|}\hat{h}_{J_0},$$

which completes the proof of (7.3).

Now observe that

$$\mathbb{E}^{\mathcal{F}} x = \sum_{I\in\mathcal{D}_{<N}} \sum_{K\in\mathcal{B}_I} a_K \varepsilon_K \frac{|K|}{|I|}\hat{h}_I = \sum_{I\in\mathcal{D}_{<N}} \frac{\langle \hat{h}_I, x\rangle}{|I|}\hat{h}_I = \hat{B}\hat{A}x.$$

We define $\mathcal{C} = \mathcal{D} \setminus \bigcup_{I\in\mathcal{D}_{<N}} \mathcal{B}_I$ and split the function x into two parts accordingly:

$$\begin{aligned}\|x\|_{X(\mathbf{r})} &= \Big\| s \mapsto \int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} \sum_{K\in\mathcal{B}_I} r_K(u) a_K h_K(s) + \sum_{K\in\mathcal{C}} r_K(u) a_K h_K(s)\Big|\, \mathrm{d}u \Big\|_X\\ &= \Big\| s \mapsto \int_0^1\int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} r_I(u) \sum_{K\in\mathcal{B}_I} a_K h_K(s) + \sum_{K\in\mathcal{C}} r_K(v) a_K h_K(s)\Big|\, \mathrm{d}u\,\mathrm{d}v \Big\|_X,\end{aligned}$$

where we again replaced $r_K(u)$ by $r_I(u)$ in the first sum. Now we use Lemma 4.3, Jensen's inequality for conditional expectations and Proposition 4.1 (iv) to obtain

$$\|x\|_{X(\mathbf{r})} \geq \Big\| s \mapsto \int_0^1\int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} r_I(u) \sum_{K\in\mathcal{B}_I} a_K (\mathbb{E}^{\mathcal{F}} h_K)(s) + \sum_{K\in\mathcal{C}} r_K(v) a_K (\mathbb{E}^{\mathcal{F}} h_K)(s)\Big|\, \mathrm{d}u\,\mathrm{d}v \Big\|_X.$$

According to (7.3), we have $\mathbb{E}^{\mathcal{F}} h_K = \varepsilon_K \frac{|K|}{|I|}\hat{h}_I$ in the first sum and $\mathbb{E}^{\mathcal{F}} h_K = 0$ in the second sum. Hence, it follows that

$$\begin{aligned}\|x\|_{X(\mathbf{r})} &\geq \Big\| s \mapsto \int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} r_I(u) \Big(\sum_{K\in\mathcal{B}_I} \varepsilon_K a_K \frac{|K|}{|I|}\Big) \hat{h}_I(s)\Big|\, \mathrm{d}u \Big\|_X\\ &= \Big\| s \mapsto \int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} r_I(u) \frac{\langle \hat{h}_I, x\rangle}{|I|} \sum_{K\in\mathcal{B}_I} \varepsilon_K h_K(s)\Big|\, \mathrm{d}u \Big\|_X.\end{aligned}$$

Now we swap back $r_I(u)$ with $r_K(u)$ and obtain

$$\|x\|_{X(\mathbf{r})} \geq \Big\| s \mapsto \int_0^1 \Big| \sum_{I\in\mathcal{D}_{<N}} \frac{\langle \hat{h}_I, x\rangle}{|I|} \sum_{K\in\mathcal{B}_I} r_K(u)\varepsilon_K h_K(s)\Big|\, \mathrm{d}u \Big\| = \|\hat{B}\hat{A}x\|_{X(\mathbf{r})} = \|\hat{A}x\|_{X(\mathbf{r})}.$$

By extending both operators continuously from H_0 to Y, we obtain $\hat{A}, \hat{B} \colon Y \to Y$ as defined in (7.1), where both series converge in norm. □

Theorem 7.3. *Let $Y \in \mathcal{HH}_0(\delta)$. Suppose that $(\tilde{h}_I)_{I\in\mathcal{D}}$ is a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system for some family $(\kappa_I)_{I\in\mathcal{D}}$ of scalars in $(0,1]$ that satisfies*

$$\sigma := \sum_{I\in\mathcal{D}} (1 - \varkappa_I) < 1.$$

Define the operators $A, B\colon Y \to Y$ by

$$Bx = \sum_{I\in\mathcal{D}} \frac{\langle h_I, x\rangle}{\|h_I\|_2^2}\tilde{h}_I \qquad \text{and} \qquad Ax = \sum_{I\in\mathcal{D}} \frac{\langle \tilde{h}_I, x\rangle}{\|\tilde{h}_I\|_2^2}h_I, \qquad x\in Y.$$

Then we have $AB = I_Y$, and the operators A and B satisfy

$$\|B\| = 1 \qquad \text{and} \qquad \|A\| \le \frac{1}{\mu}\cdot\frac{1+3\sigma}{1-\sigma},$$

where $\mu = |\{\tilde{h}_{[0,1)} \neq 0\}|$.

Proof. Using Lemma 5.6 and Lemma 4.5, we can find a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ and a Haar multiplier $R\colon Y\to Y$ with $\|R\|\le 1$ such that $\tilde{h}_I = R\hat{h}_I$, $I\in\mathcal{D}$. We know from Proposition 7.1 that the operators $\hat{B}, \hat{A}\colon Y\to Y$ given by

$$\hat{B}x = \sum_{I\in\mathcal{D}} \frac{\langle h_I, x\rangle}{|I|}\hat{h}_I \qquad \text{and} \qquad \hat{A}x = \sum_{I\in\mathcal{D}} \frac{\langle \hat{h}_I, x\rangle}{|I|}h_I$$

satisfy $\hat{A}\hat{B} = I_Y$ and $\|\hat{A}\| = \|\hat{B}\| = 1$. For $I\in\mathcal{D}$, let $\mathcal{B}_I\subset\mathcal{D}$ denote the Haar support of $\tilde{h}_I$, and define $M\colon Y\to Y$ as the linear extension of $Mh_I = m_I h_I$, where $m_I = |I|/|\mathcal{B}_I^*|$, $I\in\mathcal{D}$. We will show that the operator M is bounded. To this end, let $n\in\mathbb{N}_0$ and $I\in\mathcal{D}$ and observe that repeatedly exploiting Definition 5.1 (ii) yields

$$|\mathcal{B}_I^*| \ge \Big(\prod_{J\in\mathcal{D}:J\supsetneq I} \varkappa_J\Big)\frac{|\mathcal{B}_{[0,1)}^*|}{2^n} \ge \Big(1 - \sum_{J\in\mathcal{D}}(1-\varkappa_J)\Big)\frac{|\mathcal{B}_{[0,1)}^*|}{2^n} = (1-\sigma)\mu|I|.$$

Moreover, we clearly have $|\mathcal{B}_I^*| \le \mu|I|$ for all $I\in\mathcal{D}$. Hence,

$$1-\sigma \le \frac{|\mathcal{B}_I^*|}{\mu|I|} \le 1, \qquad I\in\mathcal{D}. \tag{7.4}$$

Note that $m_{[0,1)} = 1/\mu$. Now let $I\in\mathcal{D}\setminus\{[0,1)\}$ and consider

$$|m_I - m_{\pi(I)}| = \left|\frac{|I|}{|\mathcal{B}_I^*|} - \frac{2|I|}{|\mathcal{B}_{\pi(I)}^*|}\right|.$$

Using the inequalities $|\mathcal{B}_I^*| \le \frac{1}{2}|\mathcal{B}_{\pi(I)}^*|$ and $|\mathcal{B}_{\pi(I)}^*| \le \frac{2}{\varkappa_{\pi(I)}}|\mathcal{B}_I^*|$, we obtain

$$0 \le \frac{|I|}{|\mathcal{B}_I^*|} - \frac{2|I|}{|\mathcal{B}_{\pi(I)}^*|} \le (1-\varkappa_{\pi(I)})\frac{|I|}{|\mathcal{B}_I^*|}.$$

Together with (7.4), this yields

$$|m_I - m_{\pi(I)}| \le \frac{1-\varkappa_{\pi(I)}}{\mu(1-\sigma)}, \qquad I\in\mathcal{D}\setminus\{[0,1)\}.$$

Thus, by Lemma 4.5, we have

$$\begin{aligned}\|M\| &\le \frac{1}{\mu}\Big(1 + \frac{2}{1-\sigma}\sum_{I\in\mathcal{D}\setminus\{[0,1)\}}(1-\varkappa_{\pi(I)})\Big)\\ &= \frac{1}{\mu}\Big(1+\frac{4\sigma}{1-\sigma}\Big) = \frac{1}{\mu}\cdot\frac{1+3\sigma}{1-\sigma} < \infty.\end{aligned}$$

Finally, observe that $B = R\hat{B}$ and $A = M\hat{A}R$ because we have for all $x\in H$:

$$M\hat{A}Rx = M\Big(\sum_{I\in\mathcal{D}}\frac{\langle\hat{h}_I, Rx\rangle}{|I|}h_I\Big) = \sum_{I\in\mathcal{D}}\frac{\langle R\hat{h}_I, x\rangle}{|\mathcal{B}_I^*|}h_I = \sum_{I\in\mathcal{D}}\frac{\langle\tilde{h}_I, x\rangle}{|\mathcal{B}_I^*|}h_I = Ax.$$

Thus, we conclude that $\|B\| \le \|R\|\|\hat{B}\| \le 1$ and $\|A\| \le \|M\|\|\hat{A}\|\|R\| \le \frac{1}{\mu}\cdot\frac{1+3\sigma}{1-\sigma}$, as claimed. □

8. Stabilization of Haar multipliers

The key ingredient for proving our main results is the observation that any bounded Haar multiplier on a Haar system Hardy space Y can be reduced to a *stable* Haar multiplier, i.e., a Haar multiplier whose entries have very small variation (see Proposition 8.4). Such a stable Haar multiplier is in turn close to cI_Y for some scalar value c. The stabilization is achieved by utilizing randomized faithful Haar systems.

Let $Y \in \mathcal{HH}_0(\delta)$, and let $D\colon Y \to Y$ denote a bounded Haar multiplier. Then its entries $(d_I)_{I\in\mathcal{D}}$ are defined as

$$d_I = \frac{\langle h_I, Dh_I\rangle}{|I|}, \qquad I \in \mathcal{D}.$$

In the following, we denote the product measure on $\{\pm1\}^{\mathcal{D}}$ of the normalized uniform measure on $\{\pm1\}$ by $\mathbb{P}$. Moreover, for fixed $\varepsilon = (\varepsilon_J)_{J\in\mathcal{D}} \in \{\pm1\}^{\mathcal{D}}$ and $n \in \mathbb{N}_0$, we define

$$\mathbb{P}_n^{\varepsilon}(\cdot) = \mathbb{P}(\cdot \mid \{(\theta_J)_{J\in\mathcal{D}} : \theta_J = \varepsilon_J,\ J \in \mathcal{D}\setminus\mathcal{D}_n\})$$

and denote the corresponding conditional expectation and variance by $\mathbb{E}_n^{\varepsilon}$ and $\mathbb{V}_n^{\varepsilon}$.

Definition 8.1. Let $n \in \mathbb{N}_0$, let Γ be a subset of $[0,1)$, and let $\varepsilon = (\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm1\}^{\mathcal{D}}$ be a sequence of signs. Then we define

$$r_n^{\Gamma} = \sum_{\substack{K\in\mathcal{D}_n\\ K\subset\Gamma}} h_K \qquad \text{and} \qquad r_n^{\Gamma}(\varepsilon) = \sum_{\substack{K\in\mathcal{D}_n\\ K\subset\Gamma}} \varepsilon_K h_K.$$

Remark 8.2. Let $(n_I)_{I\in\mathcal{D}}$ be a fixed strictly increasing sequence of non-negative integers. Then, given any sequence of signs $\theta \in \{\pm1\}^{\mathcal{D}}$, we can construct a faithful Haar system $(\hat{h}_I(\theta))_{I\in\mathcal{D}}$ relative to the frequencies $(n_I)_{I\in\mathcal{D}}$ by putting

$$\hat{h}_I(\theta) = r_{n_I}^{\Gamma_I(\theta)}(\theta) = \sum_{\substack{K\in\mathcal{D}_{n_I}\\ K\subset\Gamma_I(\theta)}} \theta_K h_K, \qquad I \in \mathcal{D},$$

where the sets $\Gamma_I(\theta)$ are defined as

$$\Gamma_{[0,1)}(\theta) = [0,1) \qquad \text{and} \qquad \Gamma_{I^\pm}(\theta) = \{\hat{h}_I(\theta) = \pm1\}, \quad I \in \mathcal{D}.$$

Note that for every $I \in \mathcal{D}$, the set $\Gamma_I(\theta)$ only depends on the signs $(\theta_K : K \in \mathcal{D}_{<n_I})$, whereas $\hat{h}_I(\theta)$ also depends on $(\theta_K : K \in \mathcal{D}_{n_I})$.

The next result is our main probabilistic lemma, which will be essential for proving our stabilization result. We use the technique from [29, Lemma 5.3]—for convenience, we provide a detailed proof.

Lemma 8.3. *Let $Y \in \mathcal{HH}_0(\delta)$. Given a bounded Haar multiplier $D\colon Y \to Y$ and a strictly increasing sequence of non-negative integers $(n_I)_{I\in\mathcal{D}}$, we define the random variables*

$$X_I(\theta) = \langle r_{n_I}^{\Gamma_I(\theta)}, D r_{n_I}^{\Gamma_I(\theta)}\rangle, \qquad I \in \mathcal{D},\ \theta \in \{\pm1\}^{\mathcal{D}},$$

where the sets $\Gamma_I(\theta)$ are defined as in Remark 8.2 with respect to $(n_I)_{I\in\mathcal{D}}$. Then for every $\varepsilon \in \{\pm1\}^{\mathcal{D}}$ and $I \in \mathcal{D}$, we have

$$\mathbb{E}_{n_I}^{\varepsilon} X_{I^\pm} = \frac{1}{2}\langle r_{n_{I^\pm}}^{\Gamma_I(\varepsilon)}, D r_{n_{I^\pm}}^{\Gamma_I(\varepsilon)}\rangle \qquad \textit{and} \qquad \mathbb{V}_{n_I}^{\varepsilon} X_{I^\pm} \le \frac{1}{4}2^{-n_I}\|D\|^2|I|.$$

Proof. Let $(d_K)_{K\in\mathcal{D}}$ denote the entries of the Haar multiplier D, and fix $I \in \mathcal{D}$ and $\theta \in \{\pm1\}^{\mathcal{D}}$. Then we have

$$X_{I^\pm}(\theta) = \sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset\Gamma_{I^\pm}(\theta)}} \sum_{\substack{L\in\mathcal{D}_{n_{I^\pm}}\\ L\subset\Gamma_{I^\pm}(\theta)}} \theta_K\theta_L\langle h_K, Dh_L\rangle = \sum_{\substack{K\in\mathcal{D}_{I^\pm}\\ K\subset\Gamma_{I^\pm}(\theta)}} d_K|K| = \sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\theta)}} \sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J^{\pm\theta_J}}} d_K|K|.$$

Taking the conditional expectation yields

$$\mathbb{E}^\varepsilon_{n_I} X_{I^\pm} = \sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\varepsilon)}} \mathbb{E}^\varepsilon_{n_I}\Big(\sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J^{\pm\theta_J}}} d_K|K|\Big) = \frac{1}{2}\sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset\Gamma_I(\varepsilon)}} d_K|K| = \frac{1}{2}\langle r^{\Gamma_I(\varepsilon)}_{n_{I^\pm}}, Dr^{\Gamma_I(\varepsilon)}_{n_{I^\pm}}\rangle,$$

which proves the first identity.

Next, we calculate the variance. Exploiting the independence of $(\theta_J : J \in \mathcal{D}_{n_I}, J \subset \Gamma_I(\varepsilon))$, we obtain

$$\mathbb{V}^\varepsilon_{n_I} X_{I^\pm} = \sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\varepsilon)}} \mathbb{V}^\varepsilon_{n_I}\Big(\sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J^{\pm\theta_J}}} d_K|K|\Big) = \sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\varepsilon)}} \frac{1}{4}\Big(\sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J^+}} d_K|K| - \sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J^-}} d_K|K|\Big)^2.$$

By Lemma 4.4, we have

$$|d_K||K| = \langle h_K, Dh_K\rangle \le \|h_K\|_{Y^*}\|h_K\|_Y\|D\| = \|D\||K|$$

for all $K \in \mathcal{D}$. Hence, we obtain

$$\mathbb{V}^\varepsilon_{n_I} X_{I^\pm} \le \sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\varepsilon)}} \frac{1}{4}\Big(\sum_{\substack{K\in\mathcal{D}_{n_{I^\pm}}\\ K\subset J}} \|D\||K|\Big)^2 \le \frac{1}{4}\sum_{\substack{J\in\mathcal{D}_{n_I}\\ J\subset\Gamma_I(\varepsilon)}} |J|^2\|D\|^2 = \frac{1}{4}2^{-n_I}\|D\|^2|\Gamma_I(\varepsilon)|. \qquad \square$$

Proposition 8.4. *Let $Y \in \mathcal{HH}_0(\delta)$, let $D\colon Y \to Y$ be a bounded Haar multiplier with entries $(d_I)_{I\in\mathcal{D}}$, and suppose that $c \in \Lambda(D)$. Given a sequence of positive real numbers $(\eta_I)_{I\in\mathcal{D}}$, there exists another bounded Haar multiplier $D^{\mathrm{stab}}\colon Y \to Y$ such that D^{stab} projectionally factors through D with constant 1 and error 0 and such that its entries $(d^{\mathrm{stab}}_I)_{I\in\mathcal{D}}$ satisfy*

$$|d^{\mathrm{stab}}_{[0,1)} - c| \le \eta_{[0,1)} \qquad \text{and} \qquad |d^{\mathrm{stab}}_{I^\pm} - d^{\mathrm{stab}}_I| \le \eta_I, \qquad I \in \mathcal{D}.$$

Moreover, if we additionally assume that there exists a $\delta > 0$ such that $d_I \ge \delta$ for all $I \in \mathcal{D}$, then we also have $d^{\mathrm{stab}}_I \ge \delta$ for all $I \in \mathcal{D}$.

Proof. In the following, we will first select a strictly increasing sequence of non-negative integers $(n_I)_{I\in\mathcal{D}}$ and then construct a faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ relative to the frequencies $(n_I)_{I\in\mathcal{D}}$ by choosing a sequence of signs $\varepsilon \in \{\pm1\}^{\mathcal{D}}$ and putting $\hat{h}_I = \hat{h}_I(\varepsilon)$, $I \in \mathcal{D}$, as defined in Remark 8.2.

Before beginning the construction, we make the following observation. Given a subset $\mathcal{A} \subset \mathcal{D}$, let $\mathbb{F}(\mathcal{A})$ denote the countable set of all finite unions of intervals in $\mathcal{A}$. For any $\Gamma \in \mathbb{F}(\mathcal{D})$, the sequence $(\langle r^\Gamma_n, Dr^\Gamma_n\rangle)_{n\in\mathbb{N}}$ is bounded because $\|r^\Gamma_n\|_Y \le 1$ and, by Corollary 7.2, $\|r^\Gamma_n\|_{Y^*} \le 1$. Thus, by Cantor's diagonalization argument, we can find an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that for each $\Gamma \in \mathbb{F}(\mathcal{D})$, the sequence $(\langle r^\Gamma_n, Dr^\Gamma_n\rangle)_{n\in\mathcal{N}}$ converges to some real number α_Γ, and since $c \in \Lambda(D)$, we can ensure that $\alpha_{[0,1)} = c$.

We now begin our inductive construction. First, we can find $n_{[0,1)} \in \mathcal{N}$ such that

$$|\langle r_n, Dr_n\rangle - c| < \eta_{[0,1)} \quad \text{and} \quad 2^{-n}\|D\|^2 < \eta^2_{[0,1)}, \qquad n \in \mathcal{N},\ n \ge n_{[0,1)}. \tag{8.1}$$

This completes the initial step.

Now let $I \in \mathcal{D} \setminus \{[0,1)\}$ and suppose that we have already constructed the numbers n_J, $J < I$, such that $(n_J)_{J<I}$ is strictly increasing. By the definition of $\mathcal{N}$, we can find $n_I \in \mathcal{N}$ such that $n_I > n_J$ for all $J < I$ and

$$|\langle r_n^\Gamma, Dr_n^\Gamma\rangle - \alpha_\Gamma| < \eta_I, \qquad \Gamma \in \mathbb{F}(\mathcal{D}_{n_{\pi(I)}+1}),\ n \in \mathcal{N},\ n \geq n_I \tag{8.2}$$

as well as

$$2^{-n}\|D\|^2 < \eta_I^2, \qquad n \geq n_I. \tag{8.3}$$

Next, we construct the sequence of signs $\varepsilon = (\varepsilon_K)_{K\in\mathcal{D}}$. Let $I \in \mathcal{D}$ and suppose that we have already chosen the signs $(\varepsilon_K : K \in \mathcal{D}_{<n_I})$ (or no signs yet, if $I = [0,1)$). Now let $\varepsilon^I \in \{\pm 1\}^{\mathcal{D}}$ be any sequence of signs such that $\varepsilon_K^I = \varepsilon_K$ for all $K \in \mathcal{D}_{<n_I}$ and consider the random variables

$$X_{I^\pm}(\theta) = \langle r_{n_{I^\pm}}^{\Gamma_{I^\pm}(\theta)}, Dr_{n_{I^\pm}}^{\Gamma_{I^\pm}(\theta)}\rangle, \qquad \theta \in \{\pm 1\}^{\mathcal{D}}.$$

Using Lemma 8.3 together with (8.1) and (8.3) yields

$$\mathbb{E}_{n_I}^{\varepsilon^I} X_{I^\pm} = \frac{1}{2}\langle r_{n_{I^\pm}}^{\Gamma_I(\varepsilon^I)}, Dr_{n_{I^\pm}}^{\Gamma_I(\varepsilon^I)}\rangle \qquad \text{and} \qquad \mathbb{V}_{n_I}^{\varepsilon^I} X_{I^\pm} < \frac{1}{4}\eta_I^2 |I|.$$

By Chebyshev's inequality, we conclude that

$$\mathbb{P}_{n_I}^{\varepsilon^I}(|X_{I^+} - \mathbb{E}_{n_I}^{\varepsilon^I} X_{I^+}| \geq \eta_I \text{ or } |X_{I^-} - \mathbb{E}_{n_I}^{\varepsilon^I} X_{I^-}| \geq \eta_I)$$

is bounded from above by

$$\frac{1}{\eta_I^2}\mathbb{V}_{n_I}^{\varepsilon^I} X_{I^+} + \frac{1}{\eta_I^2}\mathbb{V}_{n_I}^{\varepsilon^I} X_{I^-} \leq \frac{1}{\eta_I^2}\cdot 2 \cdot \frac{1}{4}\eta_I^2 |I| \leq \frac{1}{2} < 1.$$

Thus, we can choose signs $(\varepsilon_K : K \in \mathcal{D}_{n_I})$ such that for any $\theta \in \{\pm 1\}^{\mathcal{D}}$ with $\theta_K = \varepsilon_K$, $K \in \mathcal{D}_{\leq n_I}$, we have

$$\left|\langle r_{n_{I^\pm}}^{\Gamma_{I^\pm}(\theta)}, Dr_{n_{I^\pm}}^{\Gamma_{I^\pm}(\theta)}\rangle - \frac{1}{2}\langle r_{n_{I^\pm}}^{\Gamma_I(\theta)}, Dr_{n_{I^\pm}}^{\Gamma_I(\theta)}\rangle\right| < \eta_I. \tag{8.4}$$

The signs $(\varepsilon_K : K \in \mathcal{D}_{<n_J} \setminus \mathcal{D}_{\leq n_I})$, where $J = \iota^{-1}(\iota(I)+1)$, can be chosen arbitrarily: We put $\varepsilon_K = 1$ for $K \in \mathcal{D}_{<n_J} \setminus \mathcal{D}_{\leq n_I}$. This concludes the construction of the signs.

Now we proceed to analyze the properties of our construction. For $I \in \mathcal{D}$, we write $\Gamma_I = \Gamma_I(\varepsilon)$, where ε denotes the sequence of signs that we have just chosen. Observe that we have $n_I, n_{I^\pm} \in \mathcal{N}$ and $n_I < n_{I^\pm}$ for all $I \in \mathcal{D}$, and moreover, $\Gamma_I \in \mathbb{F}(\mathcal{D}_{n_{\pi(I)}+1})$ for all $I \in \mathcal{D} \setminus \{[0,1)\}$. Combining (8.1) with (8.2) yields

$$|\langle r_{n_{I^\pm}}^{\Gamma_I}, Dr_{n_{I^\pm}}^{\Gamma_I}\rangle - \langle r_{n_I}^{\Gamma_I}, Dr_{n_I}^{\Gamma_I}\rangle| \leq |\langle r_{n_{I^\pm}}^{\Gamma_I}, Dr_{n_{I^\pm}}^{\Gamma_I}\rangle - \alpha_{\Gamma_I}| + |\alpha_{\Gamma_I} - \langle r_{n_I}^{\Gamma_I}, Dr_{n_I}^{\Gamma_I}\rangle| < 2\eta_I$$

for all $I \in \mathcal{D}$. Together with (8.4), applied to $\theta = \varepsilon$, this implies that

$$\left|\langle r_{n_{I^\pm}}^{\Gamma_{I^\pm}}, Dr_{n_{I^\pm}}^{\Gamma_{I^\pm}}\rangle - \frac{1}{2}\langle r_{n_I}^{\Gamma_I}, Dr_{n_I}^{\Gamma_I}\rangle\right| < 2\eta_I, \qquad I \in \mathcal{D}.$$

Now observe that

$$\langle r_{n_I}^{\Gamma_I}(\varepsilon), Dr_{n_I}^{\Gamma_I}(\varepsilon)\rangle = \sum_{\substack{K\in\mathcal{D}_{n_I}\\ K\subset\Gamma_I}} \langle h_K, Dh_K\rangle = \langle r_{n_I}^{\Gamma_I}, Dr_{n_I}^{\Gamma_I}\rangle, \qquad I \in \mathcal{D}.$$

Thus, the faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$ defined by $\hat{h}_I = \hat{h}_I(\varepsilon) = r_{n_I}^{\Gamma_I}(\varepsilon)$, $I \in \mathcal{D}$, satisfies

$$|\langle \hat{h}_{[0,1)}, D\hat{h}_{[0,1)}\rangle - c| < \eta_{[0,1)}, \tag{8.5}$$

$$\left|\langle \hat{h}_{I^\pm}, D\hat{h}_{I^\pm}\rangle - \frac{1}{2}\langle \hat{h}_I, D\hat{h}_I\rangle\right| < 2\eta_I, \qquad I \in \mathcal{D}. \tag{8.6}$$

Now we use the operators $\hat{A}, \hat{B} \colon Y \to Y$, defined as in Proposition 7.1 with respect to our newly constructed faithful Haar system $(\hat{h}_I)_{I\in\mathcal{D}}$, and we define $D^{\text{stab}} = \hat{A}D\hat{B}$. We have $\hat{A}\hat{B} = I_Y$ and $\|\hat{A}\| = \|\hat{B}\| = 1$. Since D is diagonal with respect to $(h_I)_{I\in\mathcal{D}}$ and the functions $(\hat{h}_I)_{I\in\mathcal{D}}$ have pairwise disjoint Haar supports, it follows that $\langle \hat{h}_I, D\hat{h}_J\rangle = 0$ for all $I \neq J$. Thus, we have

$$D^{\mathrm{stab}} h_I = \hat{A}D\hat{B}h_I = \sum_{J\in\mathcal{D}} \frac{\langle \hat{h}_J, D\hat{h}_I\rangle}{|J|} h_J = \frac{\langle \hat{h}_I, D\hat{h}_I\rangle}{|I|} h_I, \quad I \in \mathcal{D},$$

and so D^{stab} is also diagonal with respect to $(h_I)_{I\in\mathcal{D}}$. By (8.5) and (8.6), the entries $d_I^{\mathrm{stab}} = \langle \hat{h}_I, D\hat{h}_I\rangle/|I|$, $I \in \mathcal{D}$, satisfy

$$|d_{[0,1)}^{\mathrm{stab}} - c| < \eta_{[0,1)} \qquad \text{and} \qquad |d_{I^\pm}^{\mathrm{stab}} - d_I^{\mathrm{stab}}| < 4\eta_I/|I|, \quad I \in \mathcal{D}.$$

The desired conclusion is obtained by replacing η_I by $\eta_I|I|/4$ for each $I \in \mathcal{D}$ before the construction begins.

The additional statement follows since the inequalities $\langle h_K, Dh_K\rangle \geq \delta|K|$, $K \in \mathcal{D}$, imply that

$$\langle \hat{h}_I, D\hat{h}_I\rangle = \sum_{\substack{K\in\mathcal{D}_{n_I}\\ K\subset\Gamma_I}} \langle h_K, Dh_K\rangle \geq \delta \sum_{\substack{K\in\mathcal{D}_{n_I}\\ K\subset\Gamma_I}} |K| = \delta|I|, \qquad I \in \mathcal{D}. \qquad \square$$

This stabilization result enables us to prove the main result Theorem 3.6.

Proof of Theorem 3.6. Proof of (i). Fix $\eta > 0$ and let $(\eta_I)_{I\in\mathcal{D}}$ be a sequence of positive real numbers with

$$\sum_{I\in\mathcal{D}} \eta_I \leq \frac{\eta}{8}.$$

By applying Proposition 8.4 to D and $(\eta_I)_{I\in\mathcal{D}}$, we obtain a bounded Haar multiplier $D^{\mathrm{stab}}\colon Y \to Y$ and operators $\hat{A}, \hat{B}\colon Y \to Y$ such that $\hat{A}\hat{B} = I_Y$, $\|\hat{A}\|\|\hat{B}\| \leq 1$ and $D^{\mathrm{stab}} = \hat{A}D\hat{B}$, and such that the entries $(d_I^{\mathrm{stab}})_{I\in\mathcal{D}}$ of D^{stab} satisfy

$$|d_{[0,1)}^{\mathrm{stab}} - c| \leq \frac{\eta}{2} \qquad \text{and} \qquad |d_I^{\mathrm{stab}} - d_{\pi(I)}^{\mathrm{stab}}| \leq \eta_{\pi(I)}, \quad I \in \mathcal{D}\setminus\{[0,1)\}. \tag{8.7}$$

We will show that $\|cI_Y - D^{\mathrm{stab}}\| \leq \eta$. Note that $d_{[0,1)}^{\mathrm{stab}} I_Y - D^{\mathrm{stab}}$ is a Haar multiplier with entries $(d_{[0,1)}^{\mathrm{stab}} - d_I^{\mathrm{stab}})_{I\in\mathcal{D}}$, which satisfy

$$|||(d_{[0,1)}^{\mathrm{stab}} - d_I^{\mathrm{stab}})_{I\in\mathcal{D}}||| \leq 2 \sum_{I\in\mathcal{D}\setminus\{[0,1)\}} |d_I^{\mathrm{stab}} - d_{\pi(I)}^{\mathrm{stab}}| \leq 4\sum_{I\in\mathcal{D}} \eta_I \leq \frac{\eta}{2}.$$

Thus, by Lemma 4.5, we have $\|d_{[0,1)}^{\mathrm{stab}} I_Y - D^{\mathrm{stab}}\| \leq \eta/2$, and hence, using (8.7),

$$\|cI_Y - \hat{A}D\hat{B}\| \leq \eta.$$

Proof of (i)–(iii). The remaining statements follow immediately from (i) combined with Remark 2.5, Remark 2.6 and Remark 3.5. $\square$

9. Strategically supporting systems and strategically reproducible bases

In this section, we provide an overview of the framework of strategically supporting systems and strategically reproducible Schauder bases introduced in [20, 27]. We explain how this framework can be used to diagonalize operators and to reduce operators with large diagonal to operators with large *positive* or large *negative* diagonal. Moreover, we extend the definitions so that they can be utilized in the context of the primary factorization property. We need the following definitions.

Definition 9.1. Suppose that $(e_j)_{j=1}^\infty$ is a Schauder basis for a Banach space E and let $(e_j^*)_{j=1}^\infty$ denote the biorthogonal functionals. Then we say that $(e_j^*)_{j=1}^\infty$ is a *weak* Schauder basis*. In this case, for any $e^* \in E^*$, we have the following unique expansion:

$$e^* = \text{w}^*\text{-}\sum_{j=1}^\infty \langle e^*, e_j\rangle e_j^*,$$

where the above series converges in the weak* topology. From now on, we will always indicate weak* convergence as above.

Definition 9.2. Let E and F be Banach spaces. Let $(x_j)_{j=1}^\infty$, $(y_j)_{j=1}^\infty$, $(x_j^*)_{j=1}^\infty$ and $(y_j^*)_{j=1}^\infty$ be sequences in E, F, E^* and F^*, respectively, and let $C > 0$.

- ▷ We say that $(x_j)_{j=1}^\infty$ and $(y_j)_{j=1}^\infty$ are *impartially C-equivalent* if for any sequence of scalars $(a_j)_{j=1}^\infty$ with $a_j \neq 0$ for at most finitely many j, we have

$$\frac{1}{\sqrt{C}}\Big\|\sum_{j=1}^\infty a_j y_j\Big\|_F \leq \Big\|\sum_{j=1}^\infty a_j x_j\Big\|_E \leq \sqrt{C}\Big\|\sum_{j=1}^\infty a_j y_j\Big\|_F.$$

- ▷ We say that $(x_j)_{j=1}^\infty$ is *C-norm-dominated* by $(y_j)_{j=1}^\infty$ if whenever $\sum_{j=1}^\infty a_j y_j$ converges in norm for a scalar sequence $(a_j)_{j=1}^\infty$, then $\sum_{j=1}^\infty a_j x_j$ converges in norm and

$$\Big\|\sum_{j=1}^\infty a_j x_j\Big\|_E \leq C\Big\|\sum_{j=1}^\infty a_j y_j\Big\|_F.$$

- ▷ We say that $(x_j^*)_{j=1}^\infty$ is *C-weak*-dominated* by $(y_j^*)_{j=1}^\infty$ if whenever $\text{w}^*\text{-}\sum_{j=1}^\infty a_j y_j^*$ converges weak* for a scalar sequence $(a_j)_{j=1}^\infty$, then $\text{w}^*\text{-}\sum_{j=1}^\infty a_j x_j^*$ converges weak* and

$$\Big\|\text{w}^*\text{-}\sum_{j=1}^\infty a_j x_j^*\Big\|_{E^*} \leq C\Big\|\text{w}^*\text{-}\sum_{j=1}^\infty a_j y_j^*\Big\|_{F^*}.$$

In [20], T. Kania and the first named author introduced strategically supporting systems in dual pairs of Banach spaces. We will now provide the definition for the special case of a Banach space E and its dual E^* on the one hand, while also expanding the concept to accommodate *projectional factors* on the other hand.

Definition 9.3. Let $(e_j)_{j=1}^\infty$ denote a Schauder basis for the Banach space E and let $(e_j^*)_{j=1}^\infty$ denote the biorthogonal functionals. We say that $((e_j, e_j^*))_{j=1}^\infty$ *is C-strategically supporting (in $E \times E^*$)* if for all $\eta > 0$ and all partitions N_1, N_2 of $\mathbb{N}$ there exists $i \in \{1, 2\}$ and

$$\begin{aligned}
&\exists \text{ finite } E_1 \subset N_i\ \exists(\lambda_j^1)_j, (\mu_j^1)_j \in \mathbb{R}^{E_1}\ \forall(\varepsilon_j^1)_j \in \{\pm 1\}^{E_1}\\
&\exists \text{ finite } E_2 \subset N_i\ \exists(\lambda_j^2)_j, (\mu_j^2)_j \in \mathbb{R}^{E_2}\ \forall(\varepsilon_j^2)_j \in \{\pm 1\}^{E_2}\\
&\quad\vdots\\
&\exists \text{ finite } E_k \subset N_i\ \exists(\lambda_j^k)_j, (\mu_j^k)_j \in \mathbb{R}^{E_k}\ \forall(\varepsilon_j^k)_j \in \{\pm 1\}^{E_k}\\
&\quad\vdots
\end{aligned}$$

such that if we define

$$x_k = \sum_{j\in E_k} \varepsilon_j^k \lambda_j^k e_j \quad \text{and} \quad x_k^* = \sum_{j\in E_k} \varepsilon_j^k \mu_j^k e_j^*, \qquad k \in \mathbb{N}, \tag{9.1}$$

we have that

(i) $(x_k)_{k=1}^\infty$ is $\sqrt{C+\eta}$-norm-dominated by $(e_k)_{k=1}^\infty$;
(ii) $(x_k^*)_{k=1}^\infty$ is $\sqrt{C+\eta}$-weak*-dominated by $(e_k^*)_{k=1}^\infty$,
and $\sum_{k=1}^\infty \langle x_k^*, x\rangle e_k$ converges in norm for all $x \in E$;
(iii) $1 \leq \langle x_k^*, x_k\rangle = \sum_{j\in E_k} \lambda_j^k \mu_j^k \leq 1+\eta$, $k \in \mathbb{N}$;
(iv) $\lambda_j^k \mu_j^k \geq 0$, $k \in \mathbb{N}$, $j \in E^k$.

If additionally $(x_k^*)_{k=1}^\infty$ is biorthogonal to $(x_k)_{k=1}^\infty$, then we say that $((e_j, e_j^*))_{j=1}^\infty$ is C-strategically supporting (in $E \times E^*$) *with projectional factors.*

Remark 9.4. Note that Definition 9.3 contains a phrase with infinitely many quantifiers. For a formal definition of this kind of notation, we refer to [45, Section 2]. Moreover, this condition can be understood to mean that one of two player has a winning strategy in an *infinite game.* In fact, we will define *strategically reproducible* Schauder bases in terms of such an infinite two-player game in Definition 9.8, which should be interpreted in the same manner (see [45, 12, 33]).

The following proposition states that if a system consisting of a Schauder basis and its biorthogonal functionals is strategically supporting, then any bounded linear operator with large diagonal can be reduced to an operator with large positive or large negative diagonal.

Proposition 9.5. *Let $(e_j)_{j=1}^\infty$ denote a Schauder basis for a Banach space E, and let $(e_j^*)_{j=1}^\infty$ denote the biorthogonal functionals. Suppose that $((e_j, e_j^*))_{j=1}^\infty$ is C-strategically supporting in $E \times E^*$ (with projectional factors) and let $T\colon E \to E$ denote a bounded linear operator which has δ-large diagonal with respect to $(e_j)_{j=1}^\infty$. Then the following statements are true:*

- ▷ *For every $\gamma > 0$, there exists a bounded linear operator $S\colon E \to E$ with either δ-large positive or δ-large negative diagonal with respect to $(e_j)_{j=1}^\infty$ such that S (projectionally) factors through T with constant $C+\gamma$ and error 0.*
- ▷ *If T is diagonal with respect to $(e_j)_{j=1}^\infty$, then in the case of projectional factors, S can also be chosen to be diagonal with respect to $(e_j)_{j=1}^\infty$.*

Proof. Suppose that for some $\delta > 0$ the bounded linear operator $T\colon Y \to Y$ has δ-large diagonal with respect to $(e_j)_{j=1}^\infty$, i.e., $|\langle e_j^*, Te_j\rangle| \geq \delta$ for all $j \in \mathbb{N}$, and let $\gamma > 0$. Define $N_1 = \{j \in \mathbb{N} : \langle e_j^*, Te_j\rangle \geq \delta\}$, and $N_2 = \{j \in \mathbb{N} : \langle e_j^*, Te_j\rangle \leq -\delta\}$. Then, since T has δ-large diagonal, N_1, N_2 is a partition of $\mathbb{N}$.

Exploiting that $((e_j, e_j^*))_{j=1}^\infty$ is C-strategically supporting, we first obtain an index $i \in \{1, 2\}$, determining that our systems $(x_k)_{k=1}^\infty$, $(x_k^*)_{k=1}^\infty$ will be supported in N_i. From now on, we will assume that $i = 1$ and note that the case $i = 2$ is obtained by replacing T with $-T$. Next, we obtain a finite set $E_1 \subset N_i$ and $(\lambda_j^1)_j, (\mu_j^1)_j \in \mathbb{R}^{E_1}$ and we are now free to pick the signs $(\varepsilon_j^1)_j \in \{\pm 1\}^{E_1}$. To this end, let $(\theta_j^1 : j \in E_1)$ be an independent family of random variables taking the values ± 1, each with probability $1/2$, and let $\mathbb{E}$ denote the expectation. Then, since $E_1 \subset N_1$, Definition 9.3 (iii) and (iv) yield

$$\mathbb{E}\Big\langle \sum_{j\in E_1} \theta_j^1 \mu_j^1 e_j^*, T\Big(\sum_{j\in E_1} \theta_j^1 \lambda_j^1 e_j\Big)\Big\rangle = \sum_{j\in E_1} \mu_j^1 \lambda_j^1 \langle e_j^*, Te_j\rangle \geq \delta \sum_{j\in E_1} \mu_j^1 \lambda_j^1 \geq \delta,$$

and hence, we can find signs $(\varepsilon_j^1)_j \in \{\pm 1\}^{E_1}$ such that

$$x_1 = \sum_{j\in E_1} \varepsilon_j^1 \lambda_j^1 e_j \quad \text{and} \quad x_1^* = \sum_{j\in E_1} \varepsilon_j^1 \mu_j^1 e_j^* \qquad \text{satisfy} \qquad \langle x_1^*, x_1\rangle \geq \delta.$$

Proceeding in this manner, we obtain sequences $(x_k)_{k=1}^\infty$ and $(x_k^*)_{k=1}^\infty$, defined according to (9.1), which satisfy Definition 9.3 (i)–(iv) as well as the additional condition

$$\langle x_k^*, Tx_k\rangle \geq \delta, \qquad k \in \mathbb{N}. \tag{9.2}$$

If we assume that $((e_j, e_j^*))_{j=1}^\infty$ is C-strategically supporting in $E \times E^*$ *with projectional factors*, then we get additionally that $(x_k^*)_{k=1}^\infty$ is biorthogonal to $(x_k)_{k=1}^\infty$.

We will now define the canonical operators $A, B\colon E \to E$ given by

$$Bx = \sum_{k=1}^\infty \langle e_k^*, x\rangle x_k \qquad \text{and} \qquad Ax = \sum_{k=1}^\infty \langle x_k^*, x\rangle e_k, \qquad x \in E, \tag{9.3}$$

and note that they are well-defined since the above series converge in norm. Exploiting that $(e_k^*)_{k=1}^\infty$ is a weak* Schauder basis, together with (ii), we have for all $x^* \in E^*$ and $x \in E$:

$$\begin{aligned}|\langle x^*, Ax\rangle| &= \Big|\sum_{k=1}^\infty \langle x_k^*, x\rangle\langle x^*, e_k\rangle\Big| = \Big|\Big\langle \text{w}^*\text{-}\sum_{k=1}^\infty \langle x^*, e_k\rangle x_k^*, x\Big\rangle\Big|\\ &\leq \Big\|\text{w}^*\text{-}\sum_{k=1}^\infty \langle x^*, e_k\rangle x_k^*\Big\|_{E^*}\|x\|_E \leq \sqrt{C+\gamma}\|x^*\|_{E^*}\|x\|_E.\end{aligned}$$

The above estimate together with (i) yields

$$\|A\|, \|B\| \leq \sqrt{C+\gamma}. \tag{9.4}$$

We now define the bounded linear operator $S\colon E \to E$ by putting $S = ATB$. Firstly, note that by (9.4), S factors through T with constant $C+\gamma$ and error 0. Secondly, using (9.3) and (9.2), we also obtain that S has δ-large positive diagonal with respect to $(e_j)_{j=1}^\infty$:

$$\langle e_j^*, Se_j\rangle = \langle e_j^*, ATx_j\rangle = \langle x_j^*, Tx_j\rangle \geq \delta, \qquad j \in \mathbb{N}. \tag{9.5}$$

If we assume that $((e_j, e_j^*))_{j=1}^\infty$ is C-strategically supporting in $E \times E^*$ *with projectional factors*, then $(x_k)_{k=1}^\infty$ and $(x_k^*)_{k=1}^\infty$ are biorthogonal. Hence, we obtain that $AB = I_E$, i.e., S projectionally factors through T with constant $C+\gamma$ and error 0. Moreover, in this case, if T is diagonal with respect to $(e_j)_{j=1}^\infty$, then (9.5) implies that S is also diagonal with respect to $(e_j)_{j=1}^\infty$. □

The notion of a *strategically reproducible* Schauder basis was introduced in [27]. Before presenting the definition, we establish the following terminology.

Definition 9.6. Let E be a Banach space. By $\operatorname{cof}(E)$, we denote the set of cofinite-dimensional closed subspaces of E, while $\operatorname{cof}_{w^*}(E^*)$ denotes the set of cofinite-dimensional weak* closed subspaces of E^*.

Remark 9.7. In fact, we will always use the following characterization of $\operatorname{cof}(E)$ and $\operatorname{cof}_{w^*}(E^*)$: A subspace $W \subset E$ is in $\operatorname{cof}(E)$ if and only if there are there are $N \in \mathbb{N}_0$ and $x_1^*, \dots, x_N^* \in E^*$ such that

$$W = \{x_1^*, \dots, x_N^*\}_\perp.$$

Similarly, a subspace $G \subset E^*$ is in $\operatorname{cof}_{w^*}(E^*)$ if and only if there are $M \in \mathbb{N}_0$ and $x_1, \dots, x_M \in E$ such that

$$G = \{x_1, \dots, x_M\}^\perp.$$

Details can be found in [50, Lemma 5.1.7].

Definition 9.8. Let E be a Banach space with a normalized Schauder basis $(e_j)_{j=1}^\infty$ and associated biorthogonal functionals $(e_j^*)_{j=1}^\infty$, and fix positive constants $C \geq 1$ and $\eta > 0$.

Consider the following two-player game between player (I) and player (II):

Before the first turn player (I) is allowed to choose a partition of $\mathbb{N} = N_1 \cup N_2$. For $k \in \mathbb{N}$, turn k is played out in three steps.

Step 1: Player (I) chooses $\eta_k > 0$, $W_k \in \operatorname{cof}(E)$, and $G_k \in \operatorname{cof}_{w^*}(E^*)$.

Step 2: Player (II) chooses $i_k \in \{1, 2\}$, a finite subset E_k of N_{i_k} and sequences of non-negative real numbers $(\lambda_j^k)_{j\in E_k}$, $(\mu_j^k)_{j\in E_k}$ satisfying

$$1 - \eta < \sum_{j\in E_k} \lambda_j^k \mu_j^k < 1 + \eta.$$

Step 3: Player (I) chooses $(\varepsilon_j^k)_{j\in E_k}$ in $\{\pm 1\}^{E_k}$.

We say that player (II) has a winning strategy in the game $\operatorname{Rep}_{(E,(e_j))}(C, \eta)$ if he can force the following properties on the result:

For all $k \in \mathbb{N}$, we set

$$x_k = \sum_{j\in E_k} \varepsilon_j^k \lambda_j^k e_j \qquad \text{and} \qquad x_k^* = \sum_{j\in E_k} \varepsilon_j^k \mu_j^k e_j^* \tag{9.6}$$

and demand:

(i) the sequences $(x_k)_{k=1}^\infty$ and $(e_k)_{k=1}^\infty$ are impartially $(C + \eta)$-equivalent,
(ii) the sequences $(x_k^*)_{k=1}^\infty$ and $(e_k^*)_{k=1}^\infty$ are impartially $(C + \eta)$-equivalent,
(iii) for all $k \in \mathbb{N}$, we have $\operatorname{dist}(x_k, W_k) < \eta_k$, and
(iv) for all $k \in \mathbb{N}$, we have $\operatorname{dist}(x_k^*, G_k) < \eta_k$.

We say that $(e_j)_{j=1}^\infty$ is *C-strategically reproducible in E* if for every $\eta > 0$, player (II) has a winning strategy in the game $\operatorname{Rep}_{(E,(e_j))}(C, \eta)$. If player (II) can additionally guarantee that $(x_k^*)_{k=1}^\infty$ is biorthogonal to $(x_k)_{k=1}^\infty$, then we say that $(e_j)_{j=1}^\infty$ is C-strategically reproducible in E *with projectional factors*.

Before proceeding, we make the following observations.

Remark 9.9. Let $(e_j)_{j=1}^\infty$ denote a Schauder basis of a Banach space E with basis constant $\lambda \geq 1$. In the proof of [27, Theorem 3.12], it is noted that if $(e_j)_{j=1}^\infty$ is C-strategically reproducible in E, then player (I) can always force the following outcome on the winning strategy of player (II): For any $\eta > 0$, in addition to (i)–(iv) in Definition 9.8, the sequence $(x_k^*)_{k=1}^\infty$ is summably close to some block sequence $(\tilde{x}_k^*)_{k=1}^\infty$ of $(e_j^*)_{j=1}^\infty$, i.e.,

$$\sum_{k=1}^\infty \|x_k^* - \tilde{x}_k^*\|_{X^*} < \infty. \tag{9.7}$$

In this case, the operator $A\colon E \to E$ given by

$$Ax = \sum_{k=1}^\infty \langle x_k^*, x\rangle e_k, \qquad x \in E$$

is well-defined and satisfies

$$\|A\| \leq \lambda\sqrt{C + \eta}.$$

For more details, we refer to Lemma 5.2.1 and the proof of Theorem 5.2.3 in [50].

On the other hand, observe that even without (9.7), condition (i) implies that the operator $B\colon E \to E$ given by

$$Bx = \sum_{k=1}^\infty \langle e_k^*, x\rangle x_k, \qquad x \in E$$

is well-defined and satisfies

$$\|B\| \leq \sqrt{C + \eta}.$$

Finally, note that if $(x_k)_{k=1}^\infty$ and $(x_k^*)_{k=1}^\infty$ were constructed according to a winning strategy with projectional factors, i.e., $(x_k^*)_{k=1}^\infty$ is biorthogonal to $(x_k)_{k=1}^\infty$, then we additionally have $AB = I_E$.

The next proposition states that if a Schauder basis of a Banach space is strategically reproducible, then any bounded linear operator can be diagonalized with respect to the given basis, and moreover, it is possible to preserve a large (positive) diagonal when diagonalizing. This result is is implicitly contained in the proof of [27, Theorem 3.12]. Nonetheless, we state this diagonalzation step as a separate result since we will need it in the proof of Theorem 11.1.

Proposition 9.10. *Let E be a Banach space with a normalized Schauder basis $(e_j)_{j=1}^\infty$ with basis constant λ. Let $C \geq 1$ and suppose that $(e_j)_{j=1}^\infty$ is C-strategically reproducible in E (with projectional factors). Moreover, let $T\colon E \to E$ denote a bounded linear operator and let $\eta > 0$. Then the following assertions are true:*

(i) There exists a bounded linear operator $D\colon E \to E$ which is diagonal with respect to $(e_j)_{j=1}^\infty$ such that D (projectionally) factors through T with constant $\lambda(C+\eta)$ and error η.

(ii) If we additionally assume that T has δ-large (positive) diagonal with respect to $(e_j)_{j=1}^\infty$ for some $\delta > 0$, then the above diagonal operator D can be chosen so that D also has δ-large (positive) diagonal with respect to $(e_j)_{j=1}^\infty$.

Proof. Proposition 9.10 (i) follows from the proof of [27, Theorem 3.12] together with the discussion in Remark 9.9. More precisely, at the beginning of the proof of [27, Theorem 3.12], we replace player (I)'s choice of N_1 and N_2 by, say, $N_1 = \mathbb{N}$ and $N_2 = \emptyset$, since in (i) we do not assume that T has large diagonal. Moreover, in the last step of the nth turn, instead of using a probabilistic argument to pick the signs $(\varepsilon_i^{(n)})_{i\in E_n}$, player (I) simply chooses $\varepsilon_i^{(n)} = 1$ for all $i \in E_n$. Then we follow the proof of [27, Theorem 3.12] until the conditions of [27, Lemma 3.14] are checked, and we conclude by [27, Lemma 3.14] and Remark 9.9.

In order to prove Proposition 9.10 (ii), we repeat the argument from [27, Theorem 3.12], and thereby we directly obtain a diagonal operator D which has $(1-\eta)\delta$-large (positive) diagonal. By rescaling with the factor $(1-\eta)^{-1}$, we obtain the result as claimed in (ii). □

Remark 9.11. Fix $\lambda, C_r \geq 1$ and suppose that for every $k \in \mathbb{N}$, E_k is a Banach space with a normalized Schauder basis $(e_{k,j})_{j=1}^\infty$ which is C_r-strategically reproducible in E_k and whose basis constant is bounded by λ. Then by [27, Proposition 7.5], the sequence $(e_{k,j})_{j=1}^\infty$, enumerated as $(\tilde{e}_m)_{m=1}^\infty$ according to Remark 2.9, is a C_r-strategically reproducible Schauder basis of $\ell^p((E_k)_{k=1}^\infty)$ whose basis constant is bounded by λ. By inspecting the proof of [27, Proposition 7.5], it is clear that $(\tilde{e}_m)_{m=1}^\infty$ is even C_r-strategically reproducible *with projectional factors* if the same is true for all bases $(e_{k,j})_{j=1}^\infty$, $k \in \mathbb{N}$. Thus, Proposition 9.10 can be used to diagonalize any bounded linear operator $T\colon \ell^p((E_k)_{k=1}^\infty) \to \ell^p((E_k)_{k=1}^\infty)$.

For $p = \infty$, however, we cannot use Proposition 9.10 to diagonalize an operator on $\ell^\infty((E_k)_{k=1}^\infty)$. Instead, we have to extract an analogous result from the proof of [28, Theorem 3.9]. First, we give the definition of a diagonal operator on $\ell^\infty((E_k)_{k=1}^\infty)$.

Definition 9.12. For each $k \in \mathbb{N}$, let E_k be a Banach space with a Schauder basis $(e_{k,j})_{j=1}^\infty$. Denote $Z = \ell^\infty((E_k)_{k=1}^\infty)$. A bounded linear operator $D\colon Z \to Z$ is called *diagonal (with respect to $(e_{k,j})_{k,j=1}^\infty$)* if it is of the form $D(x_k)_{k=1}^\infty = (D_k x_k)_{k=1}^\infty$, $(x_k)_{k=1}^\infty \in Z$, where for each $k \in \mathbb{N}$, $D_k\colon E_k \to E_k$ is a bounded diagonal operator with respect to $(e_{k,j})_{j=1}^\infty$.

Proposition 9.13. *For each $k \in \mathbb{N}$, let E_k be a Banach space with a normalized Schauder basis $(e_{k,j})_{j=1}^\infty$, and denote $Z = \ell^\infty((E_k)_{k=1}^\infty)$. Assume that $(E_k)_{k=1}^\infty$ is uniformly asymptotically curved with respect to $(e_{k,j})_{j,k=1}^\infty$ and that there are $C_r, \lambda \geq 1$ such that*

▷ *for every $k \in \mathbb{N}$, the basis constant of $(e_{k,j})_{j=1}^\infty$ in E_k is at most λ;*

▷ *for every $k \in \mathbb{N}$, the basis $(e_{k,j})_{j=1}^\infty$ is C_r-strategically reproducible in E_k (with projectional factors).*

Then for every bounded linear operator $T\colon Z \to Z$ and every $\eta > 0$, there exists a bounded linear operator $D\colon Z \to Z$ that is diagonal with respect to $(e_{k,j})_{k,j=1}^\infty$ such that D (projectionally) factors through T with constant $\lambda(C_r + \eta)$ and error η.

Proof. Similarly to Proposition 9.10, this result is contained in the proof of [28, Theorem 3.9]. At the beginning we again replace the choice of N_1 and N_2 by $N_1 = \mathbb{N}$ and $N_2 = \emptyset$, and in *Turn n, Step 3*, player (I) chooses the signs $\varepsilon_i^{(n)} = 1$, $i \in E_n$. In the end, from [28, Proposition 4.5 (b)], we obtain a bounded linear operator $D\colon Z \to Z$ that is diagonal with respect to $(e_{k,j})_{k,j=1}^\infty$ and bounded linear operators $A, B\colon Z \to Z$ such that

$$\|D - ATB\| \leq 2\lambda\sqrt{C_r + \eta}(3 + \|T\|)\eta$$

and

$$\|A\| \leq \lambda\sqrt{C_r + \eta}, \qquad \|B\| \leq \sqrt{C_r + \eta}$$

(see [28, Remark 4.4]). Moreover, for all $(z_k)_{k=1}^\infty \in Z$, we have $A(z_k)_{k=1}^\infty = (A_k z_k)_{k=1}^\infty$ and $B(z_k)_{k=1}^\infty = (B_k z_k)_{k=1}^\infty$, where for each $k \in \mathbb{N}$, $A_k, B_k\colon E_k \to E_k$ are bounded linear operators of the same form as $A, B\colon E \to E$ in Remark 9.9. In particular, if all bases are C_r-strategically reproducible *with projectional factors*, then we have $A_k B_k = I_{E_k}$ for all $k \in \mathbb{N}$ and thus $AB = I_Z$. □

10. Strategic properties of the Haar system

Next, we are going to prove that for every Haar system Hardy space $Y \in \mathcal{HH}_0(\delta)$, the system $((h_I/\|h_I\|_Y, h_I^*))_{I\in\mathcal{D}}$ is 2-strategically supporting in $Y \times Y^*$, and under the assumption that $(r_n)_{n=0}^\infty$ is weakly null, the normalized Haar basis of Y is 2-strategically reproducible. The following lemma will be used in the proofs of both properties.

Lemma 10.1. *Let $Y \in \mathcal{HH}_0(\delta)$ and let $\eta > 0$. Then there exist $\sigma_0, \rho_0 > 0$ such that for all $0 < \sigma \leq \sigma_0$ and $0 < \rho \leq \rho_0$, the following holds:*

Let $(\tilde{h}_I)_{I\in\mathcal{D}}$ be a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system for some family $(\varkappa_I)_{I\in\mathcal{D}}$ of real numbers in $(0, 1]$ that satisfies

$$\sum_{I\in\mathcal{D}} (1 - \varkappa_I) \leq \sigma.$$

Moreover, for every $I \in \mathcal{D}$, let $\mathcal{B}_I \subset \mathcal{D}$ denote the Haar support of $\tilde{h}_I$, and suppose that $|\mathcal{B}_{[0,1)}^*| \geq \frac{1}{2} - \rho$. Put

$$x_I = \sqrt{2}\frac{\tilde{h}_I}{\|h_I\|_Y} \qquad \text{and} \qquad x_I^* = \frac{1}{\sqrt{2}}\frac{|I|}{|\mathcal{B}_I^*|} \cdot \frac{\tilde{h}_I}{\|h_I\|_{Y^*}} \tag{10.1}$$

for every $I \in \mathcal{D}$. Then the following assertions are true:

- *(i) The sequence $(x_I)_{I\in\mathcal{D}}$ is impartially $(2 + \eta)$-equivalent to $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ in Y; in particular, $(x_I)_{I\in\mathcal{D}}$ is $\sqrt{2 + \eta}$-norm-dominated by $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$.*
- *(ii) The sequence $(x_I^*)_{I\in\mathcal{D}}$ is impartially $(2 + \eta)$-equivalent to $(h_I/\|h_I\|_{Y^*})_{I\in\mathcal{D}}$ in Y^*.*
- *(iii) The sequence $(x_I^*)_{I\in\mathcal{D}}$ is $\sqrt{2 + \eta}$-weak*-dominated by $(h_I/\|h_I\|_{Y^*})_{I\in\mathcal{D}}$, and for every $x \in Y$, the series $\sum_{I\in\mathcal{D}} \langle x_I^*, x\rangle h_I/\|h_I\|_Y$ converges in norm.*

Proof. If $\sigma < 1$, then by Theorem 7.3, the operators $A, B\colon Y \to Y$ defined by

$$Bx = \sum_{I\in\mathcal{D}} \frac{\langle h_I, x\rangle}{\|h_I\|_2^2}\tilde{h}_I \qquad \text{and} \qquad Ax = \sum_{I\in\mathcal{D}} \frac{\langle \tilde{h}_I, x\rangle}{\|\tilde{h}_I\|_2^2} h_I$$

satisfy

$$\|B\| = 1 \qquad \text{and} \qquad \|A\| \le \frac{1}{\mu} \cdot \frac{1+3\sigma}{1-\sigma},$$

where $\mu = |\mathcal{B}^*_{[0,1)}| \ge \frac{1}{2} - \rho$. In particular, by choosing σ_0 and ρ_0 small enough, we can ensure that

$$\|B\| = 1 \qquad \text{and} \qquad \|A\| \le \sqrt{2(2+\eta)}. \tag{10.2}$$

Moreover, we know that $AB = I_Y$ and hence $B^*A^* = I_{Y^*}$, as well as

$$Bh_I = \tilde{h}_I, \quad A\tilde{h}_I = h_I \qquad \text{and} \qquad B^*\tilde{h}_I = \frac{|\mathcal{B}^*_I|}{|I|}h_I, \quad A^*h_I = \frac{|I|}{|\mathcal{B}^*_I|}\tilde{h}_I \tag{10.3}$$

for all $I \in \mathcal{D}$. Combining our estimates (10.2) with the identities in (10.1) and (10.3) yields (i) and (ii).

Next, if $(a_I)_{I\in\mathcal{D}}$ is a scalar sequence and $x^* \in Y^*$ is such that

$$x^* = \text{w}^*\text{-}\sum_{I\in\mathcal{D}} a_I \frac{h_I}{\|h_I\|_{Y^*}}, \tag{10.4}$$

then, since $A^* \colon Y^* \to Y^*$ is weak*-to-weak* continuous, it follows that

$$A^*x^* = \sqrt{2} \cdot \text{w}^*\text{-}\sum_{I\in\mathcal{D}} a_I x_I^* \tag{10.5}$$

is weak* convergent, and thus, combining (10.4) and (10.5) with (10.2) yields

$$\sqrt{2} \cdot \Big\|\text{w}^*\text{-}\sum_{I\in\mathcal{D}} a_I x_I^*\Big\|_{Y^*} \le \|A^*\|\|x^*\|_{Y^*} = \|A\|\Big\|\text{w}^*\text{-}\sum_{I\in\mathcal{D}} a_I \frac{h_I}{\|h_I\|_{Y^*}}\Big\|_{Y^*}.$$

Finally, for every $x \in Y$, using (10.1) and (10.3), we see that the series

$$\sum_{I\in\mathcal{D}} \langle x_I^*, x\rangle \frac{h_I}{\|h_I\|_Y} = \frac{1}{\sqrt{2}} \sum_{I\in\mathcal{D}} \Big\langle \frac{h_I}{\|h_I\|_{Y^*}}, Ax\Big\rangle \frac{h_I}{\|h_I\|_Y} = \frac{1}{\sqrt{2}}Ax$$

converges in norm. □

In the following, we will always identify the index $k \in \mathbb{N}$ from Definition 9.3 with a dyadic interval I via the bijection $\iota|_{\mathcal{D}} \colon \mathcal{D} \to \mathbb{N}$. Thus, the partition $\mathbb{N} = N_1 \cup N_2$ will correspond to a partition $\mathcal{D} = \mathcal{A}_1 \cup \mathcal{A}_2$. Moreover, instead of finite subsets E_k of N_1 or N_2, we will construct finite subsets $\mathcal{B}_I$ of $\mathcal{A}_1$ or $\mathcal{A}_2$, and the real numbers λ_j^k, μ_j^k and signs ε_j^k, $j \in E_k$, will be denoted as λ_L^I, μ_L^I and ε_L^I, $L \in \mathcal{B}_I$.

Theorem 10.2. *Let $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ denote the normalized Haar basis of $Y \in \mathcal{HH}_0(\delta)$, and let $(h_I^*)_{I\in\mathcal{D}}$ denote the biorthogonal functionals. Then the system $((h_I/\|h_I\|_Y, h_I^*))_{I\in\mathcal{D}}$ is 2-strategically supporting in $Y \times Y^*$ with projectional factors.*

Proof. Let $\eta > 0$ and let $\mathcal{A}_1$, $\mathcal{A}_2$ be a partition of $\mathcal{D}$. Thus, $\limsup(\mathcal{A}_1) \cup \limsup(\mathcal{A}_2) = [0,1)$, and hence, we have either

$$|\limsup(\mathcal{A}_1)| \ge \frac{1}{2} \qquad \text{or} \qquad |\limsup(\mathcal{A}_2)| \ge \frac{1}{2}.$$

If the former inequality is true, we define $i = 1$, and we put $i = 2$ otherwise. In any case, we obtain

$$|\limsup(\mathcal{A}_i)| \ge \frac{1}{2}.$$

By [27, Lemma 4.4], we can find a subset $\mathcal{A} \subset \mathcal{A}_i$ such that

- ▷ $\mathcal{G}_n(\mathcal{A})$ is finite and $\mathcal{G}_n(\mathcal{A}) \subset \mathcal{G}_n(\mathcal{A}_i)$ for all $n \in \mathbb{N}_0$,
- ▷ $|\limsup(\mathcal{A})| \ge \frac{1}{2} - \rho$,

where $\rho > 0$ is a small number, to be determined later. We will write $S = \limsup(\mathcal{A})$. Moreover, fix $0 < \sigma < 1$ to be determined later, and let $(\varkappa_I)_{I\in\mathcal{D}}$ be a sequence of real numbers in $(0,1)$ such that

$$\sum_{I\in\mathcal{D}}(1-\varkappa_I) \le \sigma.$$

Define $\varkappa_I' = 1 - \varkappa_I$ for all $I \in \mathcal{D}$.

Now let $I \in \mathcal{D}$ and assume that we have already picked strictly increasing frequencies $(n_J)_{J<I}$ and chosen collections $\mathcal{B}_J \subset \mathcal{G}_{n_J}(\mathcal{A})$ and positive real numbers $(\lambda_L^J)_{L\in\mathcal{B}_J}, (\mu_L^J)_{L\in\mathcal{B}_J} \in \mathbb{R}^{\mathcal{B}_J}$ for all $J < I$. Moreover, suppose that the sequences of signs $(\varepsilon_L^J)_{L\in\mathcal{B}_J} \in \{\pm1\}^{\mathcal{B}_J}$, $J < I$, are given, thus determining the newly constructed systems $(x_J)_{J<I}$ and $(x_J^*)_{J<I}$ in (9.1), i.e.,

$$x_J = \sum_{L\in\mathcal{B}_J} \varepsilon_L^J \lambda_L^J \frac{h_L}{\|h_L\|_Y} \qquad \text{and} \qquad x_J^* = \sum_{L\in\mathcal{B}_J} \varepsilon_L^J \mu_L^J h_L^*.$$

In addition, we define the auxiliary L^∞-normalized Haar system $(\tilde{h}_J)_{J<I}$ by putting

$$\tilde{h}_J = \sum_{L\in\mathcal{B}_J} \varepsilon_L^J h_L, \qquad J \in \mathcal{D},\ J < I.$$

We now describe the construction of $\mathcal{B}_I$ and $(\lambda_L^I)_{L\in\mathcal{B}_I}, (\mu_L^I)_{L\in\mathcal{B}_I} \in \mathbb{R}^{\mathcal{B}_I}$, which in turn determine x_I and x_I^*. Firstly, we pick $\mathcal{B}_I \subset \mathcal{A}$ as follows:

$$\begin{aligned} &\mathcal{B}_I = \mathcal{G}_{n_I}(\mathcal{A}), && \text{if } I = [0,1),\\ &\mathcal{B}_I = \big\{L \in \mathcal{G}_{n_I}(\mathcal{A}) : L \subset \{\tilde{h}_{\pi(I)} = \pm 1\}\big\}, && \text{if } I = J^{\pm} \text{ for some } J \in \mathcal{D}, \end{aligned}$$

where $n_I \in \mathbb{N}_0$ is chosen sufficiently large such that $n_I > n_J$ for all $J < I$ and such that

$$|\mathcal{B}_I^* \cap S| > (1 - \varkappa_I'/2)|\mathcal{B}_I^*|. \tag{10.6}$$

Secondly, we define the non-negative real numbers

$$\lambda_L^I = \sqrt{2}\cdot\frac{\|h_L\|_Y}{\|h_I\|_Y} \qquad \text{and} \qquad \mu_L^I = \frac{1}{\sqrt{2}}\frac{|I|}{|\mathcal{B}_I^*|}\cdot\frac{\|h_L\|_{Y^*}}{\|h_I\|_{Y^*}}, \qquad L \in \mathcal{B}_I.$$

Now we fix $(\varepsilon_L^I)_{L\in\mathcal{B}_I} \in \{\pm1\}^{\mathcal{B}_I}$ and observe that

$$x_I = \sum_{L\in\mathcal{B}_I} \varepsilon_L^I \lambda_L^I \frac{h_L}{\|h_L\|_Y} \qquad \text{and} \qquad x_I^* = \sum_{L\in\mathcal{B}_I} \varepsilon_L^I \mu_L^I h_L^*$$

is in accordance with (9.1). Recall that by Lemma 4.4, we have $\|h_K\|_Y\|h_K\|_{Y^*} = |K|$, $K \in \mathcal{D}$, and thus, we obtain

$$\langle x_I^*, x_I\rangle = \sum_{L\in\mathcal{B}_I} \lambda_L^I \mu_L^I = 1,$$

which shows condition (iii) in Definition 9.3.

We record the following identities relating $\tilde{h}_I$ with x_I and x_I^*:

$$x_I = \frac{\sqrt{2}}{\|h_I\|_Y}\tilde{h}_I \qquad \text{and} \qquad x_I^* = \frac{1}{\sqrt{2}}\frac{|I|}{|\mathcal{B}_I^*|}\cdot\frac{\tilde{h}_I}{\|h_I\|_{Y^*}}.$$

We will now discuss essential properties of our construction. First, let $I \in \mathcal{D}\setminus\{[0,1)\}$. Exploiting the nestedness of the dyadic intervals, one can show that $\mathcal{B}_I^* \cap S = \{\tilde{h}_{\pi(I)} = \pm1\} \cap S$, and thus, by (10.6) and [27, Lemma 4.5 (i)], we have

$$|\mathcal{B}_I^*| \ge |\mathcal{B}_I^* \cap S| = |\{\tilde{h}_{\pi(I)} = \pm1\} \cap S| \ge (1 - \varkappa_{\pi(I)}')\frac{|\mathcal{B}_{\pi(I)}^*|}{2} = \varkappa_{\pi(I)}\frac{|\mathcal{B}_{\pi(I)}^*|}{2}.$$

Moreover, by the definition of $\mathcal{B}_{I^{\pm}}$, we have $\operatorname{supp}(\tilde{h}_{I^{\pm}}) \subset \{\tilde{h}_I = \pm 1\}$ for all $I \in \mathcal{D}$. Thus, it follows that $(\tilde{h}_I)_{I\in\mathcal{D}}$ is a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system. If ρ and σ are chosen sufficiently small, then by Lemma 10.1, $(x_I)_{I\in\mathcal{D}}$ is $\sqrt{2+\eta}$-norm-dominated by $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ and $(x_I^*)_{I\in\mathcal{D}}$ is $\sqrt{2+\eta}$-weak*-dominated by $(h_I^*)_{I\in\mathcal{D}}$, and moreover, $\sum_{I\in\mathcal{D}}\langle x_I^*, x\rangle h_I/\|h_I\|_Y$ converges in norm for all $x \in Y$. Finally, $(x_I^*)_{I\in\mathcal{D}}$ is clearly biorthogonal to $(x_I)_{I\in\mathcal{D}}$. ☐

Lemma 10.3. *Let $Y \in \mathcal{HH}_0(\delta)$ and assume that the sequence of Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in Y. Moreover, suppose that $\mathcal{A} \subset \mathcal{D}$ has finite generations. For every sequence of signs $\varepsilon = (\varepsilon_K)_{K\in\mathcal{D}} \in \{\pm1\}^{\mathcal{D}}$ and every $n \in \mathbb{N}_0$, put*

$$\tilde{r}_n(\varepsilon) = \sum_{K\in\mathcal{G}_n(\mathcal{A})} \varepsilon_K h_K.$$

Then for every $y \in Y$ and $y^* \in Y^*$, we have

$$\lim_{n\to\infty} \sup_{\varepsilon\in\{\pm1\}^{\mathcal{D}}} |\langle \tilde{r}_n(\varepsilon), y\rangle| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sup_{\varepsilon\in\{\pm1\}^{\mathcal{D}}} |\langle y^*, \tilde{r}_n(\varepsilon)\rangle| = 0. \tag{10.7}$$

Proof. Let $\eta > 0$. Since H_0 is dense in Y, we can find $z \in H_0$ such that $\|y - z\|_Y \le \eta$. Then for all sufficiently large $n \in \mathbb{N}_0$ and all $\varepsilon \in \{\pm1\}^{\mathcal{D}}$, we have $\langle \tilde{r}_n(\varepsilon), z\rangle = 0$ and hence

$$|\langle \tilde{r}_n(\varepsilon), y\rangle| = |\langle \tilde{r}_n(\varepsilon), y - z\rangle| \le \eta$$

because $\|\tilde{r}_n(\varepsilon)\|_{Y^*} \le 1$ by Corollary 7.2.

In order to prove the second equality in (10.7), we first fix $y^* \in Y^*$ and $\varepsilon \in \{\pm1\}^{\mathcal{D}}$. By Lemma 5.5, there exists another collection $\hat{\mathcal{A}} \subset \mathcal{D}$ with finite generations such that $\mathcal{G}_n^*(\hat{\mathcal{A}}) = [0,1)$ and $\mathcal{G}_n(\mathcal{A}) \subset \mathcal{G}_n(\hat{\mathcal{A}})$ for all $n \in \mathbb{N}_0$, and there exists a bounded Haar multiplier $R\colon Y \to Y$ with $\|R\| \le 1$ such that for every $n \in \mathbb{N}_0$ and $K \in \mathcal{G}_n(\hat{\mathcal{A}})$, we have

$$Rh_K = \begin{cases} h_K, & K \in \mathcal{G}_n(\mathcal{A}),\\ 0, & K \in \mathcal{G}_n(\hat{\mathcal{A}}) \setminus \mathcal{G}_n(\mathcal{A}).\end{cases}$$

Now let $(\hat{h}_I)_{I\in\mathcal{D}}$ be the faithful Haar system defined as in Remark 5.4 with respect to the collection $\hat{\mathcal{A}}$ and the sequence of signs ε. Since $(\hat{h}_I)_{I\in\mathcal{D}}$ is faithful, we know from Proposition 7.1 that the operator $\hat{B}\colon Y \to Y$ defined as the linear extension of $\hat{B}h_I = \hat{h}_I$, $I \in \mathcal{D}$, is bounded. Now we can write

$$\tilde{r}_n(\varepsilon) = R\Big(\sum_{K\in\mathcal{G}_n(\hat{\mathcal{A}})} \varepsilon_K h_K\Big) = R\Big(\sum_{I\in\mathcal{D}_n} \hat{h}_I\Big) = R\hat{B}r_n, \qquad n \in \mathbb{N}_0,$$

and since $(r_n)_{n=0}^\infty$ is weakly null in Y, it follows that

$$\lim_{n\to\infty} |\langle y^*, \tilde{r}_n(\varepsilon)\rangle| = 0. \tag{10.8}$$

Clearly, there exists a sequence $\varepsilon \in \{\pm1\}^{\mathcal{D}}$ that satisfies $|\langle y^*, \tilde{r}_n(\varepsilon)\rangle| = \sup_{\theta\in\{\pm1\}^{\mathcal{D}}} |\langle y^*, \tilde{r}_n(\theta)\rangle|$ for all $n \in \mathbb{N}_0$. Hence, by applying (10.8) to this sequence, we obtain the second equality in (10.7). ☐

Theorem 10.4. *Let $Y \in \mathcal{HH}_0(\delta)$ and suppose that the sequence of Rademacher functions $(r_n)_{n=0}^\infty$ is weakly null in Y. Then the normalized Haar basis $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ of Y is 2-strategically reproducible in Y with projectional factors.*

Proof. Fix $\eta > 0$. In the following, we will describe a winning strategy for player (II) in the game $\operatorname{Rep}_{(Y,(h_I/\|h_I\|_Y))}(2,\eta)$. Before the game starts, player (I) chooses a partition $\mathcal{D} = \mathcal{A}_1 \cup \mathcal{A}_2$. Thus, $\limsup(\mathcal{A}_1) \cup \limsup(\mathcal{A}_2) = [0,1)$, and hence, we have

$$|\limsup(\mathcal{A}_1)| \ge \frac{1}{2} \qquad \text{or} \qquad |\limsup(\mathcal{A}_2)| \ge \frac{1}{2}.$$

We may assume without loss of generality that $|\limsup(\mathcal{A}_1)| \geq \frac{1}{2}$. By [27, Lemma 4.4], we can find a subset $\mathcal{A} \subset \mathcal{A}_1$ such that

▷ $\mathcal{G}_n(\mathcal{A})$ is finite and $\mathcal{G}_n(\mathcal{A}) \subset \mathcal{G}_n(\mathcal{A}_1)$ for all $n \in \mathbb{N}_0$,
▷ $|\limsup(\mathcal{A})| \geq \frac{1}{2} - \rho$,

where $\rho > 0$ is a small number, to be determined later. We will write $S = \limsup(\mathcal{A})$. Moreover, fix $0 < \sigma < 1$ to be determined later, and let $(\varkappa_I)_{I\in\mathcal{D}}$ be a family of real numbers in $(0, 1)$ such that

$$\sum_{I\in\mathcal{D}}(1 - \varkappa_I) \leq \sigma.$$

Define $\varkappa'_I = 1 - \varkappa_I$ for all $I \in \mathcal{D}$.

Now consider turn $I \in \mathcal{D}$. Let $(n_J)_{J<I}$ denote a strictly increasing sequence of natural numbers and suppose that for each $J < I$, player (II) has already chosen a finite set $\mathcal{B}_J \subset \mathcal{G}_{n_J}(\mathcal{A})$ and non-negative real numbers $(\lambda^J_L)_{L\in\mathcal{B}_J}$, $(\mu^J_L)_{L\in\mathcal{B}_J}$. Moreover, suppose that player (I) has chosen sequences of signs $(\varepsilon^J_L)_{L\in\mathcal{B}_J} \in \{\pm1\}^{\mathcal{B}_J}$, $J < I$, thus determining the systems $(x_J)_{J<I}$ and $(x^*_J)_{J<I}$ defined in (9.6), i.e.,

$$x_J = \sum_{L\in\mathcal{B}_J} \varepsilon^J_L\lambda^J_L \frac{h_L}{\|h_L\|_Y} \qquad\text{and}\qquad x^*_J = \sum_{L\in\mathcal{B}_J} \varepsilon^J_L\mu^J_L \frac{h_L}{\|h_L\|_{Y^*}}.$$

In addition, we define the auxiliary L^∞-normalized Haar system $(\tilde{h}_J)_{J<I}$ by putting

$$\tilde{h}_J = \sum_{L\in\mathcal{B}_J} \varepsilon^J_L h_L, \qquad J \in \mathcal{D},\ J < I.$$

At the beginning of turn I, player (I) chooses $\eta_I > 0$ and spaces $W_I \in \operatorname{cof}(Y)$ and $G_I \in \operatorname{cof}_{w^*}(Y^*)$. Then player (II) may choose a subset $\mathcal{B}_I$ of either $\mathcal{A}_1$ or $\mathcal{A}_2$ and finite sequences of non-negative real numbers $(\lambda^I_L)_{L\in\mathcal{B}_I}, (\mu^I_L)_{L\in\mathcal{B}_I} \in \mathbb{R}^{\mathcal{B}_I}$. In this winning strategy, player (II) picks $\mathcal{B}_I \subset \mathcal{A}$ as follows:

$$\begin{aligned} &\mathcal{B}_I = \mathcal{G}_{n_I}(\mathcal{A}), && \text{if } I = [0, 1),\\ &\mathcal{B}_I = \big\{L \in \mathcal{G}_{n_I}(\mathcal{A}) : L \subset \{\tilde{h}_{\pi(I)} = \pm1\}\big\}, && \text{if } I = J^\pm \text{ for some } J \in \mathcal{D}, \end{aligned}$$

where $n_I \in \mathbb{N}_0$ is chosen sufficiently large such that $n_I > n_J$ for all $J < I$ and such that the following two conditions are satisfied:

(i) $|\mathcal{B}^*_I \cap S| > (1 - \varkappa'_I/2)|\mathcal{B}^*_I|$.
(ii) For every $\varepsilon \in \{\pm1\}^{\mathcal{D}}$, the following two inequalities hold for $h = \sum_{L\in\mathcal{B}_I} \varepsilon_L h_L$:

$$\operatorname{dist}_Y(h, W_I) \leq \tilde{\eta}_I \qquad\text{and}\qquad \operatorname{dist}_{Y^*}(h, G_I) \leq \tilde{\eta}_I, \tag{10.9}$$

where

$$\tilde{\eta}_{[0,1)} = \eta_{[0,1)} \cdot \frac{1}{\sqrt{2}}|S| \qquad\text{and}\qquad \tilde{\eta}_I = \eta_I \cdot \frac{|I|}{\sqrt{2}} \min\bigg(1, \varkappa_{\pi(I)}\frac{|\mathcal{B}^*_{\pi(I)}|}{2|I|}\bigg),$$

for all $I \in \mathcal{D} \setminus \{[0, 1)\}$.

We have to show that such a number n_I exists. According to [27, Lemma 4.5 (ii)], condition (i) is satisfied for all sufficiently large n_I. In order to prove that (ii) is also satisfied for all large enough n_I, first recall that by Remark 9.7, there are $N, M \in \mathbb{N}_0$ and $y^*_1, \dots, y^*_N \in Y^*$ as well as $y_1, \dots, y_M \in Y$ such that

$$W_I = \{y^*_1, \dots, y^*_N\}_\perp \qquad\text{and}\qquad G_I = \{y_1, \dots, y_M\}^\perp.$$

Given $h \in H_0$, the inequalities (10.9) are certainly satisfied if $|\langle y^*_j, h\rangle|$ is sufficiently small for all $j \in \{1, \dots, N\}$ and $|\langle h, y_j\rangle|$ is sufficiently small for all $j \in \{1, \dots, M\}$. Thus, it suffices to prove the following assertion: Put $\Gamma = \{\tilde{h}_{\pi(I)} = \pm1\}$ if $I = J^\pm$ for some $J \in \mathcal{D}$

(and $\Gamma = [0, 1)$ if $I = [0, 1)$) and, in addition, for $n \in \mathbb{N}_0$, put $\tilde{\mathcal{G}}_n = \{L \in \mathcal{G}_{n_{\pi(I)}+1+n}(\mathcal{A}) : L \subset \Gamma\}$ and

$$\tilde{r}_n(\varepsilon) = \sum_{K\in\tilde{\mathcal{G}}_n} \varepsilon_K h_K, \qquad \varepsilon \in \{\pm 1\}^{\mathcal{D}}.$$

Then for every $y^* \in Y^*$ and $y \in Y$, we claim that

$$\lim_{n\to\infty} \sup_{\varepsilon\in\{\pm1\}^{\mathcal{D}}} |\langle y^*, \tilde{r}_n(\varepsilon)\rangle| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sup_{\varepsilon\in\{\pm1\}^{\mathcal{D}}} |\langle \tilde{r}_n(\varepsilon), y\rangle| = 0.$$

But this follows by applying Lemma 10.3 to $\tilde{\mathcal{A}} := \bigcup_{n=0}^{\infty} \tilde{\mathcal{G}}_n$ since $(r_n)_{n=0}^{\infty}$ is weakly null in Y. Thus, (i) and (ii) are satisfied if n_I is chosen sufficiently large.

After the set $\mathcal{B}_I$ is selected, player (II) chooses the non-negative real numbers

$$\lambda_L^I = \sqrt{2} \cdot \frac{\|h_L\|_Y}{\|h_I\|_Y} \qquad \text{and} \qquad \mu_L^I = \frac{1}{\sqrt{2}} \frac{|I|}{|\mathcal{B}_I^*|} \cdot \frac{\|h_L\|_{Y^*}}{\|h_I\|_{Y^*}}, \qquad L \in \mathcal{B}_I.$$

Recall that by Lemma 4.4, we have $\|h_K\|_Y \|h_K\|_{Y^*} = |K|$ for all $K \in \mathcal{D}$, so we obtain

$$\sum_{L\in\mathcal{B}_I} \lambda_L^I \mu_L^I = 1.$$

Next, player (I) chooses signs $(\varepsilon_L^I)_{L\in\mathcal{B}_I} \in \{\pm1\}^{\mathcal{B}_I}$. Observe that

$$x_I = \sum_{L\in\mathcal{B}_I} \varepsilon_L^I \lambda_L^I \frac{h_L}{\|h_L\|_Y} \qquad \text{and} \qquad x_I^* = \sum_{L\in\mathcal{B}_I} \varepsilon_L^I \mu_L^I h_L^*$$

is in accordance with (9.6). Then the next turn begins.

Now we assume that the game is completed, and we record the following identities relating $\tilde{h}_I$ to x_I and x_I^*:

$$x_I = \frac{\sqrt{2}}{\|h_I\|_Y} \tilde{h}_I \qquad \text{and} \qquad x_I^* = \frac{1}{\sqrt{2}} \frac{|I|}{|\mathcal{B}_I^*|} \cdot \frac{\tilde{h}_I}{\|h_I\|_{Y^*}}, \qquad I \in \mathcal{D}.$$

Like in the proof of Theorem 10.2, it follows from the above condition (i) that

$$|\mathcal{B}_I^*| \geq \varkappa_{\pi(I)} \frac{|\mathcal{B}_{\pi(I)}^*|}{2}, \qquad I \in \mathcal{D} \setminus \{[0, 1)\}, \tag{10.10}$$

and since $\operatorname{supp}(\tilde{h}_{I^\pm}) \subset \{\tilde{h}_I = \pm 1\}$ for all $I \in \mathcal{D}$, this implies that $(\tilde{h}_I)_{I\in\mathcal{D}}$ is a $(\varkappa_I)_{I\in\mathcal{D}}$-faithful Haar system. If ρ and σ are chosen sufficiently small, then by Lemma 10.1, $(x_I)_{I\in\mathcal{D}}$ is impartially $(2+\eta)$-equivalent to $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ in Y and $(x_I^*)_{I\in\mathcal{D}}$ is impartially $(2+\eta)$-equivalent to $(h_I/\|h_I\|_{Y^*})_{I\in\mathcal{D}} = (h_I^*)_{I\in\mathcal{D}}$ in Y^*. Furthermore, the inequalities (10.9), Lemma 4.4 and Proposition 4.1 (i) imply that $\operatorname{dist}_Y(x_I, W_I) \leq \eta_I$ for all $I \in \mathcal{D}$, and together with $|S| \leq |\mathcal{B}_{[0,1)}^*|$ and (10.10), we also have $\operatorname{dist}_{Y^*}(x_I^*, G_I) \leq \eta_I$ for all $I \in \mathcal{D}$. Finally, it is obvious that $(x_I^*)_{I\in\mathcal{D}}$ is biorthogonal to $(x_I)_{I\in\mathcal{D}}$. □

11. Proofs of Theorem 3.1 and Theorem 3.2

Our proof of Theorem 3.1 is based on the following result, which adds to the framework of strategically reproducible bases developed in [27]. More precisely, the result transfers [27, Theorem 3.12 and Theorem 7.6] to the setting of the primary factorization property: It allows us to reduce the primary factorization property to the primary *diagonal* factorization property with respect to a Schauder bases which is strategically reproducible with projectional factors.

Theorem 11.1. *Let E be a Banach space with a normalized Schauder basis $(e_j)_{j=1}^\infty$ with basis constant $\lambda \geq 1$. Let $C_r, C \geq 1$ and suppose that $(e_j)_{j=1}^\infty$ is C_r-strategically reproducible in E with projectional factors and that E has the C-primary diagonal factorization property with respect to $(e_j)_{j=1}^\infty$. Let Z be one of the following spaces:*

(i) $Z = E$
(ii) $Z = \ell^p(E)$ for some $1 \leq p < \infty$
(iii) $Z = \ell^\infty(E)$ if E is asymptotically curved with respect to $(e_j)_{j=1}^\infty$.

Then Z has the $\lambda C_r C$-primary factorization property, and hence, $\mathcal{M}_Z$ is the unique maximal ideal of $\mathcal{B}(Z)$. In particular, the spaces in (ii) and (iii) are primary.

Proof. Case (i). Let $T\colon E \to E$ be a bounded linear operator, and let $\eta > 0$. By Proposition 9.10 (i), there exists a bounded linear operator $D\colon E \to E$, which is diagonal with respect to $(e_j)_{j=1}^\infty$, such that D projectionally factors through T with constant $\lambda(C_r + \eta)$ and error η. Recall that by Remark 2.7, we also have that $I_E - D$ projectionally factors through $I_E - T$ with constant $\lambda(C_r + \eta)$ and error η. Moreover, by the hypothesis, we know that the identity I_E factors either through D or through $I_E - D$ with constant $C + \eta$ (and error 0). Thus, using Remark 2.5 and Remark 2.6, we conclude that for sufficiently small η, the identity I_E either factors through T or through $I_E - T$ with constant

$$\frac{\lambda(C_r + \eta)(C + \eta)}{1 - (C + \eta)\eta},$$

and this converges to $\lambda C_r C$ as $\eta \to 0$.

Case (ii). For each $k \in \mathbb{N}$, let $(e_{k,j})_{j=1}^\infty$ be a copy of $(e_j)_{j=1}^\infty$ in the kth component of $\ell^p(E)$. We know from Remark 9.11 that the sequence $(e_{k,j})_{j,k=1}^\infty$, enumerated as $(\tilde{e}_m)_{m=1}^\infty$ according to Remark 2.9, is a Schauder basis of $\ell^p(E)$ whose basis constant is bounded by λ, and it is C_r-strategically reproducible in $\ell^p(E)$ with projectional factors. Thus, by (i), it suffices to show that $\ell^p(E)$ has the C-primary diagonal factorization property with respect to $(\tilde{e}_m)_{m=1}^\infty$.

Let $D\colon \ell^p(E) \to \ell^p(E)$ be a bounded linear operator that is diagonal with respect to $(\tilde{e}_m)_{m=1}^\infty$. Then for each $k \in \mathbb{N}$, by restricting D to the kth component of $\ell^p(E)$, we obtain a bounded linear operator $D_k\colon E \to E$ that is diagonal with respect to $(e_j)_{j=1}^\infty$. By the hypothesis, the sets

$$\mathcal{N}_1 = \{k \in \mathbb{N} : I_E \text{ factors through } D_k \text{ with constant } C^+\},$$
$$\mathcal{N}_2 = \{k \in \mathbb{N} : I_E \text{ factors through } I_E - D_k \text{ with constant } C^+\}$$

satisfy $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathbb{N}$, so at least one of these sets is infinite. We assume without loss of generality that $\mathcal{N}_1$ is infinite. For each $k \in \mathcal{N}_1$, there exist bounded linear operators $A_k, B_k\colon E \to E$ such that

$$I_E = A_k D_k B_k \qquad \text{and} \qquad \|A_k\|\|B_k\| \leq C + \eta.$$

By rescaling, we may assume that $\|A_k\| \leq \sqrt{C + \eta}$ and $\|B_k\| \leq \sqrt{C + \eta}$. For $k \in \mathbb{N} \setminus \mathcal{N}_1$, we put $A_k = B_k = 0$. Now write $\mathcal{N}_1 = \{n_1 < n_2 < n_3 < \dots\}$ and define the operators $Q, R\colon \ell^p(E) \to \ell^p(E)$ by $Q(x_k)_{k=1}^\infty = (x_{n_k})_{k=1}^\infty$ as well as $R(x_k)_{k=1}^\infty = (y_k)_{k=1}^\infty$ where $y_{n_k} = x_k$ for all $k \in \mathbb{N}$ and $y_k = 0$ if $k \notin \mathcal{N}_1$. Note that $\|Q\|, \|R\| \leq 1$. Finally, define $A, B\colon \ell^p(E) \to \ell^p(E)$ by

$$A(x_k)_{k=1}^\infty = (A_k x_k)_{k=1}^\infty \qquad \text{and} \qquad B(x_k)_{k=1}^\infty = (B_k x_k)_{k=1}^\infty$$

and observe that $\|A\|, \|B\| \leq \sqrt{C + \eta}$ and $I_{\ell^p(E)} = QADBR$, so $I_{\ell^p(E)}$ factors through D with constant $C + \eta$ (and error 0).

Case (iii). Let $T\colon \ell^\infty(E) \to \ell^\infty(E)$ be a bounded linear operator, and let $\eta > 0$. Like in the proof for case (i), we first diagonalize the operator T. Here, however, instead of using Proposition 9.10 (i), we apply Lemma 6.5 and then Proposition 9.13 to obtain a bounded linear operator $D\colon \ell^\infty(E) \to \ell^\infty(E)$ which is diagonal with respect to $(e_{k,j})_{j,k=1}^\infty$ such that D projectionally factors through T with constant $\lambda(C_r + \eta)$ and error η. Next, like in the proof of (ii), we see that $\ell^\infty(E)$ has the C-primary diagonal factorization property with respect to $(e_{k,j})_{j,k=1}^\infty$. The rest of the proof is the same as for (i). □

Proof of Theorem 3.1. Since the sequence $(r_n)_{n=1}^\infty$ is weakly null in Y, we know from Theorem 10.4 that the normalized Haar basis $(h_I/\|h_I\|_Y)_{I\in\mathcal{D}}$ of Y is 2-strategically reproducible in Y with projectional factors. Moreover, Theorem 3.6 (iii) implies that Y has the 2-primary diagonal factorization property with respect to the Haar basis. Thus, the statements in Theorem 3.1 follow from Theorem 11.1. □

Proof of Theorem 3.2. First, we show that the Haar basis of Y (or, analogously, of any space Y_k, $k \in \mathbb{N}$) has the $2/\delta$-diagonal factorization property. Let $\delta > 0$, and let $D\colon Y \to Y$ be a bounded Haar multiplier with δ-large diagonal. Let $\gamma > 0$. Since by Theorem 10.2, the system $((h_I/\|h_I\|_Y, h_I^*))_{I\in\mathcal{D}}$ is strategically supporting in $Y \times Y^*$, we can apply Proposition 9.5, which yields a bounded Haar multiplier $\tilde{D}\colon Y \to Y$ with either δ-large *positive* or δ-large *negative* diagonal such that $\tilde{D}$ factors through D with constant $2+\gamma$ (and error 0). Hence, we have $|c| \geq \delta$ for all $c \in \Lambda(\tilde{D})$. Now Theorem 3.6 (ii) implies that the identity I_Y factors through $\tilde{D}$ with constant $(1/\delta)^+$ (and error 0). Thus, we have proved that I_Y factors through D with constant $(2/\delta)^+$.

Case (i). Since the sequence $(r_n)_{n=1}^\infty$ is weakly null in Y, $k \in \mathbb{N}$, we know from Theorem 10.4 that the normalized Haar basis 2-strategically reproducible in Y. Hence, if $\delta > 0$ and $T\colon Y \to Y$ is a bounded linear operator with δ-large diagonal with respect to the Haar basis, then by Proposition 9.10, for every $\eta > 0$, there exists a bounded Haar multiplier $D\colon Y \to Y$ with δ-large diagonal such that D factors through T with constant $2 + \eta$ and error η. Thus, the $4/\delta$-factorization property of the Haar basis of Y follows from $2/\delta$-diagonal factorization property together with Remark 2.5 and Remark 2.6. Alternatively, it also fallows from the strategic reproducibility and the diagonal factorization property using [27, Theorem 3.12].

Case (ii). We know that the Haar basis is 2-strategically reproducible and has the $2/\delta$-diagonal factorization property in each space Y_k, $k \in \mathbb{N}$. By [27, Lemma 7.3], it follows that $(h_{k,I})_{k\in\mathbb{N},I\in\mathcal{D}}$ has the $2/\delta$-diagonal factorization property in $\ell^p((Y_k)_{k=1}^\infty)$ for $1 \leq p < \infty$. Thus, the $4/\delta$-factorization property follows like in Case (i), again using Proposition 9.10 for diagonalization (see Remark 9.11). Alternatively, it follows from [27, Theorem 7.6].

Case (iii). In this case, the $4/\delta$-factorization property follows from [28, Theorem 3.9]. □

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