Bull Braz Math Soc, New Series (2026) 57:16
https://doi.org/10.1007/s00574-026-00497-4

Bifurcations in the Family of Billiards Associated with the Curvature Flow

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Received: 9 May 2025 / Accepted: 14 January 2026

Abstract

We describe some dynamical properties of one parameter families of billiards on convex curves (ovals) which are deformed by the curvature (curve-shortening) flow. We obtain the bifurcations of the period two orbits and some special non-Birkhoff orbits, the normal periodic orbits. We prove the destruction of non-convex caustics of the ellipse by deforming it through the curvature flow. This work has some results from the doctoral thesis Damasceno (Família de aplicações bilhares geradas pelo fluxo de curvature. Tese de Doutorado, UNICAMP, 2011) and Salazar (Região de Instabilidade em Bilhares. tese de Doutorado, UFMG, 2019).

Keywords Family · Curvature · Flow · Billiard

1 Introduction

This work aims to describe some dynamical properties of the family of billiard maps associated with a one-parameter family of curves satisfying the curvature flow or the curve shortening flow.

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This flow was studied in a series of papers by Gage (1983), Gage and Hamilton (1986), and Grayson (1989) that were published in the late 1980 s. These works deal mainly with the long-time behavior of regular closed plane curves which deform in the direction of the curvature vectors. The curves generally shrink to a point, but they become increasingly “round". This geometrical feature is proved by considering a *normalized flow* such that the enclosed area is constant equals to π. In this case, the curvature converges uniformly to one. Grayson proved that any simple closed curve becomes convex and then evolves as a family of convex curves approximating the circle of radius one.

Therefore it is natural to ask what type of changes occur in the billiard maps associated with this family of curves. In other words, changes in the dynamical properties can be observed as the curve evolves by the curvature flow.

Billard maps generally may have very complicated behavior, since it may contain chaotic regions, the Birkhoff instability regions. On the other hand, billiards on circles are trivial in the sense that the phase space is foliated by invariant circles and the map restricted to each circle is a rotation. Hence as the curves evolves the dynamics should tend to a less (although not monotonically) complicated behavior.

The main result of this paper is analogous to Theorem 1.1 of Damasceno et al. (2017) for homotopically trivial invariant curves around the elliptic period 2 orbit of the ellipse. The caustics corresponding to such invariant curves are co-focal hyperboles:

Theorem 1 *The normalized curvature flow breaks all resonant hyperbolic caustics of the billiard map on the ellipse.*

We also study the bifurcations of another type of periodic orbits, the normal periodic orbits $NP(2n)$ for a family of ovals C_t satisfying the normalized curve shortening flow.

Similar to the period two orbits, these periodic orbits start perpendicular to the boundary curve and after some reflections, they hit the boundary again orthogonally. Thus, after the reflection they return tracing back the same polygonal, arriving to the initial point.

There is a geometric condition for the existence of these orbits in terms of curves similar to evolutes. We use these curves to show in Corollary 2, that, as the boundary curve evolves by the curvature flow, these orbits disappear gradually.

In other words, in Theorem we prove:

If $X(t, s)$ is a family of convex curves satisfying the curvature flow, then for any integer $N > 1$, there is a real number T such that, for every $t > T$, the billiard map associated with the curve does not have a normal periodic orbit of period $2n \leq 2N$.

This also may be interpreted as *the long term breaking of a family of invariant rotational resonant curve around elliptic diameters*.

The paper is organized as follows: in Sect. 2 we describe the main properties of the curvature flow and briefly introduce the billiard map.

In Sect. 3, we discuss the example of the ellipse. In Sect. 4, the bifurcations of period two orbits (diameters) and the normal periodic orbits are considered. The last section is dedicated to the proof of Theorem 1.1.

Part of the results of this paper are contained in the Thesis of the first author (Damasceno 2011) and in the Thesis of the third author (Salazar 2019), both written in Portuguese.

2 The Curve Shortening Flow and Billiards

2.1 The Curve Shortening Flow

The main references for this section are Gage (1993), Gage and Hamilton (1986) and Gage (1983) and, for an exposition of the Curve Shortening Fllow, Chou and Zhu (2001).

We consider family of regular simple, closed curves in $\mathbb{R}^2$, with Euclidean metric, $X(t, u)$ that satisfies the curve shortening flow, with initial condition a strctly $\gamma(u) = X(t, u)$, convex close curve, of class C^r, $r \geq 2$:

$$\frac{\partial X}{\partial t}(t, u) = k(t, u)N(t, u)$$

Where $k(t, u)$ is the curvature and $N(t, u)$ is the normal (unitary) vector pointing inward. Observe that u is not necessarily the arc-length parameter.

The main results of this flow are the following:

Theorem 2 (Gage–Hamilton) *The curve shortening flow preserves convexity and shrinks any closed simple convex curve to a point.*

In Gage and Hamilton (1986), the authors prove in Theorem 3.2.1 that, that if the curvature is unifomly bounded and d the initial condition $X(\cdot, 0)$ is embedded, then each $X(\cdot, t)$ is embedded for $t \in (0, \omega)$, Biographies : then each $X(\cdot, t)$ is embedded, for $t \in (0, \omega)$. In Theorem 4.4.1 they prove the solution of the Cauchy problem, for $t \in [0, \omega)$ of the curvature equation—which is a quasilinear parabolic PDE, and finally, in Theorem 4.3.1 the solution obtained have area bounded away from 0 and the curvature is uniformly bounded in $\mathbb{S}^1 \times [0, \omega)$. The geometric arguments and estimates follow ideas of previous works by Gage, see Gage (1993) and Gage (1983).

Real analyticity of the curvature equation, for $t > 0$, and consequently of the family $X(t, u)$, $t > 0$, is obtained using a theorem by Angenent, see Angenent (1990b).

Theorem 3 (Grayson) (The Main Theorem in Grayson 1989) *Starting with any closed curve, it becomes convex before it shrinks to a point.*

To obtain ω, the maximal time of existe of a solution, let $A(t) = \frac{1}{2} \int_{\gamma(t)} xdy - ydx$ be the area enclosed by the curve $\gamma(t)$. Then $A'(t) = -2\pi$.

This result tells two things:

(a) The existence time τ of the solution is given by $\tau = \frac{A(0)}{2\pi}$;
(b) there is a natural normalization of the flow by taking a homothety $Y(t, u) = \frac{1}{\sqrt{\tau - 2t}} X(t, u)$ such that the area of the region enclosed by the curve $Y(t, .)$ is constant equal to π.

There is also a time normalization given by $\bar{t} = \frac{1}{2} ln[\frac{\tau}{\tau - t}]$, so that $t \to \tau$ implies $\bar{t} \to \infty$.

For the normalized flow, the area enclosed by the curves is constant, equal to π, and Gage and Hamilton prove that the family converges uniformly to a circle of radius 1.

In this work, we consider the *Normalized flow*. The normalization consists of a time t reparametrization, a translation, and a homothety, so the dynamical properties of the billiards maps do not change.

We state some facts that will be useful later.

Convex curves may be parametrized by the angle θ that its tangent makes with a fixed direction, denoting by $T(\theta)$ the unitary tangent vector and by $N(\theta)$ its normal vector pointing inwards, we get the Frenet's formula:

$$T'(\theta) = N(\theta) \quad N'(\theta) = -T(\theta)$$

See, for example, Flanders (1968).

Throughout this paper, g' will always denote the derivative of a function $g(t,\theta)$ with respect to the variable θ and g_t will denote the derivative with respect to t. Assuming that the domain U bounded by X contains the origin $O = (0,0)$ we let $h(\theta) = -\langle X(\theta), N(\theta)\rangle$ be the support function with respect to O. Then $X(\theta) = h'(\theta)T(\theta) - h(\theta)N(\theta)$ and $h''(\theta) = R(\theta) - h(\theta)$, for $R(\theta) = \frac{1}{k(\theta)}$, the radius of curvature.

Using this representation, the equation for the normalized flow, with t the normalized time, is

$$X_t(t,\theta) = [h'(t,\theta) - k'(t,\theta)]T(\theta) + [k(t,\theta) - h(t,\theta)]N(\theta)$$

As for the normalized geometric quantities we have the following evolution equations:

$$k_t(t,\theta) = [k(t,\theta)]^2 k''(t,\theta) + [k(t,\theta)]^3 - k(t,\theta)$$

$$h_t(t,\theta) = h(t,\theta) - k(t,\theta)$$

For bifurcation problems, the parameter $\theta \in \mathbb{S}^1$ is more convenient because it provides a uniform domain for all curves. This is the parametrization used in the work of Gage–Hamilton, and it simplifies some equations.

One of the results that Gage and Hamilton have proved in Gage and Hamilton (1986), Theorem 5.7.1, is the uniform exponential decay of the derivatives of the curvature:

Theorem 4 *For every $n \geq 1$ and $0 < \alpha < 1$, there exists a constant $C(n)$ such that*:

$$\|k^{(n)}\|_\infty \leq C(n)e^{-2\alpha t}$$

Moreover, for any $\epsilon > 0$, there is $C(\epsilon) > 0$ and $t_\epsilon > 0$ such that

$$|k(\theta,t) - 1| < C(\epsilon)e^{-(2-\epsilon)t}, \forall t > t_\epsilon$$

Analogously estimates follows for the radius of curvature $R(\theta)$.

2.2 Strictly Convex Billiards

The billiard map is obtained from the free motion of a point particle on the planar region U being reflected elastically at the impacts on its boundary X. The trajectories are polygonals on this planar region. Since the motion is free inside the region, it is completely determined by the points of impact at X and the direction of the motion. This direction is measured by the angle between the line and the tangent of the curve.

Therefore, to each oval parametrized by the angle θ of a tangent vector to a fixed direction, we associated a billiard map B from the cylinder $\mathbb{S}^1 \times (0, \pi)$ into itself, defined in the following way: given a point $x \in X$, let $\rho_{(x,\phi)}$ be the ray through x making a angle ϕ with the vector tangent at x. The billiard map assigns to (x, ϕ) the point (x_1, ϕ_1), where x_1 is the intersection of $\rho_{(x,\phi)}$ with the curve X, and ϕ_1 is the reflected angle between $\rho_{(x,\phi)}$ and the tangent to X at x_1.

If the curve X is parametrized by the arc-length s, then the billiard map is a monotone twist map and has a generating function: $L(s, s') = ||X(s') - X(s)||$ where $|| \cdot ||^2 = \langle \cdot, \cdot \rangle$, the euclidean inner product. This means:

$$-\frac{\partial L}{\partial s} = \cos(\phi)$$

$$\frac{\partial L}{\partial s'} = \cos(\phi_1).$$

3 Bifurcations of Normal Periodic Points Along the Curvature Flow

3.1 Period Two Orbits

Let $X(t, \theta)$ be a family of simple convex plane curves parametrized by θ, the angle between its tangent vector and a fixed direction. Suppose that X satisfies the normalized curve shortening equation.

It is well known that the period two orbits are the critical points of $\ell(t, \theta) = \|X(t, \theta + \pi) - X(t, \theta)\|$.

In other words, the value θ corresponds to a periodic orbit of period two if and only if $f(t, \theta) = \langle T(\theta), X(t, \theta + \pi) - X(t, \theta)\rangle = -h'(t, \theta + \pi) - h'(t, \theta) = 0$

The classification of period two orbits is also well-known, (see in Dias Carneiro et al. (2007)):

- Hyperbolic: $\ell(t, \theta) - [R(t, \theta) + R(t, \theta + \pi)] > 0$ or $\ell(t, \theta) - [R(t, \theta) + R(t, \theta + \pi)] < 0$ and $[\ell(t, \theta) - R(t, \theta)][\ell(t, \theta) - R(t, \theta + \pi)] < 0$
- Elliptic: $\ell(t, \theta) - [R(t, \theta) + R(t, \theta + \pi)] < 0$ and $[\ell(t, \theta) - R(t, \theta)][\ell(t, \theta) - R(t, \theta + \pi)] > 0$
- Parabolic: $\ell(t, \theta) - [R(t, \theta) + R(t, \theta + \pi)] = 0$ or $\ell(t, \theta) - [R(t, \theta) + R(t, \theta + \pi)] < 0$ and $[\ell(t, \theta) - R(t, \theta)][\ell(t, \theta) - R(t, \theta + \pi)] = 0$

Remark 1 For sufficiently large t and for every θ, $\ell(t, \theta) - R(t, \theta) > 0$.

Indeed, let $w(t) = \int_{\mathbb{S}^1} log(h(t,\theta))d\theta$.

Then

$$w'(t) = \int_{\mathbb{S}^1} \frac{(h(t,\theta) - k(t,\theta)}{h(t,\theta)} d\theta = 2\pi - \int_{\mathbb{S}^1} \frac{k(t,\theta)}{h(t,\theta)} d\theta$$

Substituting

$$\frac{k(t,\theta)}{h(t,\theta)} = \frac{[h(t,\theta) - k(t,\theta)]^2}{h(t,\theta)k(t,\theta)} + 2 - \frac{h(t,\theta)}{k(t,\theta)}$$

and using the fact $\int_{\mathbb{S}^1} \frac{h(t,\theta)}{k(t,\theta)} d\theta = 2Area(U) = 2\pi$ to obtain

$$w'(t) = -\int_{\mathbb{S}^1} \frac{[h(t,\theta) - k(t,\theta)]^2}{h(t,\theta)k(t,\theta)} d\theta < 0$$

So $w(t)$ is strictly decreasing and bounded from below. Hence $w'(t)$ converges to 0. Since $k(t,\theta)$ converges uniformly to 1 we obtain that $h(t,\theta)$ also converges uniformly to 1.

As a consequence we obtain $\ell(t,\theta) - R(t,\theta) = h(t,\theta+\pi) + h(t,\theta) - R(t,\theta)$ converges to 1.

Hence, by assuming $t > 0$ large enough, the above conditions may be formulated in a simplified form:

- Hyperbolic: $\ell(t,\theta) - [R(t,\theta) + R(t,\theta+\pi)] > 0$
- Elliptic: $\ell(t,\theta) - [R(t,\theta) + R(t,\theta+\pi)] < 0$
- Parabolic: $\ell(t,\theta) - [R(t,\theta) + R(t,\theta+\pi)] = 0$

Therefore, the dynamical properties of the period two orbits, and their bifurcations are obtained by analyzing the curve $f(t,\theta) = 0$.

Let us assume that θ_0 is a period two orbit for the curve $X(t_0,\theta)$, that is: $f(t_0,\theta_0) = \langle X(t_0,x_0) - X(t_0,x_0), T(\theta)\rangle = 0 = -[h'(t_0,\theta_0+\pi) + h'(t_0,\theta_0)]$. Observe that if $f'(t_0,\theta_0) = \ell(t,\theta) - [R(t,\theta) + R(t,\theta+\pi)] \neq 0$ then by the Implicit function theorem there is a C^r family of nondegenerate period-two orbits $(t,\theta(t))$, Hyperbolic or Elliptic according to the sign of $f'(t_0,\theta_0)$.

The *parabolic* period-two orbits are defined by:

$f(t_0,\theta_0) = 0$ and $f'(t,\theta) = h(t,\theta) + h(t,\theta+\pi) - [R(t,\theta) + R(t,\theta+\pi)]$

Recall the equation of the evolute of the curve $X(t,\cdot)$: $E(t,\theta) = X(t,\theta) + R(t,\theta)N(\theta) = h'(t,\theta)T(\theta) + [R(t,\theta) - h(t,\theta)]N(\theta)$.

The following difference characterizes the period-two orbits:

$$E(t,\theta+\pi) - E(t,\theta) = f(t,\theta)T(\theta) + f'(t,\theta)N(\theta)$$

A parabolic period-two orbit is characterized by $E(t_0,\theta_0+\pi) - E(t_0,\theta_0) = 0$. ($f'(t,\theta)$ measures the speed of crossing of the two points).

$$\begin{aligned} f_t(t,\theta) &= -h'(t,\theta+\pi) - h'(t,\theta) + k'(t,\theta+\pi) + k'(t,\theta) \\ &= -f(t,\theta) + k'(t,\theta+\pi) + k'(t,\theta) \end{aligned} \tag{1}$$

Integrating, we obtain:

$$f(t,\theta) = e^{t_0 - t} f(t_0,\theta) + e^{t_0 - t}\left\{\int_{t_0}^{t} e^s[k'(s,\theta+\pi) + k'(s,\theta)]ds\right\}$$

Hence if $\int_{t_0}^{t} e^s[k'(s,\theta_0+\pi) + k'(s,\theta_0)]ds \neq 0$ it follows that $f(t,\theta) \neq 0$ for $t > t_0$ and θ near θ_0 and there is no period two orbit for the curve $X(t,\theta)$. A typical bifurcation of a parabolic periodic point.

Proposition 5 *Let θ_0 be a parabolic orbit of period two for the billiard associated with the curve $X(t_0,\cdot)$. Let us assume that there exists an integer $n > 1$ such that $k^{(n)}(t_0,\theta_0) + k^{(n)}(t_0,\theta_0+\pi) \neq 0$.*

Then there is a neighborhood of (t_0,θ_0) such that, any periodic orbit of period two in $V_0 - \{(t_0,\theta_0)\}$ is non-degenerate, elliptic or hyperbolic.

Proof Observe that the non-flatness condition is satisfied for $t_0 > 0$ due to the analycity of the family $X(t,\theta)$.

If $n = 2$, then $k''(s,\theta_0) + k''(s,\theta_0+\pi) \neq 0$ for $s > t_0$ small. Hence

$$f'(t,\theta_0) = e^{t_0 - t} f'(t_0,\theta_0) + e^{t_0 - t}\left\{\int_{t_0}^{t} e^s[k''(s,\theta_0+\pi) + k''(s,\theta_0)]ds\right\} \neq 0$$

This implies that any period two orbit for $X(t,.)$ near (t_0,θ_0) is hyperbolic or elliptic.

For general n, the argument is similar. Let $n > 2$ be the smallest integer such that $k^{(n)}(t_0,\theta_0) + k^{(n)}(t_0,\theta_0+\pi) \neq 0$

By the Taylor expansion we get

$$k''(t_0,\theta) + k''(t_0,\theta+\pi) = [k^{(n)}(t_0,\theta_0) + k^{(n)}(t_0,\theta_0+\pi)]\frac{(\theta-\theta_0)^n}{n!} + O(n)$$

Hence we also get an isolated zero at θ_0 and there is a neighborhood V_0 of (t_0,θ_0) such that $k''(t,\theta) + k''(t,\theta+\pi) \neq 0$ in $V - (t_0.\theta_0)$.

By applying the same reasoning as above, we get $f'(t,\theta) \neq 0$ and the conclusion of the statement holds: any periodic orbit of period two in $V_0 - (t_0,\theta_0)$ is non-degenerate, elliptic or hyperbolic. □

If $\forall t \geq t_0$ such that $f(t,\theta_0) = 0$, we have $k'(t,\theta_0+\pi) + k'(t,\theta_0) = 0$, then the line $\theta = \theta_0, t \geq t_0$ defines a family of parabolic period-two orbits.

In other words the point $E(t,\theta_0+\pi) = E(t,\theta_0)$ moves in the chord from $X(t,\theta_0)$ to $X(t,\theta_0+\pi)$.

Since $R(t,\theta) \to 1$ uniformly as $t \to \infty$ the width $h(t,\theta_0+\pi) + h(t,\theta_0) \to 2$ and $E(t,\theta_0)$ converges to the origin, the middle point of the chord.

Example 1 Curve of constant width.

As an example of the above evolution of diameters, let us consider a curve of constant width. In other words, a closed simple convex curve such that any point of type $(\theta, \frac{\pi}{2})$

is a diameter. This means that in the phase space of the billiard map, the straight line $\Gamma\{\phi = \frac{\pi}{2}\}$ is invariant.

The condition of constant width is equivalent to:

$f(t,\theta) = \langle X(\theta+\pi) - X(\theta), T(\theta)\rangle = 0$ or $h(\theta) + h(\theta+\pi) = d, \forall\theta \in [0, 2\pi]$, d a positive number.

If $f_t(0,\theta_0) \neq 0$, for some θ_0, then for $t > 0$ small, the curve $X(t,\theta)$ is not of constant width anymore. Or, the invariant line is destroyed.

Hence, let us assume that the initial curve of constant width also satisfies $f_t(0,\theta_0) = k'(0,\theta+\pi) + k'(0,\theta) = 0, \forall\theta$.

We claim that such a curve is already the circle of radius one, that is the stationary curve. To see this, we recall that $h''(\theta) = R(\theta) - h(\theta)$, hence, for curves of constant width we have $R(\theta+\pi) - h(\theta+\pi) + R(\theta) - h(\theta) = 0$. If this is the case, $R(0,\theta+\pi) + R(0,\theta) = d$ and, using $k'(0,\theta+\pi) + k'(0,\theta) = 0$, we obtain that $X(0,\theta)$ is a circle with area π.

The conclusion is: in the family deformed by the curvature flow starting in a curve of constant width, which is not the circle of radius one, the dynamic of the billiard map immediately gets more complicated. By Angement's theorem (Angenent 1990a), if there is no separatrix connection the we get positivity of topological entropy due to the presence of non-Birkhoff periodic orbits and an instability region around the period two orbits.

These special periodic orbits are the subject of the next Section.

3.2 The Family $NP(2n)$ of Normal Periodic Points

If the diameters are not contained in an invariant curve (either a resonant rotational invariant curve or a separatrix), then there is a Birkhoff instability region D that contains them. We recall that D is an invariant region, homeomorphic to an annulus, that does not contain any rotational invariant curve, whose boundary is a Lipschitz graph. The topological dynamics in such invarinant region is well known, see for instance Le Calvez (1987), but, in the case of billiard maps the region D has special features due to its symmetry with respect to the involution $\tau(\theta,\varphi) = (\theta, \pi - \varphi)$, $\tau(D) = D$. This region contains a special collection of periodic points, called *Normal periodic points*, closely related to the family of period two orbits.

The normal periodic points are non-Birkhoff orbits and are of particular interest because they give a partial codification of the instability region D, by just considering orbits that cross the curve $\Gamma = Fix(\tau)$. Γ be the orthogonal wavefront, that is, the subset of initial conditions $(\theta, \frac{\pi}{2})$, and the symmetric region D is formed by invariant subsets contained in the complement of the elliptic island set. In Dias Carneiro et al. (2007) it is proved that, under certain generic conditions, billiards on ovals have only a finite number of periodic orbits, all non-degenerate, and at least one diameter is hyperbolic. Moreover, the invariant curves of these hyperbolic points have a transversal intersection. See Salazar (2019).

Definition 1 (*Normal periodic orbits*) We say that θ_0 is a normal periodic orbit for a billiard map B if $n \geq 1$ is the smallest integer such that there is a point $\theta_1 \neq \theta_0$ such that $B^n(\theta_0, \frac{\pi}{2}) = (\theta_1, \frac{\pi}{2})$.

It follows immediately from the definition that $(\theta_0, \frac{\pi}{2})$ is a periodic point of period $2n$. The trajectory of this periodic orbit starts orthogonal to the boundary curve, and after $n-1$ reflections, it hits the boundary again orthogonally, retracing its path in the opposite direction.

For example, the diameters $NP(2) = B(\Gamma) \cap \Gamma \neq \emptyset$, but the set $NP(2n)$ may be empty for some n. For example, $NP(4) = \emptyset$ in the case of an ellipse with small eccentricity.

In this Section, we show that as the family of curves evolves as $t \to \infty$ by the curvature flow, these orbits gradually disappear. This is done by describing some geometric necessary conditions for the existence of normal periodic points.

Since $B^{-1} = \tau \circ B \circ \tau$, if $z \in \Gamma$ and n is the smallest positive integer such that $B^{2n}(z) = z$, then $B^n(z) = B^{-n}(z) = \tau \circ B^n(z)$, that is, $B^n(z) \in \Gamma$.

The symmetric invariant region D may be characterized as the smallest invariant annulus that contains the closure of $\cup B^n(\Gamma)$. If there is a hyperbolic diameter σ, then $D = \bar{W^u(\sigma)} \cup [\cup_i U_i]$, with U_i a periodic disk. Also, an important feature is that $W^u(\sigma) \cap \Gamma$ is contained in the homoclinic set. That is the reason why in the sequel we study the singularities of the projections $\pi(B^n(\Gamma), n \neq 1$.

Proposition 6 *Suppose that for every positive integer n, $B^n(\Gamma)$ is a graph over Γ. Then $B(\Gamma) = \Gamma$ and the boundary of the billiard table is a curve of constant width.*

Proof The hypothesis gives $B^n(\Gamma) \cap \Gamma = NP(2)$. Let us assume, by contradiction, that $B(\Gamma) \neq \Gamma$. Let us consider the curve $B^2(\Gamma)$, which by hypothesis is a graph over Γ. By denoting $I = [\sigma_0, \sigma_1] \subset \Gamma$ the maximal interval between two distinct period two points such that $B^2(I) \cap I = \{\sigma_0, \sigma_1\}$, we have that the curve $B^2(I) \cup R \circ B^2(I)$ bounds a disc which is invariant under B^2 and contains I. Since B^2 is area-preserving, there is an intersection between $B^2(I)$ and the interior of I which is a point in $NP(2n)$, with $n > 1$. This is a contradiction. □

We now describe the geometric conditions for the existence of normal periodic points in terms of the envelope of families of rays (wavefronts) that is related to the critical points on the projection $\pi(B^n(\Gamma)$. This may be considered a generalization of the relation between the position of the evolute of the curve and the existence of normal periodic orbits of period 4.

Firstly, we obtain some estimates on the evolution of *Chords* between two points of a path along an orbit.

Let $v_\phi = \cos(\phi)T(\theta_0) + \sin(\phi)N(\theta_0)$,$0 \leq \phi \leq \frac{\pi}{2}$ and $X(\theta_1) = h'(\theta_1)T(\theta_1) - h(\theta_1)N(\theta_1)$ with $[X(\theta_1) - X(\theta_0)] \wedge v_\phi = 0$, then

$\langle X(\theta_1) - X(\theta_0), v_\phi \rangle = h'(\theta_1)\cos(\theta_1 - \theta_0 - \phi) + h(\theta_1)\sin(\theta_1 - \theta_0 - \phi) - \cos(\phi)h'(\theta_0) + \sin(\phi)h(\theta_0)$.

Integration by parts gives:

$\int_{\theta_0}^{\theta_1} \cos(\theta - \theta_0 - \phi)R(\theta)d\theta = \int_{\theta_0}^{\theta_1} \cos(\theta - \theta_0 - \phi)[h''(\theta) + h(\theta)]d\theta$
$= \langle X(\theta_1) - X(\theta_0), v_\phi \rangle = \|X(\theta_1) - X(\theta_0)\|$ the length of the chord.

Observe that for $\theta_1 = \theta_0 + \pi, \phi = \frac{\pi}{2}$ we obtain the *width* $\omega(\theta_0) = h(\theta_0 + \pi) + h(\theta_0)$. Here we generalize formula proved in the Lemma 3.9 of Chou and Zhu (2001) for the width.

By following that proof, which we reproduce here, for the sake of completeness, we obtain an estimate for the chord in terms of the *entropy* of the curve, $\mathcal{E} = \frac{1}{2\pi}\int_0^{2\pi} \log k(\theta)d\theta$, thus obtaining the following estimate:

Claim: There is a constant C, such that $L(\theta_0, \phi) \geq Ce^{-\mathcal{E}(X)}$.

Proof By Jensen's inequality, $\log L(\theta_0, \phi) = \log \int_{\theta_0}^{\theta_1} |\cos(\theta - \theta_0 - \phi)|R(\theta)d\theta$
$\geq \frac{1}{\theta_1-\theta_0}\int_{\theta_0}^{\theta_1} \log|\cos(\theta-\theta_0-\phi)|d\theta - \frac{1}{\theta_1-\theta_0}\int_{\theta_0}^{\theta_1}\log[k(\theta)]d\theta - \frac{1}{\theta_1-\theta_0}\log(\theta_1-\theta_0)$.
Analogously, by integrating from θ_1 to $\theta_0 + 2\pi$ we obtain:
$\|X(\theta_1) - X(\theta_0)\| = \int_{\theta_1}^{\theta_0+2\pi} \cos(\theta - \theta_1 - \phi_1)R(\theta)d\theta$, for $v_{\phi_1} = v_{\phi+\pi}$.
$\log L(\theta_1, \phi + \pi) = \log \int_{\theta_1}^{\theta_0+2\pi} |\cos(\theta - \theta_1 + \phi + \pi)|R(\theta)d\theta$
$\geq \frac{1}{\theta_0+2\pi-\theta_1}\int_{\theta_1}^{\theta_0+2\pi} \log|\cos(\theta - \theta_1 + \phi)|d\theta - \frac{1}{\theta_0+2\pi-\theta_1}\int_{\theta_1}^{\theta_0+2\pi}\log[k(\theta)]d\theta-$
$\frac{1}{\theta_0+2\pi-\theta_1}\log(\theta_0 + 2\pi - \theta_1)$.
By adding the two estimates, we obtain:
$2\log[\|X(\theta_1) - X(\theta_0)\| \geq C_0 - \frac{2}{\pi}\mathcal{E}(X)$, C_0 independent of the curve. □

The uniform boundedness from below of the chord follows from the fact (Gage and Hamilton 1986 or Lemma 3.8 of Chou and Zhu 2001) that the entropy of the curve $X(., t)$ *decreases monotonically* along the normalized curvature flow, that $\frac{d}{dt}\mathcal{E}(X(., t) < 0$ and the lower boundedness of the curvature for the normalized flow.

We now use the above estimate to show that given an integer $n > 1$, there is $T_n > 0$ large enough, such that the set of normal periodic points $NP(2n)$ of the curve $X(t, 0)$ is empty, for $t > T_n$.

For $n = 1$, we consider the intersection between the normal ray and the curve. This intersection is the point θ_1 that satisfies: $\langle X(t, \theta_1) - X(t, \theta), T(\theta)\rangle = 0$.

By convexity, $\frac{\partial}{\partial\theta_1}\langle X(\theta_1) - X(\theta), T(\theta)\rangle = R(\theta_1)\langle T(\theta_1), T(\theta)\rangle \neq 0$. So this equation defines implicitly a C^r function $A_1 : \mathbb{S}^1 \to \mathbb{S}^1$ by $\theta_1 = A_1(\theta) = \pi_1 \circ B$.

Proposition 7 *A_1 is a diffeomorphism if and only if the evolute of X is contained in the region bounded by X.*

Proof Indeed, $A_1'(\theta) = -\frac{\ell(t,\theta)-R(t,\theta)}{R(A_1)\langle T(A_1),T(\theta)\rangle}$. Hence $A_1'(\theta) = 0$ if and only if $\ell(t, \theta) - R(t, \theta) = 0$.

The statement follows from the fact that $X(\theta) + R(\theta)N(\theta)$ is the equation of the evolute of X.

□

Observe that $Fix(A_1) = \emptyset$ and the period two orbits of A_1 correspond exactly to $NP(2)$, period two orbits of the billiard map.

Proposition 8 *$A_1(t, \theta)$ converges uniformly as $t \to \infty$ to $\pi + \theta$(antipodalfunction).*

Proof This follows from the uniform converge of curvature k and the support function h to 1.

Recall that, denoting $x_0 = R(\theta)$, $x_1 = R(A_1(t, \theta))\sin(\phi_1(t, \theta))$, ϕ_1 is the reflected angle with respect to the tangent $T(A_1(t, \theta))$, so

$$\begin{pmatrix} A_1'(\theta) \\ \phi_1'(\theta) \end{pmatrix} = \frac{1}{x_1} \begin{pmatrix} x_0 - l_0 \\ l_0 - x_0 - x_1 \end{pmatrix}$$

$l_0 = \|X(A_1(t,\theta)) - X(t,\theta)\| = \int_\theta^{A_1(t,\theta)} \sin(\xi - \theta) R(t,\xi) d\xi =$
$-\cos[A_1(t,\theta) - \theta] R(A_1(t,\theta)) + R(t,\theta) + \int_\theta^{A_1((t,\theta)} \cos(\xi - \theta) R'(t,\xi) d\xi.$
Since $\cos[A_1(t,\theta) - \sin(\theta)] = \sin(\phi_1(t,\theta))$, we obtain
$\frac{x_0 - l_0}{x_1} = 1 - \frac{1}{x_1} \int_\theta^{A_1(t,\theta)} \cos(\xi - \theta) R'(t,\xi) d\xi.$

Using the uniform exponential decay of R' we obtain the exponential decay of $|A_1'(t,\theta) - 1|$, as $t \to \infty$, since $\frac{1}{x_1}$ is bounded.

Moreover, $\phi_1'(t,\theta) = 1 - A_1'(t,\theta)$ implies the exponential decay to 0 of this term. □

Corollary 1 *For every t sufficiently large $NP(4) = \emptyset$.*

Proof The existence of a point in $NP(4)$, which is not in a period-two orbit, implies the existence of a critical pointof A_1. Indeed, if $B^2(\theta_0, \frac{\pi}{2}) = (\theta_2, \frac{\pi}{2})$, for $\theta_2 \neq \theta_0$), then $B(\theta_0, \frac{\pi}{2}) = \tau \circ B(\theta_2, \frac{\pi}{2})$. That is: $A_1(\theta_0) = A_1(\theta_2)$ which implies the existence of a critical point ζ, $A_1'(\zeta) = 0$.

The length of a regular arc of the evolute is the difference of the radius of curvature, $\ell_E(\theta_a, \theta_b) = |R(\theta_a) - R(\theta_b)| \leq |R'(\zeta)||\theta_a - \theta_b|$. Since the point of the evolute corresponding to the vertice of maximal curvature is always inside the region bounded by the curve X, the uniform exponential decay of R' imply that for t large enough, the evolute of the curve $X(t,\theta)$ is entirely contained in the region bounded by X.

Thus, concluding that the orthogonal wave front focalizes in the interior of the plane region $U(t)$. In this case A_1 is a diffeomorphism and $NP(4) = \emptyset$. □

We now proceed for higher periods, $n > 2$ to show that for $t > 0$ large enough $NP(2n) = \emptyset$.

Let $B^m(\theta, \frac{\pi}{2}) = (A_m(\theta), \phi_m(\theta))$ and denote $x_m = R(A_m(\theta)) \sin(\phi_m(\theta))$, $l_m = \|X(A_{m+1}) - X(A_m)\|$.

$$DB(A_m(\theta), \phi_m(\theta)) = \frac{1}{x_{m+1}} \begin{pmatrix} l_m - x_m & l_m \\ l_m - x_m - x_{m+1} & l_m - x_{m+1} \end{pmatrix}$$

By induction,

$$\begin{pmatrix} A_{m+1}'(\theta) \\ \phi_{m+1}'(\theta) \end{pmatrix} = \frac{1}{x_{m+1}} \begin{pmatrix} l_m - x_m & l_m \\ l_m - x_m - x_{m+1} & l_m - x_{m+1} \end{pmatrix} \begin{pmatrix} A_m'(\theta) \\ \phi_m'(\theta) \end{pmatrix}$$

Therefore, $A_{m+1}'(\theta) = 0$, for $m \geq 1$ if and only if $l_m = \frac{x_m}{A_m'(\theta) + \phi_m'(\theta)} A_m'(\theta)$.

As in the case $n = 1$ we stablish the relatioshionship between the critical points of A_{m+1} and envelopes.

The reflected angle at $X(t, A_m(t,\theta))$, denoted by $\phi_m(t,\theta)$, defines a wave front

$$\upsilon_{\phi_m(t,\theta)} = \cos(\phi_m(t,\theta)) T(A_m(t,\theta)) + \sin(\phi_m(t,\theta)) N(A_m(t,\theta))$$

This wave front focalizes in the envelope of the family of lines $X(A_m(t,\theta)) + \lambda \upsilon_{\phi_m(\theta)}$, which is defined by $E'_m(\theta) \wedge \upsilon_{\phi_m(t,\theta)} = 0$. Equivalently, for $x_m = R(t, A_m(\theta))\sin(\phi_m(t,\theta))$,

$$E_m(t,\theta) = X(A_m(t,\theta)) + \frac{x_m}{A'_m(t,\theta) + \phi'_m(t,\theta)} A'_m(t,\theta)\upsilon_{\phi_m(t,\theta)}.$$

We conclude that if $A'_m(t,\theta) = 0$ with $A'_j(t,\theta) = 0 for j = 1, \dots, m-1$ if and only if the $E_m(t,\theta) = X(A_{m+1}(t,\theta))$ an intersection point between the envelope and the curve.

Therefore, the inexistence of a normal periodic point is connected with the fact that the envelope is entirely contained in the region bounded by the curve X.

Proposition 9 *For t sufficiently large, the envelope E_m is entirely contained in the interior of the region $U(t)$ bounded by the curve $X(t,.)$.*

Proof If we let $\beta_m(t,\theta) = \frac{\phi'_m(t,\theta)}{A'_m(t,\theta)}$, then

$$\beta_m(t,\theta) = 1 - x_m \frac{1 - \beta_{m-1}(t,\theta)}{l_{m-1} - x_{m-1} - l_{m-1}\beta_{m-1}(t,\theta)}.$$

We obtain, by induction, $lim_{t\to\infty}\beta_m(t,\theta) = 0$, since $lim_{t\to\infty} l_m = 2$ and $lim_{t\to\infty} x_m = 1$.

The conclusion follows from the expression

$$E_m(t,\theta) = X(A_m(t,\theta)) + \frac{x_m}{1 + \beta_m(t,\theta)} \upsilon_{\phi_m(t,\theta)}.$$

□

Let us assume that $NP(2n) \neq \emptyset$ we are going to show that there is j such that A_j is not a diffeomorphism.

We consider two cases: $n = 2k$ and $n = 2k + 1$.

For the case $n = 2k$, we have $B^{4k}(x) = x$ or $\tau \circ B^{2k} \circ B^{2k}(x) = x$, assume that k is the smallest such integer.

Since $\tau \circ B \circ \tau = B^{-1}$ and $y = B^{2k}(x) \neq x$, we have $\tau \circ B^k(B^k(x)) = y$ or $\tau(B^k(x)) = B^k(y)$, that is, $A_k = p_1 \circ B^k$ is not injective, so there is a point $z \in \Gamma$ such that $A'_k(z) = 0$.

In the second case, $B^{4k+2}(x) = x$ or $B^{4k}(B^2(x)) = x \in \Gamma = Fix(\tau)$.

Then $\tau \circ B^{2k} \circ B^{2k}(B^2(x)) = x$ implies $\tau \circ B^{2k}(B^2(x) = B^{2k}(x)$. Since $B^{2k}(B^2 x)) \neq B^{2k}(x)$, this implies, the existence of a point z such that $A'_{2k}(z) = 0$.

Theorem 10 *Given a positive integer j, there exists a positive number t_j such that for all $t > t_j$ the function $\theta \mapsto A_m(t,\theta)$ is a diffeomorphism for $m = 1, 2, \dots, j$.*

Corollary 2 *For all $t > t_j$, obtained in the previous theorem there $NP(2n) = \emptyset$, $2 \le n \le 2j$ for the billiard maps associated to the curves $X(t,\theta)$. In particular, there is no invariant curve of period $2n$ surrounding an elliptic diameter.*

The curvature flow destroys the resonant invariant curve of type $2n$ surrounding an elliptic period-two orbit. Observe that an invariant curve of period $2n$ surrounding an elliptic period-two orbit must intersect Γ in $NP(2n)$. Therefore, there is no resonant curve of period $2n$ around an eliptical diameter.

The above result applies to the case of an ellipse with small excentricity. All the above envelopes as strictly contained in the convex region U, so $NP(2n) = \emptyset$.

Proposition 11 *Ellipses with small eccentricity, the normal periodic orbits which are not diameters have large periods.*

Proposition 12 *Let $\Gamma = \{(\theta, \frac{\pi}{2}\}$, the set of initial conditions perpendicular to the boundary curve γ. If $B^n(\Gamma)$ is a graph for every $n > 0$ then X has constant width.*

Since there is a time $t_j > 0$ such that for all $t > t_j$ the map A_n is a circle diffeomorphism with fixed points (the diameters), there exists not a periodic orbit of higher period. Thus $NP(2n) = \emptyset$.

Observe that an invariant curve of period $2n$ surrounding an elliptic period-two orbit must intersect Γ. Therefore, it must contain a point of $NP(2n)$. A contradiction.

The curvature flow destroys the resonant invariant curve of type $2n$ surrounding an elliptic period-two orbit.

3.3 Destruction of Resonant Non-convex Caustics of Ellipses

In this Section, we adapt the argument used to prove (Damasceno et al. 2017), to show that the curvature flow destroys every resonant invariant curve around the elliptic diameter, thus, producing many instability regions around these periodic orbit. We recall that such invariant curves, where every orbit is periodic, correspond to non-convex caustics (hyperbolas) of the elliptical billiard. See Chang and Freiberg (1988).

Example 2 In Salazar (2019), an elementary computation shows that the invariant curves corresponding to period four orbits around the elliptic diameter do not persist under the curvature flow. Any homotopically trivial invariant curve around the diameter σ_0 contains a normal periodic point of period four $NP(4)$. The deformation by the curvature flow has the axial symmetries of the ellipse, so the equation for $NP(4)$ is $\langle X(\epsilon, \varphi) - X(\epsilon, \frac{\pi}{2}), X'(\epsilon, \varphi)\rangle = 0$.

For $X(\epsilon, \varphi) = (\cosh(\mu_\epsilon(\varphi))\cos(\varphi), \sinh(\mu_\epsilon(\varphi))\sin(\varphi))$ with $\mu_\epsilon(\varphi) = \mu_0 + \mu_1(\varphi) + O(\epsilon^2)$, an even π-periodic function, with $\cosh(\mu_0) = a$, $\sinh(\mu_0) = b$ and $\mu_1(\varphi) = \frac{-ab}{(a^2\cos^2(\varphi)+\sin^2(\varphi))^2}$, see Damasceno et al. (2017).

For $\epsilon = 0$ and $a^2 > 2b^2$, the two orbits in $NP(4)$ are parabolic and belong to an invariant curve around the elliptic diameter. The set $NP(4)$ is defined locally as an analytic family $\epsilon \mapsto \varphi(\epsilon)$.

The final step is to show that for $\epsilon > 0$ every orbit in $NP(4)$ is *hyperbolic*. For this, one uses again the axial symmetry in order to compute $\tau(\epsilon)$, the trace of $dB^4(\varphi(\epsilon), \frac{\pi}{2})$, which is represented by $[dB^2(\varphi(\epsilon), \frac{\pi}{2})]^2$.

As a consequence, there is no invariant curve of period four around the elliptic period two orbit. It follows, by a Theorem of Angenent (1990a), that *the topological* *entropy of the billiard map is positive*. So the deformation by the curvature flow of the Billiard map of the ellipse produces a Birkhoff region of instability, with a non-trivial hyperbolic set around the elliptic diameter.

For general resonant non-convex caustics, the above argument of example is replaced by one using the generating function of $B^2 = B \circ B$.

For that, we first prove the following already known lemma on the "conjunction" of two generating functions see Mather (1987):

Lemma 1 *Let h_1 and h_2 be generating functions of two exacts area preserving twist maps F_1 and F_2 respectively. Suppose that, $\frac{\partial^2}{\partial x_1^2}h_1(x_0, x_1) + \frac{\partial^2}{\partial x_1^2}h_2(x_1, x_2) \neq 0$. Then the composition $F_2 \circ F_1$ is generated by $h(x_0, x_2) = h_1(x_0, x_1) + h_2(x_1, x_2)$, restricted to the set $\frac{\partial}{\partial x_1}h_1(x_0, x_1) + \frac{\partial}{\partial x_1}h_2(x_1, x_2) = 0$.*

Proof By the Implicit Function Theorem, $\frac{\partial}{\partial x_1}h_1(x_0, x_1) + \frac{\partial}{\partial x_1}h_2(x_1, x_2) = 0$ defines a surface $x_1 = g(x_0, x_2)$ if it is non-empty and $\frac{\partial^2}{\partial x_1^2}h_2(x_0, x_1) + \frac{\partial^2}{\partial x_1{}^2}h_2(x_1, x_2) \neq 0$.

If this is the case, then $-\frac{\partial}{\partial x_0}h(x_0, x_2) = -[\frac{\partial}{\partial x_0}h_1(x_0, x_1) + \frac{\partial}{\partial x_1}h_1(x_0, x_1)\frac{\partial x_1}{x_0} + \frac{\partial}{\partial x_1}h_2(x_1, x_2)\frac{\partial x_1}{x_0}] = -\frac{\partial}{\partial x_0}h_1(x_0, x_1) = -y_0$, using the restriction condition.

This condition also gives

$\frac{\partial}{\partial x_2}h(x_0, x_2) = \frac{\partial}{\partial x_1}h_1(x_0, x_1)\frac{\partial x_1}{x_2} + \frac{\partial}{\partial x_1}h_2(x_1, x_2)\frac{\partial x_1}{x_2} + \frac{\partial}{\partial x_2}h_2(x_1, x_2) = y_2$. $\square$

Remark: For the billiard map the generating function is $L(s, s') = \|X(s') - X(s)\|$ we have $\frac{\partial^2 L}{\partial s'^2}(s, s') + \frac{\partial^2 L}{\partial s''^2}(s', s'') = 2\sin(\phi)[\sin(\phi)(\frac{1}{L(s,s')} + \frac{1}{L(s',s'')}) + k(s')] > 0$, where $\cos(\phi) = \langle\frac{X(s')-X(s)}{L(s,s')}, X'(s')\rangle = \langle\frac{X(s'')-X(s')}{L(s',s'')}, X'(s')\rangle$

So the generating function of B^2 is

$h(s, s'') = L(s, s') + L(s', s'')$ restricted to $\frac{\partial}{\partial s'}[L(s, s') + L(s', s'')] = 0$.

We now proceed to adapt the arguments used in Damasceno et al. (2017) to prove the following:

Theorem 13 *The deformation of an ellipse along the curvature flow destroys all periodic invariant curves of the associated billiard map around the elliptic diameter (Non-convex Resonant Caustics).*

Proof We start by considering the Melnikov potential applied to the generating function of B^2.

We are now interested in trajectories tangent to *Non-convex Caustics*

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1$$

with $b < \lambda < a$.

Write, as above, the deformed ellipse by the curvature flow as $X(\epsilon, \varphi) = (\cosh(\mu_\epsilon(\varphi))\cos(\varphi), \sinh(\mu_\epsilon(\varphi))\sin(\varphi))$ with $\mu_\epsilon(\varphi) = \mu_0 + \mu_1(\varphi) + O(\epsilon^2)$, $\cosh(\mu_0) = a$, $\sinh(\mu_0) = b$ and

$$\mu_1(\varphi) = \frac{-ab}{\left(a^2\cos^2(\varphi) + \sin^2(\varphi)\right)^2}$$

Let us consider the *Action*

$$W_\epsilon^{p,2q}(\varphi_0, \varphi_2, \ldots, \varphi_{2q}) = \sum_{j=0}^{2q-1} L_\epsilon(\varphi_j, \varphi_{j+1})$$

restricted to
$\frac{\partial}{\partial \varphi_{2j+1}}[L_\epsilon(\varphi_{2j+1}, \varphi_{2j}) + L_\epsilon(\varphi_{2j+2}, \varphi_{2j+1})] = 0$, for $j = 0, \ldots, n-1$.

By writing $L_\epsilon = L_0 + \epsilon L_1 + O(\epsilon^2)$, and the generating function for B_ϵ^2 as $h_\epsilon = h_0 + \epsilon h_1 + O(\epsilon^2)$ we calculate $h_1(\varphi, \varphi'') = L_1(\varphi, \varphi') + L_1(\varphi', \varphi'')$ restricted to $\frac{\partial}{\partial \varphi'}[L_0(\varphi, \varphi') + L_0(\varphi', \varphi'')] = 0$.

This last equation defines a smooth function $\varphi' = \Psi(\varphi, \varphi'')$, hence,we write:

$$h_1(\varphi, \varphi'') = L_1(\varphi, \Psi(\varphi, \varphi'')) + L_1(\Psi(\varphi, \varphi''), \varphi'')$$

To obtain the $(p, 2q)$-subhamonic Melnikov potential we write:

Recall that $L_1(\varphi, \varphi') = \langle \frac{X_0 \varphi') - X_0(\varphi)}{L(\varphi, \varphi')}, X_1(\varphi') - X_1(\varphi)\rangle$ where the deformed curve is $X(\epsilon, .) = X_0 + \epsilon X_1 + O(\epsilon^2)$, so $X_1(\varphi) = a\mu_1(\varphi) D^{-2} X_0(\varphi)$.

$$W_\epsilon^{p,2}(\varphi_0, \varphi_2, \ldots, \varphi_{2q}) = W_0^{p,2q} + \epsilon W_1^{p,2q} + O(\epsilon^2)$$

with

$$W_1^{p,2q} = \sum_{j=0}^{2q-1} h_1(\varphi_{2j}, \varphi_{2j+2}) = \sum_{j=0}^{2q-1} L_1(\varphi_j, \varphi_{j+1})$$

restricted to $\frac{\partial}{\partial \varphi_{2j+1}}[L_0(\varphi_{2j+1}, \varphi_{2j}) + L_0(\varphi_{2j+2}, \varphi_{2j+1})] = 0$, for $j = 0, \ldots, q-1$.

We are ready to apply the results of Ramirez-Ros et. al:

Claim 1 Lemma 10 of Pinto-de-Carvalho and Ramírez-Ros (2013) Let $p_j = \frac{X_0(\varphi') - X_0(\varphi)}{L_0(\varphi, \varphi')}$ and C_λ be the Caustic (hyperbole) tangent to the billiard trajectory inside the ellipse determined by $(X_0(\phi_j))$. Then

$$a\langle p_{j-1} - p_j, D^{-2} X_0(\phi_j)\rangle = 2\lambda$$

where $D = diag(a, 1)$, the diagonal matrix.

Claim 2 The $(p, 2q)$-subharmonic Melnikov potential is:

$$\sum_{j=0}^{2q-1} L_1(\varphi_j, \varphi_{j+1}) = 2\lambda \sum_{j=0}^{2q-1} \mu_1(\phi_j)$$

restricted to $\frac{\partial}{\partial \varphi_{2j+1}}[L_0(\varphi_{2j+1}, \varphi_{2j}) + L_0(\varphi_{2j+2}, \varphi_{2j+1})] = 0$, for $j = 0, \ldots, q-1$.

Proof
Using the above notation we write $\varphi_{2j+1} = \Psi_\epsilon(\varphi_{2j}, \varphi_{2j+2})$,

$$h_1(\varphi_{2j}, \varphi_{2j+2}) = L_1(\varphi_{2j}, \Psi_0(\varphi_{2j}, \varphi_{2j+2})) + L_1(\Psi_0(\varphi_{2j}, \varphi_{2j+2}), \varphi_{2j+2})$$

But $L_1(\varphi_j, \varphi_{j+1}) = \langle p_j, X_1(\varphi_{j+1}) - X_1(\varphi_j)\rangle$
$= ab\langle p_j, \mu_1(\varphi_{j+1})D^{-2}X_0(\varphi_{j+1}) - \mu_1(\varphi_j)D^{-2}X_0(\varphi_j)\rangle$, so proceeding as in [?], Lemma 10,

$$\sum_{j=0}^{q-1} h_1(\varphi_{2j}, \varphi_{2j+2}) = \sum_{j=0}^{q-1} L_1(\varphi_{2j}, \Psi_0(\varphi_{2j}, \varphi_{2j+2})) + L_1(\Psi_0(\varphi_{2j}, \varphi_{2j+2}), \varphi_{2j+2})$$

$$= 2\lambda \left[\sum_{j=0}^{q-1} \mu_1(\varphi_{2j}) + \sum_{j=0}^{q-1} \mu_1(\Psi_0(\varphi_{2j}, \varphi_{2j+2}))\right].$$

Claim 3 h_1 is not identically zero.

Proof: We now use Damasceno et al. (2017) and a time reparametrization so that the dynamics along an invariant RIC of type $(p, 2q)$ becomes a shift $t \mapsto t + \delta(\lambda)$. For this, we calculate the period and modulus of the corresponding elliptic function associated to hyperbolic caustic, see Chang and Freiberg (1988).

We also use the fact that, for the ellipse, Ψ_0 is an analytic function and we proceed by calculating the Laurent expansion as in Lemma 5.1 of Damasceno et al. (2017).

Here are some details:

Let $k^2(\lambda) = \frac{a^2-\lambda^2}{a^2-b^2}$, with $b < \lambda < a$, the defining parameter of the hyperbolic caustic. The caustic corresponds to a periodic orbit of type $\frac{p}{2q} \in \mathbb{Q}$ if and only if $\sqrt{1-\frac{1}{\lambda}} = cn(\frac{p}{q}K(\lambda))$, with $K(k) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-k^2\sin^2(u)}}$, $\sin(\frac{\zeta}{2}) = \lambda$.

$k = k(\lambda)$ is the *modulus* and $\delta(\lambda) = 2\int_0^{\frac{\zeta}{2}} \frac{du}{\sqrt{1-k^2\sin^2(u)}}$ is the associated *period*.

$t = F(\varphi, k) = \int_0^{\varphi} \frac{1}{\sqrt{1-k^2\sin^2 u}} du$ defines a time reparametrization such that the dynamics along an orbit associated to C_λ becomes a shift $t \mapsto t + \delta(\lambda)$.

In Lemma 5.1 of Damasceno et al. (2017) it is been proven that the function

$$\tilde{\mu}_1(t) = \frac{-ab}{(a^2cn^2(t) + b^2sn^2(t))^2}$$

can be extended to $\mathbb{C}$ as an elliptic function of order four with poles of order two at $T_\pm = [\pm\frac{\zeta}{2} + K'i] + 2K\mathbb{Z} + 2K'\mathbb{Z}$, $\zeta = 2K - \delta$.

And Laurentz expansion

$$\tilde{\mu}_1(t_\pm + \tau) = \frac{\alpha_2}{\tau^2} \pm \frac{\alpha_1}{\tau} + O(1), \tau \to 0$$

The fact that $\alpha_2 \neq 0$ implies, as in Proposition 5.3 of Damasceno et al. (2017) that Melnikov potential is not constant.

To prove this we write the sub-harmonic potential in the variable t as

$$\tilde{L}_1(t,\delta) = 2\lambda \left[\sum_{j=0}^{q-1} \mu_1(t+2j\delta) + \sum_{j=0}^{q-1} \mu_1(\Psi_0(t+2j\delta, t+(2j+2)\delta) \right].$$

Recall that, in our case, the twist condition implies that the function Ψ_0 is *analytic* with $t \mapsto \Psi_0(t+2j\delta, t+(2j+2)\delta)$ a diffeomorphism.

Therefore, according to Proposition 5.3, the poles of the complex extension of (t,δ) are defined by: either $t+2j\delta \in \mathcal{T}$ or $\Psi_0(t+2j\delta, t+(2j+2)\delta) \in \mathcal{T}$.

It remains to show that the non-vanishing of the potential implies the breaking of the invariant curves. For that, we use the action-angle coordinates (x, y), in a neighborhood of the elliptic diameter, to write the invariant curve as a graph $(x, v(x))$ and apply (Pinto-de-Carvalho and Ramírez-Ros 2013).

Let us denote the generating function of this change of coordinates Ψ as $g(x_0, x_1)$,which is independent of ϵ.

The composition $\Phi^{-1} \circ B_\epsilon^2 \circ \Phi$ is generated by $G(x, x_1, \epsilon) = -g(\varphi_2, x_1) + L_\epsilon(\varphi, \varphi_2) + g(x, \varphi)$ restricted to $\partial_1 L_\epsilon(\varphi, \varphi_2) + \partial_2 g(x, \varphi) = 0$. This equation defines $\varphi_\epsilon = \varphi(x, \varphi_2, \epsilon)$ and from $-\partial_1 g(\varphi_2, x_1) + \partial_2 L_\epsilon(\varphi_\epsilon, \varphi_2) = 0$ we obtain $\varphi_2 = \varphi_2(x, x_1, \epsilon)$.

Therefore $G(x, x_1, \epsilon)$ has the following expression:
$-g(\varphi_2(x, x_1, \epsilon), x_1) + L_\epsilon(\varphi(x, \varphi_2(x, x_1, \epsilon), \varphi_2(x, x_1, \epsilon))) + g(x, \varphi(x, \varphi_2(x, x_1, \epsilon))$.

By denoting $G_1(x, x_1) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0} L_\epsilon(\varphi, \varphi_0) = h_1(\varphi.\varphi_2)$, we obtain $G(x, x_1, \epsilon) = G_0(x, x_1) + \epsilon G_1(x, x_1) + O(\epsilon^2)$.

This gives Claim 2 and finishes the proof of the Theorem. □

Acknowledgements The authors would like to thank the anonymous referee for valuable comments and suggestions, which improved the clarity of this work.

Funding The Article Processing Charge (APC) for the publication of this research was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) (ROR identifier: 00x0ma614).

References

Angenent, S.B.: Analicity of the interface in porous medium equation. SIAM J. Math. Anal. (1990a)

Angenent, S.B.: Monotone recurrence relations, their Birkhoff orbits and topological entropy. Ergod. Theory Dyn. Syst. **10**, 15–41 (1990b)

Chang, S.J., Freiberg, R.: Elliptical Billiards an Poncelet's theorem. J. Math. Phys. **29**, 1537–1550 (1988)
Chou, K.S., Zhu, X.P.: The Curve Shortening Problem, p. x+255. Chapman & Hall/CRC, Boca Raton (2001)
Damasceno, J.G.: Família de aplicações bilhares geradas pelo fluxo de curvatura. Tese de Doutorado, UNICAMP (2011)
Damasceno, J.G., Dias-Carneiro, M.J., Ramirez-Ros, R.: The billiard inside an ellipse deformed by the curvature flow. Proc. Am. Math. Soc. **2**, 705–719 (2017)
Dias Carneiro, M.J., Oliffson Kamphorst, S., Pinto-de-Carvalho, S.: Periodic orbits of generic oval billiards. Nonlinearity **20**(10), 2453 (2007)
Flanders, H.: A proof of Minkowski's inequality for convex curves. Am. Math. Mon. **75**(6), 581–593 (1968)
Gage, M.: An isoperimetric inequality with applications to curve shortening. Duke Math. J. **50**(1), 1225–1229 (1983)
Gage, M.: Evolving plane curves by curvature in relative geometries. Duke Math. J. **50**(1), 441–466 (1993)
Gage, M., Hamilton, R.S.: The Heat equation shrinking convex plane curves. J. Differ. Geom. **23**, 69–96 (1986)
Grayson, M.: Shortening embedded curves. Ann. Math. **129**, 71–111 (1989)
Le Calvez, P.: Propriétés dynamiques des régions d'instabilité. Ann. Sci. École Norm. Sup. **20**(3), 443–464 (1987)
Mather, J.N.: Modulus of continuity for Peierls's barrier. In: Rabinowitz P.H., et al. (ed). Periodic Solutions of Hamiltonian Systems and Related Topics (Il Ciocco, 1986), vol. 209, pp. 177–202. Reidel Publishing Co., Dordrecht (1987)
Pinto-de-Carvalho, S., Ramírez-Ros, R.: Non-persistence of resonant caustics in perturbed elliptic billiards. Ergod. Theory Dyn. Syst. **6**, 1876–1890 (2013)
Salazar, C.M.: Região de Instabilidade em Bilhares. tese de Doutorado, UFMG (2019)

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