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A general approach to asymptotic elimination of aggregation functions and generalized quantifiers

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Abstract
We consider a logic with truth values in the unit interval and which uses aggregation functions instead of quantifiers, and we describe a general approach to asymptotic elimination of aggregation functions and, indirectly, of asymptotic elimination of Mostowski style generalized quantifiers, since such can be expressed by using aggregation functions. The notion of "local continuity" of an aggregation function, which we make precise in two (related) ways, plays a central role in this approach.

Keywords Finite model theory · Aggregation functions · Generalized quantifiers · Probability logic · Asymptotic probabilities

Mathematics Subject Classification 03C13 · 03B50 · 03C10 · 03C80

1 Introduction

The use of (generalized) quantifiers is a way to increase the expressivity of a logic, beyond the use of functions (often called connectives) which, for some k, assign a truth value to any k-tuple of truth values. In the context of artificial intelligence and machine learning it makes sense to use aggregation functions, such as for example the average of a sequence of numbers, to increase expressivity, for example when defining probability distributions, or when defining queries. (See e.g. [3, pages 10, 117, 382] or [5, pages 31, 54].) In the context of this article, an aggregation function is a function that takes, for some integer $k > 0$, a k-tuple $(\bar{p}_1, \ldots, \bar{p}_k)$ as input, where each $\bar{p}_i$ is a finite sequence of reals from the unit interval, and gives a real number in

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the unit interval as output. In addition, for each i, the order of the reals in $\bar{p}_i$ should not matter for the output value. (Therefore some people prefer to view each $\bar{p}_i$ as a finite multiset.) Note that we allow $\bar{p}_i$ and $\bar{p}_j$ to have different length (if $i \neq j$). To systematize the use of aggregation functions one can incorporate them into the syntax and semantics of a formal logic, just as done with quantifiers. But even if a sequence of numbers contains only the numbers '0' and '1' the average of the sequence may well be a number strictly between 0 and 1. So if we use aggregation functions in a formal logic we have to accept that the (truth) values of formulas can be other numbers than 0 or 1.

In this work we consider a quite general (formal) logic, which we call PLA^* ("probability logic with aggregation functions"), with truth values in the unit interval [0, 1] and which uses all possible aggregation functions. (The '$*$' in PLA^* is there to indicate that it is another, more general and flexible, version of the logics PLA and PLA^+ considered in [17] and [18].) The restriction of truth values to [0, 1] is partly motivated by the fact that it is often natural to see such truth values as degrees of uncertainty (e.g. probabilities), and partly because a sequence of nonnegative numbers can be "normalized" so that the resulting sequence contains only numbers in the unit interval but preserves the variation in the original sequence. Example 2.9 illustrates the expressivity of PLA^* by showing that every "stage" (in the iterative approximation) of the PageRank [2] is definable in PLA^*. The first-order quantifiers $\exists$ and $\forall$ are not lost by using aggregation functions instead of quantifiers, because $\exists$ and $\forall$ can be expressed by using the aggregation functions 'maximum' and 'minimum' (see Remark 2.10). Moreover, as we will see in Proposition 3.3, every condition which can be expressed by first-order logic extended by any generalized quantifiers in the sense of Mostowski [21, 22] can be expressed by a formula in PLA^*. (Other early work on generalized quantifiers include Lindström [19] and Hajek, Havel, and Chytil [9] about generating plausible hypotheses in the early days of artificial intelligence.)

We now consider the following context: σ is a finite and relational signature (vocabulary), D_n, $n = 1, 2, 3, \ldots$, is a sequence of finite domains (i.e. sets) such that, for every n, the cardinality of D_{n+1} is greater than the cardinality of D_n, $\mathbf{W}_n$ is a set of σ-structures with domain D_n, and $\mathbb{P}_n$ is a probability distribution on $\mathbf{W}_n$. Results have been proven, for various 2-valued logics, various sets of structures $\mathbf{W}_n$, and various sequences of probability distributions $\mathbb{P}_n$, which say that all (or some) formulas of the logic are, with high probability, equivalent to a formula without quantifiers (or that this is not the case). Examples include [4, 6, 7, 10, 12–14]. Often such a result can be used to prove a so-called logical convergence law, or even a zero-one law. This kind of result often has implications for the expressivity of the logic considered and for the existence of computationally efficient ways of estimating, on a large domain, the probability of a query defined by the logic.

Within the specified context we can ask, for $\varphi(\bar{x}) \in PLA^*$, $0 \leq c < d \leq 1$, and $\bar{a} \in (D_n)^{|\bar{x}|}$, what the probability is that the value of $\varphi(\bar{a})$ in a random $\mathcal{A} \in \mathbf{W}_n$ is in the interval (c, d). Let us call a formula of any logic *aggregation-free* if it contains no aggregation function and no quantifier. For the reasons mentioned above, it is of interest to understand under what conditions for a PLA^*-formula $\varphi(\bar{x})$ there is a PLA^*-formula $\psi(\bar{x})$ such that $\psi(\bar{x})$ is aggregation-free, or at least of smaller complexity than φ, and for all $\varepsilon > 0$ the probability that the values of the two formulas differ by at most ε (for all substitutions of parameters for the free variables $\bar{x}$) tends to 1 as $n \to \infty$; in this case we say that $\varphi(\bar{x})$ and $\psi(\bar{x})$ are asymptotically equivalent (see Definition 5.2).

Such results were proved in [17, 18] and the proofs in both articles have a common strategy. This strategy also appears to some extent in some of the proofs in [10, 13, 14], although it is not as clear because the later articles deal with quantifiers instead of aggregation functions. In a simplified description, the first step of the strategy is to prove that for some set L_0 of 0/1-valued aggregation-free formulas, if $\varphi(\bar{x}, \bar{y}) \in L_0$ then there are finitely many $\varphi_i(\bar{x}) \in L_0$ such that the following holds with probability tending to 1 as $n \to \infty$: If $\bar{a}$ is a sequence of elements from D_n of the same length as $\bar{x}$, then $\varphi_i(\bar{a})$ holds for some i, and in this case the proportion of $\bar{b}$ (among tuples of elements from D_n with the same length as $\bar{y}$) such that $\varphi(\bar{a}, \bar{b})$ holds is, with high probability, close to some number α_i depending only on φ, φ_i and the sequence of probability distributions $\mathbb{P}_n$. The second step is to use the conclusions of the first step to show that for every formula $\varphi(\bar{x})$ there is a formula $\psi(\bar{x})$ without aggregation functions and without quantifiers which is asymptotically equivalent to $\varphi(\bar{x})$.

In general, the first step seems to be intricately connected to the kind of structures and the kind of probability distributions considered. But the second step can be formulated in such a way that it makes sense in a wide variety of situations, as we will show in Assumption 5.5 and Theorem 5.9. We hope that Theorem 5.9 can be used several times in the future once its preconditions (Assumption 5.5) have been shown to hold. This avoids carrying out a similar proof again and again, as already done in [17] and [18], and also in e.g. [13] and [14], although it is less clear in the latter cases. In fact, Theorem 5.9 has been used in precisely this way in [15] and [16] (which were written after the submission of this article).

Theorem 5.9 also gives us a general understanding of conditions under which aggregation functions can be asymptotically eliminated from PLA^*-formulas. Indirectly it also gives us an understanding of when Mostowski style generalized quantifiers can be asymptoticaly eliminated. Hence our Theorem 5.9 is similar in spirit, but different in the technical approach, to some results of Kaila [12, Theorems 4.4 and 4.7].

In some contexts it may be impossible to find, for a formula $\varphi(\bar{x}) \in PLA^*$ an aggregation-free formula $\psi(\bar{x}) \in PLA^*$ which is asymptotically equivalent to $\varphi(\bar{x})$. Theorem 5.9 may still be useful and show that $\varphi(\bar{x})$ is asymptotically equivalent to a formula $\psi(\bar{x})$ which is, in some sense, simpler than $\varphi(\bar{x})$. For example, $\psi(\bar{x})$ may be a formula with only "bounded aggregation functions (or quantifiers)" when this notion makes sense (as e.g. in the situation where some relations have bounded Gaifman degree).

The discussion so far has hidden the fact that, in addition to certain preconditions stated in Assumption 5.5, a sufficient and (in general) necessary condition for asymptotically eliminating an aggregation function F in a formula $\varphi \in PLA^*$ is that F is, in a sense, *continuous* on a "local" set which is determined by some parameters that are determined by the subformulas of φ and the sequence of probability distributions $\mathbb{P}_n$.

Section 4 focuses on two continuity properties of aggregation functions that are crucial to this study. The first continuity property has the advantage that it is relatively easy to check whether an aggregation function has it. The other property is formulated so that it is easy to use in the proof of Theorem 5.9. However, as stated by Proposition 4.10, both "local" continuity properties are very closely related, possibly equivalent although we have not been able to show this. The "global" versions of the two continuity properties are indeed equivalent.

The structure of the article is as follows: Section 2 defines the notions of connective, aggregation function and the logic PLA^* that we will work with. Section 3 defines generalized quantifiers in the sense of Mostowski and shows that they can be represented by aggregation functions. Section 4 defines the two notions of "local" continuity that we consider, shows how they are related, and gives examples. Finally, Section 5 describes a general approach to asymptotic elimination of aggregation functions which is concluded by Theorem 5.9.

Notation and terminology

By $\mathbb{N}$ and $\mathbb{N}^+$ we denote the sets of nonnegative, respectively, positive integers. We denote finite sequences by $\bar{a}, \bar{b}, \ldots, \bar{x}, \bar{y}, \ldots$, where typically $\bar{x}, \bar{y}, \ldots$ denote finite sequences of distinct variables. The length of a sequence $\bar{a}$ is denoted by $|\bar{a}|$, and the set of elements occuring in $\bar{a}$ is denoted by $\text{rng}(\bar{a})$. If A is a set then $|A|$ denotes its cardinality, and we let $A^{<\omega}$ denote the set of all finite nonempty sequences of elements from A, so $A^{<\omega} = \bigcup_{n \in \mathbb{N}^+} A^n$. First-order structures are denoted by $\mathcal{A}, \mathcal{B}, \ldots$ and their domains (or universes) by $A, B, \ldots$ unless we say something else (for example, in Section 5 we consider a fixed sequence of domains D_n, $n \in \mathbb{N}^+$). We use the word *signature* (or vocabulary) in its usual first-order sense. If a signature contains only relation symbols we call it *relational*.

2 Logic with aggregation functions

In this section we define the logic that we will work with, called PLA^*. It is an extension of the logics PLA and PLA^+ considered in [17] and [18], respectively, by allowing a more flexible use of aggregation functions. Formulas of PLA^* may take any (truth) value in the unit interval $[0, 1]$ and we think of the values 0 and 1 as corresponding to "false" and "true". PLA^* uses aggregation functions instead of quantifiers to "aggregate" over a domain. In Section 3 we will see that any condition expressed by a formula of first-order logic extended by generalized quantifiers in the sense of Mostowski [22] can be expressed by a formula of PLA^*. Recall that $[0, 1]^{<\omega}$ denotes the set of all finite nonempty sequences of reals in the unit interval $[0, 1]$.

Definition 2.1 Let $k \in \mathbb{N}^+$.

(i) A function $\mathsf{C} : [0, 1]^k \to [0, 1]$ will also be called a *(k-ary) connective*.

(ii) A function $F : \left([0, 1]^{<\omega}\right)^k \to [0, 1]$ which is symmetric in the following sense will be called a *(k-ary) aggregation function*: if $\bar{p}_1, \ldots, \bar{p}_k \in [0, 1]^{<\omega}$ and, for $i = 1, \ldots, k$, $\bar{q}_i$ is a reordering of the entries in $\bar{p}_i$, then $F(\bar{q}_1, \ldots, \bar{q}_k) = F(\bar{p}_1, \ldots, \bar{p}_k)$.

The functions defined in the next definition are continuous and when restricted to $\{0, 1\}$ (as opposed to the interval $[0, 1]$) they have the usual meanings of $\neg$, $\wedge$, $\vee$, and $\rightarrow$. (The definitions correspond to the semantics of Lukasiewicz logic (see for example [1, Section 11.2], or [20]).

Definition 2.2 **(Some special continuous connectives)** Let

(1) $\neg : [0, 1] \rightarrow [0, 1]$ be defined by $\neg(x) = 1 - x$,
(2) $\wedge : [0, 1]^2 \rightarrow [0, 1]$ be defined by $\wedge(x, y) = \min(x, y)$,
(3) $\vee : [0, 1]^2 \rightarrow [0, 1]$ be defined by $\vee(x, y) = \max(x, y)$, and
(4) $\rightarrow: [0, 1]^2 \rightarrow [0, 1]$ be defined by $\rightarrow (x, y) = \min(1,\ 1 - x + y)$.

Definition 2.3 **(Some common aggregation functions)** For $\bar{p} = (p_1, \ldots, p_n) \in [0, 1]^{<\omega}$, define

(1) $\max(\bar{p})$ to be the *maximum* of all p_i,
(2) $\min(\bar{p})$ to be the *minimum* of all p_i,
(3) $\mathrm{am}(\bar{p}) = (p_1 + \ldots + p_n)/n$, so 'am' is the *arithmetic mean*, or *average*, and
(4) $\mathrm{gm}(\bar{p}) = \big(\prod_{i=1}^{n} p_i\big)^{(1/n)}$, so 'gm' is the *geometric mean.*

In the examples that will follow we will meet more aggregation functions. *For the rest of this section we fix a finite and relational signature σ.*

Definition 2.4 **(Syntax of $PLA^*(\sigma)$)** We define formulas of $PLA^*(\sigma)$, as well as the set of free variables of a formula φ, denoted $Fv(\varphi)$, as follows.

(1) For each $c \in [0, 1]$, $c \in PLA^*(\sigma)$ (i.e. c is a formula) and $Fv(c) = \emptyset$. We also let $\bot$ and $\top$ denote 0 and 1, respectively.
(2) For all variables x and y, '$x = y$' belongs to $PLA^*(\sigma)$ and $Fv(x = y) = \{x, y\}$.
(3) For every $R \in \sigma$, say of arity r, and any choice of variables $x_1, \ldots, x_r$, $R(x_1, \ldots, x_r)$ belongs to $PLA^*(\sigma)$ and $Fv(R(x_1, \ldots, x_r)) = \{x_1, \ldots, x_r\}$.
(4) If $n \in \mathbb{N}^+$, $\varphi_1, \ldots, \varphi_n \in PLA^*(\sigma)$ and $\mathsf{C} : [0, 1]^n \rightarrow [0, 1]$ is a continuous connective, then $\mathsf{C}(\varphi_1, \ldots, \varphi_n)$ is a formula of $PLA^*(\sigma)$ and its set of free variables is $Fv(\varphi_1) \cup \ldots \cup Fv(\varphi_n)$.
(5) Suppose that $\varphi_1, \ldots, \varphi_k \in PLA^*(\sigma)$, $\chi_1, \ldots, \chi_k \in PLA^*(\sigma)$, $\bar{y}$ is a sequence of distinct variables, and that $F : \big([0, 1]^{<\omega}\big)^k \rightarrow [0, 1]$ is an aggregation function. Then
$$F(\varphi_1, \ldots, \varphi_k : \bar{y} : \chi_1, \ldots, \chi_k)$$
is a formula of $PLA^*(\sigma)$ and its set of free variables is
$$\Big(\bigcup_{i=1}^{k} Fv(\varphi_i)\Big) \setminus \mathrm{rng}(\bar{y}),$$
so this construction binds the variables in $\bar{y}$.

Notation 2.5 *(i) When denoting a formula in $PLA^*(\sigma)$ by e.g. $\varphi(\bar{x})$ we assume that $\bar{x}$ is a sequence of different variables and that every variable in the formula denoted by $\varphi(\bar{x})$ belongs to $rng(\bar{x})$ (but we do not require that every variable in $rng(\bar{x})$ actually occurs in the formula).*

(ii) If all $\chi_1, \dots, \chi_k$ are the same formula χ, then we may abbreviate

$$F(\varphi_1, \dots, \varphi_k : \bar{y} : \chi_1, \dots, \chi_k) \quad \text{by} \quad F(\varphi_1, \dots, \varphi_k : \bar{y} : \chi).$$

Definition 2.6 The $PLA^*(\sigma)$-formulas described in parts (2) and (3) of Definition 2.4 are called *first-order atomic formulas*. A $PLA^*(\sigma)$-formula is called a *first-order literal* if it has the form $\varphi(\bar{x})$ or $\neg\varphi(\bar{x})$, where $\varphi(\bar{x})$ is a first-order atomic formula and $\neg$ is like in Definition 2.2 (so it corresponds to negation when truth values are restricted to 0 and 1).

Definition 2.7 (**Semantics of** $PLA^*(\sigma)$) For every $\varphi \in PLA^*(\sigma)$ and every sequence of distinct variables $\bar{x}$ such that $Fv(\varphi) \subseteq \text{rng}(\bar{x})$ we associate a mapping from pairs $(\mathcal{A}, \bar{a})$, where $\mathcal{A}$ is a finite σ-structure and $\bar{a} \in A^{|\bar{x}|}$, to $[0, 1]$. The number in $[0, 1]$ to which $(\mathcal{A}, \bar{a})$ is mapped is denoted by $\mathcal{A}(\varphi(\bar{a}))$ and is defined by induction on the complexity of formulas.

(1) If $\varphi(\bar{x})$ is a constant c from $[0, 1]$, then $\mathcal{A}(\varphi(\bar{a})) = c$.
(2) If $\varphi(\bar{x})$ has the form $x_i = x_j$, then $\mathcal{A}(\varphi(\bar{a})) = 1$ if $a_i = a_j$, and otherwise $\mathcal{A}(\varphi(\bar{a})) = 0$.
(3) For every $R \in \sigma$, of arity r say, if $\varphi(\bar{x})$ has the form $R(x_{i_1}, \dots, x_{i_r})$, then $\mathcal{A}(\varphi(\bar{a})) = 1$ if $\mathcal{A} \models R(a_{i_1}, \dots, a_{i_r})$ (where '$\models$' has the usual meaning of first-order logic), and otherwise $\mathcal{A}(\varphi(\bar{a})) = 0$.
(4) If $\varphi(\bar{x})$ has the form $\mathsf{C}(\varphi_1(\bar{x}), \dots, \varphi_k(\bar{x}))$, where $\mathsf{C} : [0, 1]^k \to [0, 1]$ is a continuous connective, then

$$\mathcal{A}\big(\varphi(\bar{a})\big) = \mathsf{C}\big(\mathcal{A}(\varphi_1(\bar{a})), \dots, \mathcal{A}(\varphi_k(\bar{a}))\big).$$

(5) Suppose that $\varphi(\bar{x})$ has the form

$$F(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_k(\bar{x}, \bar{y}) : \bar{y} : \chi_1(\bar{x}, \bar{y}), \dots, \chi_k(\bar{x}, \bar{y}))$$

where $\bar{x}$ and $\bar{y}$ are sequences of distinct variables, $\text{rng}(\bar{x}) \cap \text{rng}(\bar{y}) = \emptyset$, and $F : \big([0, 1]^{<\omega}\big)^k \to [0, 1]$ is an aggregation function. If, for every $i = 1, \dots, k$, the set $\{\bar{b} \in A^{|\bar{y}|} : \mathcal{A}(\chi_i(\bar{a}, \bar{b})) = 1\}$ is nonempty, then let

$$\bar{p}_i = \big(\mathcal{A}\big(\varphi_i(\bar{a}, \bar{b})\big) : \bar{b} \in A^{|\bar{y}|} \text{ and } \mathcal{A}\big(\chi_i(\bar{a}, \bar{b})\big) = 1\big)$$

and

$$\mathcal{A}\big(\varphi(\bar{a})\big) = F(\bar{p}_1, \dots, \bar{p}_k).$$

Otherwise let $\mathcal{A}\big(\varphi(\bar{a})\big) = 0$.

Definition 2.8 (i) Suppose that $\varphi(\bar{x}, \bar{y}) \in PLA^*(\sigma)$, $\bar{x}$ and $\bar{y}$ are sequences of distinct variables, $\text{rng}(\bar{x}) \cap \text{rng}(\bar{y}) = \emptyset$ $\mathcal{A}$ is a finite σ-structure and $\bar{a} \in A^{|\bar{x}|}$. Then $\varphi(\bar{a}, \mathcal{A})$ denotes the set $\big\{\bar{b} \in A^{|\bar{y}|} : \mathcal{A}\big(\varphi(\bar{a}, \bar{b})\big) = 1\big\}$.
(ii) If $\varphi(\bar{x}) \in PLA^*(\sigma)$, $\mathcal{A}$ is a finite σ-structure and that $\bar{a} \in A^{|\bar{x}|}$, then '$\mathcal{A} \models \varphi(\bar{a})$' means the same as '$\mathcal{A}\big(\varphi(\bar{a})\big) = 1$'.

Example 2.9 As an example of what can be expressed with $PLA^*(\sigma)$ we consider the notion of PageRank [2]. The PageRank of an internet site can be approximated in "stages" as follows (if we supress the "damping factor" for simplicity), where IN_x is the set of sites that link to x, and OUT_y is the set of sites that y link to:

$$PR_0(x) = 1/N \text{ where } N \text{ is the number of sites,}$$
$$PR_{k+1}(x) = \sum_{y \in IN_x} \frac{PR_k(y)}{|OUT_y|}.$$

It is not difficult to prove, by induction on k, that for every k the sum of all $PR_k(x)$ as x ranges over all sites is 1. Hence the sum in the definition of PR_{k+1} is less or equal to 1 (which will matter later). Let $E \in \sigma$ be a binary relation symbol representing a link. Define the aggregation function $\text{length}^{-1} : [0,1]^{<\omega} \to [0,1]$ by $\text{length}^{-1}(\bar{p}) = 1/|\bar{p}|$. Then $PR_0(x)$ is expressed by the $PLA^*(\sigma)$-formula $\text{length}^{-1}(x = x : y : \top)$.

Suppose that $PR_k(x)$ is expressed by $\varphi_k(x) \in PLA^*(\sigma)$. Note that multiplication is a continuous connective from $[0,1]^2$ to $[0,1]$ so it can be used in $PLA^*(\sigma)$-formulas. Then observe that the quantity $|OUT_y|^{-1}$ is expressed by the $PLA^*(\sigma)$-formula

$$\text{length}^{-1}\big(y = y : z : E(y,z)\big)$$

which we denote by $\psi(y)$. Let $\text{tsum} : [0,1]^{<\omega} \to [0,1]$ be the "truncated sum" defined by letting $\text{tsum}(\bar{p})$ be the sum of all entries in $\bar{p}$ if the sum is at most 1, and otherwise $\text{tsum}(\bar{p}) = 1$. Then $PR_{k+1}(x)$ is expressed by the $PLA^*(\sigma)$-formula

$$\text{tsum}\big(x = x \wedge (\varphi_k(y) \cdot \psi(y)) : y : E(y,x)\big).$$

With PLA^* we can also define all stages of the SimRank [11] in a simpler way than done in [17] with the sublogic $PLA \subseteq PLA^*$.

Remark 2.10 **(The relation to first-order logic)** From the syntax and semantics of $PLA^*(\sigma)$ and the fact the connectives $\neg, \wedge, \vee$ and $\rightarrow$ are continuous it follows that every quantifier-free first-order formula $\varphi(\bar{x})$ (over σ) is also a $PLA^*(\sigma)$-formula. Now suppose that $\varphi(\bar{x}, y)$ is a first-order formula and $\psi(\bar{x}, y)$ a $PLA^*(\sigma)$-formula such that for every finite σ-structure $\mathcal{A}$ and $\bar{a} \in A^{|\bar{x}|}$ and $b \in A$, $\mathcal{A} \models \varphi(\bar{a}, b)$ if and only if $\mathcal{A}\big(\psi(\bar{a}, b)\big) = 1$. Then, for all $\bar{a} \in A^{|\bar{x}|}$,

$$\mathcal{A} \models \exists y \varphi(\bar{a}, y) \text{ if and only if } \mathcal{A}\big(\max(\psi(\bar{a}, y) : y : \top)\big) = 1.$$

Similarly, the quantifier $\forall$ can be expressed in PLA^* by using the aggregation function min. By induction on the complexity of first-order formulas it follows that, for relational signatures and finite structures, every condition that can be expressed by first-order logic can be expressed with PLA^*.

Notation 2.11 *(***Using $\exists$ and $\forall$ as abbreviations***) Motivated by the discussion in Remark 2.10, if $\varphi(\bar{x}, \bar{y}) \in PLA^*(\sigma)$ is a formula which can only take the values 0 or 1 (e.g a boolean combination of atomic first-order formulas) then we may use*

'$\exists \bar{y} \varphi(\bar{x}, \bar{y})$' to mean the same as '$\max(\varphi(\bar{x}, \bar{y}) : \bar{y} : \top)$'*, and '$\forall \bar{y} \varphi(\bar{x}, \bar{y})$' to mean the same as* '$\min(\varphi(\bar{x}, \bar{y}) : \bar{y} : \top)$'.

3 Expressing generalized quantifiers with aggregation functions

In this section we consider generalized quantifiers in the sense of Mostowski [22] and see that they can be expressed by aggregation functions. Let σ be a relational signature.

Definition 3.1 A *quantifier aggregating k sets* is a class Q consisting of tuples $(D, X_1, \ldots, X_k)$ such that D is a set, $X_1, \ldots, X_k \subseteq D$ and the following condition holds:

If $|D| = |E|$, $X_i \subseteq D$, $Y_i \subseteq E$, $|X_i| = |Y_i|$, for $i = 1, \ldots, k$, and $(D, X_1, \ldots, X_k) \in Q$, then $(E, Y_1, \ldots, Y_k) \in Q$.

Definition 3.2 (i) Then let $FOGQ(\sigma)$ be the set of all expressions that can be obtained by adding the following construction to the definition of a first-order formula over σ:

If $\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_k(\bar{x}, \bar{y}) \in FOGQ(\sigma)$, $\bar{y}$ is a sequence of different variables, and Q is a quantifier aggregating k sets, then the following expression belongs to $FOGQ(\sigma)$:

$$Q\bar{y}(\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_k(\bar{x}, \bar{y})).$$

(ii) The semantics of first-order logic is now extended to $FOGQ(\sigma)$ as follows if $|\bar{y}| = n$:

$\mathcal{A} \models Q\bar{y}(\varphi_1(\bar{a}, \bar{y}), \ldots, \varphi_k(\bar{a}, \bar{y}))\big)$ if and only if $(A^n, X_1, \ldots, X_k) \in Q$ where, for all $i = 1, \ldots, k$, $X_i = \{\bar{b} \in A^n : \mathcal{A} \models \varphi_i(\bar{a}, \bar{b}))\}$.

Proposition 3.3 *Let $\varphi(\bar{x}) \in FOGQ(\sigma)$. Then there is $\psi(\bar{x}) \in PLA^*(\sigma)$ such that for every finite σ-structure $\mathcal{A}$ and every $\bar{a} \in A^{|\bar{a}|}$, $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{A}\big(\psi(\bar{a})\big) = 1$, and $\mathcal{A} \not\models \varphi(\bar{a})$ if and only if $\mathcal{A}\big(\psi(\bar{a})\big) = 0$.*

Proof For any $\varphi(\bar{x}) \in FOGQ(\sigma)$, finite σ-structure $\mathcal{A}$ and $\bar{a} \in A^{|\bar{x}|}$, let '$\mathcal{A}\big(\varphi(\bar{a})\big) = 1$' mean the same as '$\mathcal{A} \models \varphi(\bar{a})$' and let '$\mathcal{A}\big(\varphi(\bar{a})\big) = 0$' mean the same as '$\mathcal{A} \not\models \varphi(\bar{a})$'.

In Remark 2.10 we saw that the proposition holds if $\varphi(\bar{x})$ is a first-order formula. In the same remark we also saw that $\neg, \wedge, \vee$ and $\rightarrow$ can, with Lukasiewicz semantics, be expressed by continuous functions from $[0, 1]$ or $[0, 1]^2$ to $[0, 1]$, and $\exists$ and $\forall$ can be expressed by the aggregation functions max and min, respectively.

Hence is suffices to show the following:

Claim *Suppose that $\bar{x}$ and $\bar{y}$ are sequences of different variables, $rng(\bar{x}) \cap rng(\bar{y}) = \emptyset$, $\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_k(\bar{x}, \bar{y}) \in FOGQ(\sigma)$, and $\psi_1(\bar{x}, \bar{y}), \ldots, \psi_k(\bar{x}, \bar{y}) \in PLA^*(\sigma)$ are such that for every finite σ-structure $\mathcal{A}$, all $\bar{a} \in A^{|\bar{x}|}$, and all $\bar{b} \in A^{|\bar{y}|}$,*

$$\mathcal{A}\big(\varphi_i(\bar{a}, \bar{b})\big) = \mathcal{A}\big(\psi_i(\bar{a}, \bar{b})\big) \textit{ for all } i = 1, \ldots, k.$$

If Q is an n-ary quantifier aggregating k sets, then there is an aggregation function $F : \big([0,1]^{<\omega}\big)^k \to [0,1]$ such that for every finite σ-structure $\mathcal{A}$ and $\bar{a} \in A^{|x|}$:

$$\mathcal{A}\big(Q\bar{y}(\varphi_1(\bar{a},\bar{y}),\ldots,\varphi_k(\bar{a},\bar{y}))\big) = \mathcal{A}\big(F\big(\psi_1(\bar{a},\bar{y}),\ldots,\psi_k(\bar{a},\bar{y}) : \bar{y} : \top\big)\big), \quad (3.1)$$

where we recall that the PLA^*-formula '$\top$' (or '1') has the value 1 in every structure.

Let $F : \big([0,1]^{<\omega}\big)^k \to [0,1]$ be defined as follows: If $\bar{p}_i = (p_{i,1},\ldots,p_{i,m_i})$ for $i = 1,\ldots,k$, then

$$F(\bar{p}_1,\ldots,\bar{p}_k) = \begin{cases} 1 & \text{if } (D, X_1,\ldots,X_k) \in Q \text{ where } D = [m],\ m = \max(m_1,\ldots,m_k) \\ & \text{ and } X_i = \{j \in [m_i] : r_{i,j} = 1\}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For every finite structure $\mathcal{A}$ and every $\bar{a} \in A^{|\bar{x}|}$ we now have

$$\begin{aligned} &\mathcal{A}\big(Q\bar{y}(\varphi_1(\bar{a},\bar{y}),\ldots,\varphi_k(\bar{a},\bar{y}))\big) = 1 \iff \\ &\mathcal{A} \models Q\bar{y}(\varphi_1(\bar{a},\bar{y}),\ldots,\varphi_k(\bar{a},\bar{y})) \iff \\ &(A^{|\bar{y}|}, \varphi_1(\bar{a},\mathcal{A}),\ldots,\varphi_k(\bar{a},\mathcal{A})) \in Q \iff \\ &F(\bar{p}_1,\ldots,\bar{p}_k) = 1 \text{ where, for } i = 1,\ldots,k,\ \bar{p}_i = \big(\mathcal{A}\big(\varphi_i(\bar{a},\bar{b})\big) : \bar{b} \in A^{|\bar{y}|}\big). \end{aligned}$$

Since also

$$\mathcal{A}\big(F\big(\varphi_1(\bar{a},\bar{y}),\ldots,\varphi_k(\bar{a},\bar{y}) : \bar{y} : \top\big)\big) = F(\bar{p}_1,\ldots,\bar{p}_k)$$

the equation (3.1) follows. This proves the claim and concludes the proof of the proposition. □

4 Notions of continuity for aggregation functions

Our goal is to formulate conditions that apply to a wide variety of contexts and under which an aggregation function in a PLA^*-formula can be "asymptotically eliminated". If all aggregation functions in the formula can be asymptotically eliminated then we get a "simpler" formula which is "asymptotically equivalent" (Definition 5.2) to the original formula. One of the conditions that need (necessarily according to Remark 5.12) to be satisfied is that the aggregation function to be asymptotically eliminated, say $F : \big([0,1]^{<\omega}\big)^m \to [0,1]$, has some sort of continuity property on a subset of $\big([0,1]^{<\omega}\big)^m$.

We formulate two notions of continuity, *ct-continuity* and *up-continuity*, which are "local" versions of the "global" notions of *(strongly) admissible aggregation function*, respectively, *(strongly) admissible aggregation function sensu novo* that are considered in [17, 18]. Besides being localizations of the corresponding notions in [17, 18], the notions considered here are a little bit weaker (than the corresponding global notions in [17, 18]) also in the sense that some small conditions in the definitions of (strong) admissibility and (strong) admissibility sensu novo are not present in the corresponding localizations considered here, because we have realized that those conditions are not necessary for proving "asymptotic elimination results".

Then we show (see Proposition 4.10) that ct-continuity and up-continuity are closely related. Nevertheless we consider both notions useful because ct-continuity (with respect to some parameters) is usually easier to verify for concrete aggregation functions, while up-continuity is tailored for making the proof of "elimination of an aggregation function" (Theorem 5.9) work out.

Throughout the section we give examples of aggregation functions that are ct-continuous, respectively up-continuous, with respect to some (or all) parameters.

4.1 Convergence test continuity

In this section we define the local version of *(strong) admissibility* (used in [17, 18]) which we call *ct-continuity with respect to a sequence of parameters*. Before doing that we must define the notion of *convergence testing sequence* which generalizes a notion used by Jaeger in [10]. The intuition is that a sequence $\bar{p}_n \in [0,1]^{<\omega}$, $n \in \mathbb{N}^+$ is convergence testing for for parameters $c_1, \ldots, c_k \in [0,1]$ and $\alpha_1, \ldots \alpha_k \in [0,1]$ if the length of $\bar{p}_n$ tends to infinity as $n \to \infty$ and, as $n \to \infty$, all entries in $\bar{p}_n$ congregate ever closer to the "convergence points" in the set $\{c_1, \ldots, c_k\}$, and the proportion of entries in $\bar{p}$ which are close to c_i converges to α_i.

Definition 4.1 (Convergence testing sequence) A sequence $\bar{p}_n \in [0,1]^{<\omega}$, $n \in \mathbb{N}$, is called *convergence testing* for parameters $c_1, \ldots, c_k \in [0,1]$ and $\alpha_1, \ldots \alpha_k \in [0,1]$ if the following hold, where $p_{n,i}$ denotes the ith entry of $\bar{p}_n$:

(1) $|\bar{p}_n| < |\bar{p}_{n+1}|$ for all $n \in \mathbb{N}$.
(2) For every disjoint family of open (with respect to the induced topology on [0, 1]) intervals $I_1, \ldots I_k \subseteq [0,1]$ such that $c_i \in I_i$ for each i, there is an $N \in \mathbb{N}$ such that $\mathrm{rng}(\bar{p}_n) \subseteq \bigcup_{j=1}^{k} I_j$ for all $n \geq N$, and for every $j \in \{1, \ldots, k\}$,

$$\lim_{n\to\infty} \frac{\big|\{i : p_{n,i} \in I_j\}\big|}{|\bar{p}_n|} = \alpha_j.$$

More generally, a sequence of m-tuples $(\bar{p}_{1,n}, \ldots, \bar{p}_{m,n}) \in \big([0,1]^{<\omega}\big)^m$, $n \in \mathbb{N}$, is called *convergence testing* for parameters $c_{i,j} \in [0,1]$ and $\alpha_{i,j} \in [0,1]$, where $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, k_i\}$ and $k_1, \ldots k_m \in \mathbb{N}^+$, if for every fixed $i \in \{1, \ldots, m\}$ the sequence $\bar{p}_{i,n}$, $n \in \mathbb{N}$, is convergence testing for $c_{i,1}, \ldots, c_{i,k_i}$, and $\alpha_{i,1}, \ldots, \alpha_{i,k_i}$.

Definition 4.2 (Convergence test continuity) An aggregation function $F: \big([0,1]^{<\omega}\big)^m \to [0,1]$ is called *ct-continuous (convergence test continuous)* with respect to the sequence of parameters $c_{i,j}, \alpha_{i,j} \in [0,1]$, $i = 1, \ldots, m$, $j = 1, \ldots, k_i$, if the following condition holds:

For all convergence testing sequences of m-tuples $(\bar{p}_{1,n}, \ldots, \bar{p}_{m,n}) \in \big([0,1]^{<\omega}\big)^m$, $n \in \mathbb{N}$, and $(\bar{q}_{1,n}, \ldots, \bar{q}_{m,n}) \in \big([0,1]^{<\omega}\big)^m$, $n \in \mathbb{N}$, with the same parameters $c_{i,j}, \alpha_{i,j} \in [0,1]$, $\lim_{n\to\infty} |F(\bar{p}_{1,n}, \ldots, \bar{p}_{m,n}) - F(\bar{q}_{1,n}, \ldots, \bar{q}_{m,n})| = 0$.

Example 4.3 It is easy to see that for every $\beta \in (0,1)$ the aggregation function $\text{length}^{-\beta} : [0,1]^{<\omega} \to [0,1]$ defined by $\text{length}^{-\beta}(\bar{p}) = |\bar{a}|^{-\beta}$ is ct-continuous for all parameters $c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m \in [0,1]$ and all $m \in \mathbb{N}^+$. Also, it is not hard to prove that aggregation function 'tsum' (or "truncated sum") used in Example 2.9 is ct-continuous for all parameters $c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m \in [0,1]$ and all $m \in \mathbb{N}^+$

The proof of Proposition 6.3 in [17] proves the next result because in the case of am and gm the argument works even if some α_j is zero.

Proposition 4.4 *[17]*

(i) For all $m \in \mathbb{N}^+$ and all $c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m \in [0,1]$ the aggregation functions am and gm are ct-continuous with respect to the parameters $c_1, \ldots, c_m$, $\alpha_1, \ldots, \alpha_m$.

(ii) For all $m \in \mathbb{N}^+$, all $c_1, \ldots, c_m \in [0,1]$ and all $\alpha_1, \ldots, \alpha_m \in (0,1]$ (so $\alpha_i > 0$ for all i) the aggregation functions max and min are ct-continuous with respect to the parameters $c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m$.

4.2 Uniform point continuity

In this section we define our second notion of continuity, up-continuity, and for this we need to associate every $\bar{p} \in [0,1]^{<\omega}$ with a function $f_{\bar{p}} : [0,1] \to [0,1]$. Actually we do this in two ways, an "ordered" and an "unordered" way. Common of both ways is that if $c \in \text{rng}(\bar{p})$ and the proportion of coordinates in $\bar{p}$ that are equal to c is α, then c belongs to the range/image of $f_{\bar{p}}$ and $f^{-1}(c)$ is a union of intervals such that the sum of the lengths of the intervals is α.

Then we show how ct-continuity and up-continuity are related and give examples.

Definition 4.5 (Functional representations of sequences) Let $n \in \mathbb{N}^+$ and let $\bar{p} = (p_1, \ldots, p_n) \in [0,1]^n$. To $\bar{p}$ we associate two functions from $[0,1]$ to $[0,1]$.

(1) Define $\mathfrak{f}_{\bar{p}}$, which we call the *ordered functional representation of* $\bar{p}$, as follows: For every $a \in [0, 1/n)$, let $\mathfrak{f}_{\bar{p}}(a) = p_1$, for every $i = 1, \ldots, n-1$ and every $a \in [i/n, (i+1)/n)$, let $f(a) = p_{i+1}$ and finally let $f(1) = p_n$.

(21) Define $\mathfrak{g}_{\bar{p}}$, which we call the *unordered functional representation of* $\bar{p}$, as follows: Let $\bar{p}' = (p'_1, \ldots, p'_n)$ be a reordering of $\bar{p}$ such that, for all $i = 1, \ldots, n-1$, $p'_i \leq p'_{i+1}$ and let $\mathfrak{g}_{\bar{p}} = \mathfrak{f}_{\bar{p}'}$.

Definition 4.6 (Pseudometrics on $\big([0,1]^{<\omega}\big)^k$)

(1) First we recall the L_1 and L_∞ norms: for every (bounded and integrable) $f : [0,1] \to \mathbb{R}$ they are defined as

$$\|f\|_1 = \int_{[0,1]} |f(x)|dx \qquad \text{and} \qquad \|f\|_\infty = \sup\{|f(a)| : a \in [0,1]\}.$$

(2) For $\bar{p}, \bar{q} \in [0,1]^{<\omega}$ we define

$$\mu_1^u(\bar{p}, \bar{q}) = \|\mathfrak{g}_p - \mathfrak{g}_{\bar{q}}\|_1,$$
$$\mu_\infty^o(\bar{p}, \bar{q}) = \|\mathfrak{f}_{\bar{p}} - \mathfrak{f}_{\bar{q}}\|_\infty.$$

(3) For arbitrary $k > 1$ we can define a functions on $\big([0,1]^{<\omega}\big)^k$, also denoted μ_1^u and μ_∞^o (to avoid making notation more complicated), as follows: For all $(\bar{p}_1, \ldots, \bar{p}_k), (\bar{q}_1, \ldots, \bar{q}_k) \in \big([0,1]^{<\omega}\big)^k$ let

$$\mu_1^u\big((\bar{p}_1, \ldots, \bar{p}_k), (\bar{q}_1, \ldots, \bar{q}_k)\big) = \max\big(\mu_1^u(\bar{p}_1, \bar{q}_1), \ldots, \mu_1^u(\bar{p}_k, \bar{q}_k)\big), \text{ and}$$
$$\mu_\infty^o\big((\bar{p}_1, \ldots, \bar{p}_k), (\bar{q}_1, \ldots, \bar{q}_k)\big) = \max\big(\mu_\infty^u(\bar{p}_1, \bar{q}_1), \ldots, \mu_\infty^u(\bar{p}_k, \bar{q}_k)\big).$$

It follows that μ_1^u and μ_∞^o are symmetric and satisfy the triangle inequality so they are pseudometrics on $[0,1]^{<\omega}$. None of them is a metric since it can happen that $\mu_1^u(\bar{p}, \bar{q}) = 0$ and $\bar{p} \neq \bar{q}$. For example, if $\bar{p} = (0, 1/2, 1)$ and $\bar{q} = (0, 0, 1/2, 1/2, 1, 1)$ then $\mu_1^u(\bar{p}, \bar{q}) = 0$. Also observe that for all $\bar{p}, \bar{q} \in [0,1]^{<\omega}$, $\mu_1^u(\bar{p}, \bar{q}), \mu_\infty^o(\bar{p}, \bar{q}) \leq 1$.

Definition 4.7 (Asymptotic uniform continuity on a set) Let $F : \big([0,1]^{<\omega}\big)^k \to [0,1]$ be an aggregation function and let μ be any of the the pseudometrics defined in Definition 4.6. Also let $X \subseteq \big([0,1]^{<\omega}\big)^k$. We say that F is *asymptotically uniformly continuous on* X if for every $\varepsilon > 0$ there are n and $\delta > 0$ such that if $(\bar{p}_1, \ldots, \bar{p}_k), (\bar{q}_1, \ldots, \bar{q}_k) \in X$, $|\bar{p}_i|, |\bar{q}_i| \geq n$ for all i and $\mu_1^u\big((\bar{p}_1, \ldots, \bar{p}_k), (\bar{q}_1, \ldots, \bar{q}_k)\big) < \delta$, then $\big|F(\bar{p}_1, \ldots, \bar{p}_k) - F(\bar{q}_1, \ldots, \bar{q}_k)\big| < \varepsilon$.

Definition 4.8 (Uniform point continuity) An aggregation function $F : \big([0,1]^{<\omega}\big)^m \to [0,1]$ is called *up-continuous (uniformly point continuous) with respect to the parameters* $c_{i,j}, \alpha_{i,j} \in [0,1]$, $i = 1, \ldots, m$ and $j = 1, \ldots, k_i$, if the following two conditions hold:

(1) For all sufficiently small $\delta > 0$, F is asymptotically uniformly continuous on $X_1 \times \ldots \times X_m$ where, for each $i = 1, \ldots, m$,

$$X_i = \big\{\bar{p} \in [0,1]^{<\omega} : \mathrm{rng}(\bar{p}) \subseteq \{c_{i,1}, \ldots, c_{i,k_i}\} \text{ and, for each } j = 1, \ldots, k_i, \text{ there are between } (\alpha_{i,j} - \delta)|\bar{p}| \text{ and } (\alpha_{i,j} + \delta)|\bar{p}| \text{ coordinates in } \bar{p} \text{ which equals } c_{i,j}\big\}.$$

(2) For all $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that if, for $i = 1, \ldots, m$, $\bar{p}_i, \bar{q}_i \in [0,1]^{<\omega}$ and

(a) $|\bar{p}_i| = |\bar{q}_i| > n_0$,
(b) $\mu_\infty^o(\bar{p}_i, \bar{q}_i) < \delta$,
(c) $\mathrm{rng}(\bar{p}_i) \subseteq \{c_{i,1}, \ldots, c_{i,k_i}\}$, and
(d) for each $j = 1, \ldots, k_i$, there are between $(\alpha_{i,j} - \delta)|\bar{p}_i|$ and $(\alpha_{i,j} + \delta)|\bar{p}_i|$ coordinates in $\bar{p}_i$ which equal $c_{i,j}$,

then $|F(\bar{p}_1, \ldots, \bar{p}_m) - F(\bar{q}_1, \ldots, \bar{q}_m)| < \varepsilon$.

Example 4.9 It is straightforward to verify that, for all $m, k_1, k_2 \in \mathbb{N}^+$, and all $c_{i,j}, \alpha_{i,j} \in [0,1]$ for $i = 1, 2$, $j = 1, \dots, k_i$, the pseudometric $\mu_1^u : \left([0,1]^{<\omega}\right)^2 \to [0,1]$ is up-continuous with respect to the parameters $c_{i,j}, \alpha_{i,j} \in [0,1]$.

The next proposition relates ct-continuity and up-continuity.

Proposition 4.10 *Let* $F : \left([0,1]^{<\omega}\right)^k \to [0,1]$ *be an aggregation function.*

(i) If, for some $m, k_1, \dots, k_m \in \mathbb{N}^+$*, F is up-continuous with respect to the parameters* $c_{i,j}, \alpha_{i,j} \in [0,1]$*,* $i = 1, \dots, m$*,* $j = 1, \dots, k_i$*, then F is ct-continuous with respect to the same parameters* $c_{i,j}, \alpha_{i,j} \in [0,1]$*.*

(ii) Let $m, k_1, \dots, k_m \in \mathbb{N}^+$ *and* $c_{i,j}, \alpha_{i,j} \in [0,1]$ *for* $i = 1, \dots, m$ *and* $j = 1, \dots, k_i$*. If there is* $\delta > 0$ *such that, for all* $\alpha'_{i,j} \in (\alpha_{i,j} - \delta, \alpha_{i,j} + \delta)$*,* $i = 1, \dots, m$ *and* $j = 1, \dots, k_i$*, F is ct-continuous with respect to the parameters* $c_{i,j}, \alpha'_{i,j}$*,* $i = 1, \dots, m$ *and* $j = 1, \dots, k_i$*, then F is up-continuous with respect to the parameters* $c_{i,j}, \alpha_{i,j}$*,* $i = 1, \dots, m$ *and* $j = 1, \dots, k_i$*.*

Observe that we immediately get the following:

Corollary 4.11 *If* $F : \left([0,1]^{<\omega}\right)^m \to [0,1]$ *is an aggregation function then the following are equivalent:*

(i) F is ct-continuous with respect to every choice of parameters $c_{i,j}, \alpha_{i,j} \in [0,1]$*,* $i = 1, \dots, m$*,* $j = 1, \dots, k_i$*.*

(ii) F is up-continuous with respect to every choice of parameters $c_{i,j}, \alpha_{i,j} \in [0,1]$*,* $i = 1, \dots, m$*,* $j = 1, \dots, k_i$*.*

Proof of Proposition 4.10 In order to avoid very cluttered notation and make the ideas of the proof more evident we only prove the proposition in the case when F is unary, i.e. when $F : [0,1]^{<\omega} \to [0,1]$. The general case is proved in the same way except that we need to "book keep" more sequences and parameters.

(i) Suppose that F is up-continuous with respect to the parameters $c_1, \dots, c_k$, $\alpha_1, \dots, \alpha_k$. Let $\bar{p}_n \in [01]^{<\omega}$ and $\bar{q}_n \in [0,1]^{<\omega}$, $n \in \mathbb{N}$, be convergence testing sequences for the parameters $c_1, \dots, c_k$ and $\alpha_1, \dots, \alpha_k$. We need to show that $\lim_{n\to\infty} |F(\bar{p}_n) - F(\bar{q}_n)| = 0$. Let $I_1, \dots, I_k$ be disjoint open intervals such that $c_j \in I_j$ for $j = 1, \dots, k$. Let $\bar{p}_{n,i}$ denote the ith coordinate of $\bar{p}_n$ and similarly for $\bar{q}_{n,i}$. As $\bar{p}_n$ and $\bar{q}_n$ are convergence testing sequences for $c_1, \dots, c_k$ and $\alpha_1, \dots, \alpha_k$ it follows that there is n_0 such that if $n \geq n_0$, then $\text{rng}(\bar{p}_n), \text{rng}(\bar{q}_n) \subseteq I_1 \cup \ldots \cup I_k$, and

$$\lim_{n\to\infty} \frac{|\{i : p_{n,i} \in I_j\}|}{|\bar{p}_n|} = \lim_{n\to\infty} \frac{|\{i : q_{n,i} \in I_j\}|}{|\bar{q}_n|} = \alpha_j. \tag{4.1}$$

Since we are only considering the limit, we can assume without loss of generality that $n_0 = 1$. Define the sequences $\bar{p}'_n$ and $\bar{q}'_n$ by setting $p'_{n,i} = c_j$ if $p_{n,i} \in I_j$ (recall that different I_j are disjoint), and $q'_{n,i} = c_j$ if $q_{n,i} \in I_j$.

By condition (1) of Definition 4.8 of up-continuity, for every $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that if $n > n_0$ and $\mu_1^u(\bar{p}'_n, \bar{q}'_n) < \delta$, then $|F(\bar{p}'_n) - F(\bar{q}'_n)| < \varepsilon$. From (4.1) and the definition of $\bar{p}'_n$ and $\bar{q}'_n$ it follows that for every $\delta > 0$ we have $\mu_1^u((\bar{p}'_n, \bar{q}'_n) < \delta$ for all sufficiently large n. Hence $\lim_{n\to\infty} |F(\bar{p}'_n) - F(\bar{q}'_n)| = 0$.

It now suffices to prove that

$$\lim_{n\to\infty} |F(\bar{p}_n) - F(\bar{p}'_n)| \;=\; \lim_{n\to\infty} |F(\bar{q}_n) - F(\bar{q}'_n)| = 0.$$

We only show that the first limit is 0, since the the second limit is treated in the same way. We will see that this is a consequence of Condition (2) of Definition 4.8 of up-continuity. By the choice of $\bar{p}_n$ (as convergence testing) and construction of $\bar{p}'_n$ we have $|\bar{p}_n| = |\bar{p}'_n|$, $\lim_{n\to\infty} |\bar{p}_n| = \infty$, $\mathrm{rng}(\bar{p}'_n) \subseteq \{c_1, \ldots, c_k\}$, and for every $\delta > 0$ there is n_0 such that, $\mu^o_\infty(\bar{p}_n, \bar{p}'_n) < \delta$ if $n > n_0$ (because $I_1, \ldots, I_k$ can be chosen with diameter at most δ). From (4.1) and the construction of $\bar{p}'_n$ it follows that for every $\delta > 0$ there is n_0 such that for all $n > n_0$ and $j = 1, \ldots, k$, the number of coordinates in $\bar{p}'_n$ which equals c_j is between $(\alpha_j - \delta)|\bar{p}'_n|$ and $(\alpha_j + \delta)|\bar{p}'_n|$. As F is up-continuous with respect to $c_1, \ldots, c_k$ and $\alpha_1, \ldots, \alpha_k$, it follows from condition (2) of Definition 4.8 that $\lim_{n\to\infty} |F(\bar{p}_n) - F(\bar{p}'_n)| = 0$.

(ii) We prove the contrapositive. Suppose that F is not up-continuous with respect to the parameters c_j, α_j, $j = 1, \ldots, k$. We will show that there are arbitrarily small $\delta > 0$ and $\alpha'_j \in (\alpha_j - \delta, \alpha_j + \delta)$ such that F is not ct-continuous with respect to the parameters c_j, α'_j, $j = 1, \ldots, k$. As F is not up-continuous with respect to the mentioned parameters either condition (1) or condition (2) of Definition 4.8 of up-continuity fails for the same parameters.

First suppose that condition (1) of Definition 4.8 fails for the parameters c_j, α_j, $j = 1, \ldots, k$. Then there are arbitrarily small $\delta > 0$ such that F is not asymptotically uniformly continuous on

$$\begin{aligned} X^\delta = \big\{ &\bar{p} \in [0,1]^{<\omega} : \mathrm{rng}(\bar{p}) \subseteq \{c_1, \ldots, c_k\} \text{ and, for each } j = 1, \ldots, k, \\ &\text{there are between } (\alpha_j - \delta)|\bar{p}| \text{ and } (\alpha_j + \delta)|\bar{p}| \text{ coordinates in } \bar{p} \\ &\text{which equals } c_j \big\}. \end{aligned}$$

Let $\delta > 0$. Without loss of generality we may assume that δ is small enough so that if $\alpha_j > 0$ then $\alpha_j - \delta > 0$, and if $\alpha_j < 1$ then $\alpha_j + \delta < 1$. As F is not asymptotically uniformly continuous on X^δ, there is $\varepsilon > 0$ such that for all $m, N \in \mathbb{N}^+$ there are $\bar{p}_{m,N}, \bar{q}_{m,N} \in X^\delta$ such that $\mu^u_1(\bar{p}_{m,N}, \bar{q}_{m,N}) < 1/m$, $|\bar{p}_{m,N}|, |\bar{q}_{m,N}| > N$ and $|F(\bar{p}_{m,N}) - F(\bar{q}_{m,N})| > \varepsilon$. We now define two convergence testing sequences the parameters of which will become clear later.

Define $\bar{p}'_1 = \bar{p}_{1,1}$, $\bar{q}'_1 = \bar{q}_{1,1}$, and for $n > 1$, $\bar{p}'_n = \bar{p}_{n,N}$ and $\bar{q}'_n = \bar{q}_{n,N}$ where N is larger than $|\bar{p}'_m|$ and $|\bar{q}'_m|$ for all $m < n$. Then $|\bar{p}'_n| < |\bar{p}'_{n+1}|$, $|\bar{q}'_n| < |\bar{q}'_{n+1}|$, $\mu^u_1(\bar{p}'_n, \bar{q}'_n) < 1/n$ and $|F(\bar{p}'_n) - F(\bar{q}'_n)| > \varepsilon$ for all n. Let $\bar{p}'_{n,i}$ denote the ith coordinate of $\bar{p}'$. Since $[0,1]^k$ is a compact topological space, the sequence of k-tuples

$$\big(|\{i : \bar{p}'_{n,i} = c_1\}|/|\bar{p}'_n|, \ldots, |\{i : \bar{p}'_{n,i} = c_k\}|/|\bar{p}'_n|\big) \tag{4.2}$$

has a convergent subsequence with limit $(\alpha'_1, \ldots, \alpha'_k)$, say. Without loss of generality we may assume that the convergent subsequence is the whole sequence, so (4.2) converges to $(\alpha'_1, \ldots, \alpha'_k)$ as $n \to \infty$. By the definition of X^δ and the construction of $\bar{p}'_n$ and $\bar{q}'_n$ we have $\alpha_j - \delta \le \alpha'_j \le \alpha_j + \delta$ for all $j = 1, \ldots, k$.

We now show that both the sequence $\bar{p}'_n$ and the sequence $\bar{q}'_n$ is convergence testing for the parameters $c_1, \ldots, c_k$ and $\alpha'_1, \ldots, \alpha'_k$. We have already observed that $|\bar{p}'_n| < |\bar{p}'_{n+1}|, |\bar{q}'_n| < |\bar{q}'_{n+1}|$ so condition (1) in Definition 4.1 of convergence testing sequence is satisfied for both sequences. Since $\text{rng}(\bar{p}'_n), \text{rng}(\bar{q}'_n) \subseteq \{c_1, \ldots, c_k\}$ (because both sequences belong to X^δ) it follows that for all open intervals $I_1, \ldots, I_k$ of $[0, 1]$ (with respect to the induced topology on $[0, 1]$) such that $c_j \in I_j$ for $j = 1, \ldots, k$, $\text{rng}(\bar{p}'_n), \text{rng}(\bar{q}'_n) \subseteq \bigcup_{j=1}^k I_j$. By the choice of α'_j, $j = 1, \ldots, k$, it follows that

$$\lim_{n\to\infty} \frac{\left|\{i \leq |\bar{p}'_n| : p_{n,i} \in I_j\}\right|}{|\bar{p}'_n|} = \alpha_j. \tag{4.3}$$

Hence the sequence $\bar{p}'_n$ is convergence testing for parameters $c_1, \ldots, c_k$ and $\alpha'_1, \ldots, \alpha'_k$. Since $\mu^u_1(\bar{p}'_n, \bar{q}'_n) < 1/n$ it follows that (4.3) holds if $\bar{p}'_n$ is replaced by $\bar{q}'_n$, so $\bar{q}'_n$ is convergence testing for the same parameters. As $|F(\bar{p}'_n) - F(\bar{q}'_n)| > \varepsilon$ for all n we conclude that F is not ct-continuous with respect to the sequence of parameters $c_1, \ldots, c_k, \alpha_1, \ldots, \alpha_k$.

Now suppose that condition (2) in Definition 4.8 of up-continuity fails for the sequence of parameters $c_1, \ldots, c_k, \alpha_1, \ldots, \alpha_k$. Then there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}^+$ there are $\bar{p}_n, \bar{q}_n \in [0, 1]^{<\omega}$ such that

(a) $|\bar{p}_n| = |\bar{q}_n| > n$,
(b) $\mu^o_\infty(\bar{p}_n, \bar{q}_n) < 1/n$,
(c) $\text{rng}(\bar{p}_n) \subseteq \{c_1, \ldots, c_k\}$, and
(d) for each $j = 1, \ldots, k$, there are between $(\alpha_j - \delta)|\bar{p}_n|$, $(\alpha_j + \delta)|\bar{p}_n|$ coordinates in $\bar{p}_n$ which equal c_j, and
(e) $|F(\bar{p}_n) - F(\bar{q}_n)| \geq \varepsilon$.

Without loss of generality we can also assume that $|\bar{p}_n| < |\bar{p}_{n+1}|$ and $|\bar{q}_n| < |\bar{q}_{n+1}|$ for all n. From (c) and (d) it follows that $\bar{p}_n$, $n \in \mathbb{N}^+$, is a convergence testing sequence for the parameters $c_1, \ldots, c_k$ and $\alpha_1, \ldots, \alpha_k$. From (b) it follows that for every $j = 1, \ldots, k$ and every open interval I_j around c_j there is N such that if $n > N$ then an entry of $\bar{p}_n$ lies in I_j if and only if the corresponding entry of $\bar{q}_n$ lies in I_j. Hence $\bar{q}_n$, $n \in \mathbb{N}^+$, is also a convergence testing sequence for the parameters $c_1, \ldots, c_k$ and $\alpha_1, \ldots, \alpha_k$. From (e) we now conclude that F is not ct-continuous with respect to the parameters $c_1, \ldots, c_k$ and $\alpha_1, \ldots, \alpha_k$. □

Example 4.12 From Proposition 4.4 and Proposition 4.10 it follows that the aggregation functions 'am' (arithmetic mean) and 'gm' (geometric mean) are up-continuous with respect to all parameters, and that the aggregation functions 'max' and 'min' are, for all $m, k_1, \ldots, k_m \in \mathbb{N}^+$, all $c_{i,j} \in [0, 1]$ and all $\alpha_{i,j} \in (0, 1]$, up-continuous with respect to the parameters $c_{i,j}$ and $\alpha_{i,j}$. From earlier examples and Proposition 4.10 it follows that 'length$^{-\beta}$' (where $\beta \in (0, 1)$) and 'tsum' ("truncated sum") are up-continuous with respect to all parameters. More examples of aggregation functions that are ct-continuous and up-continuous with respect to all parameters, or possibly with the requirement that $\alpha_{i,j} > 0$ for all i and j, are found in [17] and [18].

In the next two examples we use the following notation: If $\alpha \in [0, 1]$ and $\bar{p} \in [0, 1]^{<\omega}$, then $\#(\alpha, \bar{p})$ is the number of coordinates in $\bar{p}$ which are equal to α. The remaining two examples consider aggregation functions obtained from generalized quantifiers, as discussed in Section 3, and show that rather mild conditions on a sequence of parameters imply that such aggregation functions are ct-continuous, respectively up-continuous, with respect to that sequence of parameters.

Example 4.13 In [13] Keisler and Lotfallah proved results about almost sure elimination of probability quantifiers and convergence in probability. Let $\beta \in (0, 1)$. The probability quantifier "the proportion of x satisfying ... is at least β" corresponds to the following generalized quantifier (in the sense of Definition 3.1):

$$Q = \big\{(D, X) : \ D \neq \emptyset \text{ and } |X|/|D| \geq \beta\big\}.$$

As we saw in the proof of Proposition 3.3, Q can be represented by the aggregation function $F : [0, 1]^{<\omega} \to [0, 1]$ defined as follows:

$$F(\bar{p}) = \begin{cases} 1 & \text{if } \#(1, \bar{p})/|\bar{p}| \geq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\beta > 0$ and that $c_1, \dots, c_k, \alpha_1, \dots, \alpha_k \in [0, 1]$. It is straightforward to verify that F is ct-continuous with respect to the parameters $c_1, \dots, c_k, \alpha_1, \dots, \alpha_k$ if either

(1) $c_1, \dots, c_k < 1$, or if
(2) there are $m, i_1, \dots, i_m \in \mathbb{N}^+$ such that $c_{i_1}, \dots, c_{i_m}$ is an enumeration of all c_i such that $c_i = 1$ and $\alpha_{i_1} + \dots + \alpha_{i_m} \neq \beta$.

Proposition 4.10 implies that under the same conditions F is up-continuous with respect to the parameters $c_1, \dots, c_k, \alpha_1, \dots, \alpha_k$.

Example 4.14 Consider the Rescher quantifier:

$$R = \big\{(D, X_1, X_2) : \ |X_1| \leq |X_2|\big\}.$$

It can be represented by the aggregation function $G : \big([0, 1]^{<\omega}\big)^2 \to [0, 1]$ defined as follows:

$$G(\bar{p}, \bar{q}) = \begin{cases} 1 & \text{if } \#(1, \bar{p}) \leq \#(1, \bar{q}), \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 1, 2$ and $j = 1, \dots, k_i$ let $c_{i,j}, \alpha_{i,j} \in [0, 1]$. If $c_{i,j} < 1$ for all i and j then it is easy to verify that G is ct-continuous with respect to $c_{i,j}$ and $\alpha_{i,j}$. Now suppose that $c_{1,j} = 1$ for some j or that $c_{2,j} = 1$ for some j. Let $j_1, \dots, j_s$ enumerate all j such that $c_{1,j} = 1$ and let $l_1, \dots, l_t$ enumerate all l such that $c_{2,l} = 1$. It is straightforward to verify that if

$$\alpha_{1,j_1} + \dots + \alpha_{1,j_s} \neq \alpha_{2,l_1} + \dots + \alpha_{2,l_t}$$

then G is ct-continuous with respect to $c_{i,j}$ and $\alpha_{i,j}$. (We consider the left side sum to be 0 if no $c_{1,j}$ exists which is 1, and we consider the right side sum to be 0 if no

$c_{2,j}$ exists which is 1.) Proposition 4.10 implies that under the same conditions G is up-continuous with respect to the parameters $c_{i,j}$ and $\alpha_{i,j}$.

The reader may verify that ct-continuity and up-continuity of the aggregation function which represents the Härtig quantifier $H = \{(D, X_1, X_2) : |X_1| = |X_2|\}$ can be characterized in the same way.

5 Asymptotic equivalence to basic formulas

In this section we state and prove our main result, Theorem 5.9. We begin by fixing some assumptions and notation for the rest of the section and then discuss the assumptions and results of this section.

Notation 5.1 *For each $n \in \mathbb{N}^+$, D_n is a finite set, also called a* domain*, and $|D_{n+1}| > |D_n|$ for all n. By σ we denote a finite and relational signature and $\mathbf{W}_n$ denotes a set of (not necessarily all) σ-structures with domain D_n. By $\mathbb{P}_n$ we denote a probability distribution on $\mathbf{W}_n$.*

A commonly considered context is when $D_n = \{1, \ldots, n\}$ and $\mathbf{W}_n$ is the set of all σ-structures with domain D_n. However we want to allow for more generality. For example one may be interested in σ-structures with domain D_n such that all these structures have some common properties. For example, we may restrict some relation symbols in σ to be interpreted as relations with degree[1] at most 20 (say), or we could restrict a binary relation symbol $R \in \sigma$ to be interpreted as a tree, a partial order, an equivalence relation, perhaps with some additional properties. We can, but need not, require that, for some $\sigma' \subset \sigma$ every structure in $\mathbf{W}_n$ has the same reduct to σ'. If such restrictions are imposed on the σ-structures considered with domain D_n, then the sequence of cardinalities $|D_n|$, $n \in \mathbb{N}^+$, may not contain all (sufficiently large) positive integers. Note also that we do not require that $D_n \subseteq D_{n+1}$.

With this quite general set-up our goal is to isolate properties (given by Assumption 5.5) of "simpler" 0/1-valued sublogics $L_0, L_1 \subseteq PLA^*(\sigma)$ such that if L_0 and L_1 have these properties, then a formula $\varphi(\bar{x}) \in PLA^*(\sigma)$ is asymptotically equivalent (see Definition 5.2) to an "L_0-basic formula", provided that every aggregation function in φ is up-continuous (or ct-continuous) with respect to some parameters which are determined only by $(\mathbb{P}_n : n \in \mathbb{N}^+)$ and the subformulas of φ. By an "L_0-basic formula" we mean a formula of the form $\bigwedge_{i=1}^{k} (\varphi_i(\bar{x}) \to c_i)$ where $\varphi_i \in L_0$ and $c_i \in [0, 1]$ for all i and $\lim_{n\to\infty} \mathbb{P}_n\big(\forall \bar{x} \bigvee_{i=1}^{k} \varphi_i(\bar{x})\big) = 1$. If some $\psi(\bar{x}) \in PLA^*(\sigma)$ is asymptotically equivalent to an L_0-basic formula $\varphi(\bar{x}) = \bigwedge_{i=1}^{k} (\varphi_i(\bar{x}) \to c_i)$, then, for sufficiently large n, $\mathcal{A} \in \mathbf{W}_n$ and $\bar{a} \in (D_n)^{|\bar{x}|}$, the value $\mathcal{A}(\psi(\bar{a}))$ is, with high probability, close to the value $\mathcal{A}(\varphi(\bar{a}))$ which is determined by the values $\mathcal{A}(\varphi_i(\bar{a}))$, $i = 1, \ldots, k$, where $\varphi_i \in L_0$ are 0/1-valued. In earlier work like [10, 17, 18] and [13, 14] (if the later are reformulated in the context of PLA^*) L_0 can be taken to be the set of complete consistent conjunctions of first-order literals. But this choice of L_0 is not always possible.

[1] A relation has degree at most d if the Gaifman graph (also called the primal graph) corresponding to the relation has no vertex with degree more than d.

For 0/1-valued logics such as first-order logic the notion of *almost sure/everywhere equivalence* between two formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ means that the probability that the value of $\forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is 1 tends to 1 as $n \to \infty$. Since we consider a logic with (truth) values in the unit interval [0, 1] we need to generalize the notion of almost sure equivalence and we do it as follows:

Definition 5.2 (Equivalence and asymptotic equivalence) Let $\varphi(\bar{x}), \psi(\bar{x}) \in PLA^*(\sigma)$.

(i) We say that $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *equivalent* if for for every finite σ-structure $\mathcal{A}$ and every $\bar{a} \in A^{|\bar{x}|}$, $\mathcal{A}\big(\varphi(\bar{a})\big) = \mathcal{A}\big(\psi(\bar{a})\big)$.
(ii) We say that $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *asymptotically equivalent (with respect to* $(\mathbb{P}_n : n \in \mathbb{N}^+)$) if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}_n\Big(\Big\{\mathcal{A} \in \mathbf{W}_n : \text{ for all } \bar{a} \in (D_n)^{|\bar{x}|},\ \big|\mathcal{A}\big(\varphi(\bar{a})\big) - \mathcal{A}\big(\psi(\bar{a})\big)\big| \leq \varepsilon\Big\}\Big) = 1.$$

Since we have fixed a sequence $\mathbb{P}_n, n \in \mathbb{N}^+$, of probability distributions for the rest of the section we will just say that two formulas are asymptotically equivalent, omitting "with respect to $(\mathbb{P}_n, n \in \mathbb{N}^+)$".

Notation 5.3 *If* $\varphi \in PLA^*(\sigma)$ *is a formula without free variables then*

$$\mathbb{P}_n\big(\varphi\big) = \mathbb{P}_n\big(\big\{\mathcal{A} \in \mathbf{W}_n : \mathcal{A}(\varphi) = 1\big\}\big).$$

Definition 5.4 (Special kinds of formulas)

(i) A formula in $PLA^*(\sigma)$ such that no aggregation function occurs in it is called *aggregation-free*.
(i) Let $L \subseteq PLA^*(\sigma)$. We say that L is 0/1-*valued* if for every formula $\varphi(\bar{x}) \in L$, every finite σ-structure $\mathcal{A}$, and every $\bar{a} \in A^{|\bar{x}|}$, $\mathcal{A}(\varphi(\bar{a}))$ is either 0 or 1.
(ii) Let $L \subseteq PLA^*$ and suppose that L is 0/1-valued. A formula of PLA^* is called *L-basic (formula)* if it has the form $\bigwedge_{i=1}^{k}\big(\varphi_i(\bar{x}) \to c_i\big)$ where $\varphi_i \in L$ and $c_i \in [0, 1]$ for all $i = 1, \ldots, k$, and $\lim_{n\to\infty} \mathbb{P}_n\big(\forall\bar{x} \bigvee_{i=1}^{k} \varphi_i(\bar{x})\big) = 1$.

When speaking of L-basic formulas we typically think of L as a "simple" sublogic of PLA^*. For example, L could be the set of all consistent conjunctions of first-order literals. Suppose for the moment that $L \subseteq L' \subseteq PLA^*(\sigma)$ and L is the set of all consistent conjunctions of first-order literals. If we can then prove that (with respect to $\mathbb{P}_n, n \in \mathbb{N}^+$) every $\varphi(\bar{x}) \in L'$ is asymptotically equivalent to an L-basic formula, then it means that, for large n and with high probability, the value $\mathcal{A}(\varphi(\bar{a}))$ is close to a number which is determined by the first-order literals that are satisfied by $\bar{a}$ in $\mathcal{A}$.

The intuition behind the next assumption is that for some sets of formulas $L_0, L_1 \subseteq PLA^*(\sigma)$ and all $\varphi(\bar{x}, \bar{y}) \in L_0$ there is a set $L_{\varphi(\bar{x},\bar{y})} \subseteq L_1$ of formulas defining some "allowed" conditions, and if $\chi(\bar{x}, \bar{y}) \in L_{\varphi(\bar{x},\bar{y})}$, then the fraction $|\varphi(\bar{a}, \mathcal{A}) \cap \chi(\bar{a}, \mathcal{A})|/|\chi(\bar{a}, \mathcal{A})|$ is with high probability close to a number α that depends only on the involved formulas and the sequence $\mathbb{P}_n$. The assumptions in the main results of for example [10, 14, 17, 18] imply that Assumption 5.5 below holds if L_0 is the set of all formulas which are (consistent) conjunctions of first-order literals (and more generally quantifier-free first-order formulas). In the same articles L_1 can, depending on the result, be either $\{\top\}$ (where, for any $\bar{x}$ and $\bar{y}$, we can view $\top$ as a formula $\chi(\bar{x}, \bar{y})$ which has value 1 for any $\bar{a}$ and $\bar{b}$ from any structure), the set of all complete and consistent conjunctions of identities ($x = y$) and nonidentities ($x \neq y$), or the set of all consistent conjuctions of identities and nonidentities; in the same results $L_{\varphi(\bar{x},\bar{y})} = L_1$ for every $\varphi(\bar{x}, \bar{y}) \in L_0$.

Assumption 5.5 Suppose that $L_0 \subseteq PLA^*(\sigma)$ and $L_1 \subseteq PLA^*(\sigma)$ are 0/1-valued and that the following conditions hold:

(1) For every aggregation-free $\varphi(\bar{x}) \in PLA^*(\sigma)$ there is an L_0-basic formula $\varphi'(\bar{x})$ such that φ and φ' are asymptotically equivalent.
(2) For every $m \in \mathbb{N}^+$ and all $\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_m(\bar{x}, \bar{y}) \in L_0$, there are $L_{\varphi_j(\bar{x},\bar{y})} \subseteq L_1$ for $j = 1, \ldots, m$ such that if $\chi_j(\bar{x}, \bar{y}) \in L_{\varphi_j(\bar{x},\bar{y})}$ for $j = 1, \ldots, m$, then there are $s, t \in \mathbb{N}^+$, $\theta_i(\bar{x}) \in L_0$, $\alpha_{i,j} \in [0, 1]$, for $i = 1, \ldots, s$, $j = 1, \ldots, m$, and $\chi'_i(\bar{x}) \in L_0$, for $i = 1, \ldots, t$, such that for every $\varepsilon > 0$ and n there is $\mathbf{Y}_n^\varepsilon \subseteq \mathbf{W}_n$ such that $\lim_{n\to\infty} \mathbb{P}_n(\mathbf{Y}_n^\varepsilon) = 1$ and for every $\mathcal{A} \in \mathbf{Y}_n^\varepsilon$ the following conditions hold:

$$(a)\ \mathcal{A} \models \forall\bar{x} \bigvee_{i=1}^{s} \theta_i(\bar{x}),$$

$$(b)\ \text{if } i \neq j \text{ then } \mathcal{A} \models \forall\bar{x}\neg(\theta_i(\bar{x}) \wedge \theta_j(\bar{x})),$$

$$(c)\ \mathcal{A} \models \forall\bar{x}\Big(\Big(\bigvee_{i=1}^{m} \neg\exists\bar{y}\chi_i(\bar{x}, \bar{y})\Big) \leftrightarrow \Big(\bigvee_{i=1}^{t} \chi'_i(\bar{x})\Big)\Big), \text{ and}$$

$$(d)\ \text{for all } i = 1, \ldots, s \text{ and } j = 1, \ldots, m,\ \text{if } \bar{a} \in (D_n)^{|\bar{x}|},\ \text{and } \mathcal{A} \models \theta_i(\bar{a}),$$
$$\text{then } (\alpha_{i,j} - \varepsilon)|\chi_j(\bar{a}, \mathcal{A})| \ \leq\ |\varphi_j(\bar{a}, \mathcal{A}) \cap \chi_j(\bar{a}, \mathcal{A})| \ \leq\ (\alpha_{i,j} + \varepsilon)|\chi_j(\bar{a}, \mathcal{A})|.$$

From now on we assume that Assumption 5.5 holds, that is, we have fixed two 0/1-valued sublogics $L_0, L_1 \subseteq PLA^*(\sigma)$ such that conditions (1) and (2) of Assumption 5.5 are satisfied.

Lemma 5.6 *Suppose that* $\mathsf{C} : [0, 1]^k \to [0, 1]$ *is a continuous connective. If* $\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}) \in PLA^*(\sigma)$ *and, for* $i = 1, \ldots, k$, $\varphi_i(\bar{x})$ *is asymptotically equivalent to an* L_0*-basic formula* $\psi_i(\bar{x})$, *then* $\mathsf{C}(\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}))$ *is asymptotically equivalent to an* L_0*-basic formula.*

Proof Suppose that the assumptions of the lemma are satisfied. Since C is continuous and, for $i = 1, \ldots, k$, $\varphi_i(\bar{x})$ and $\psi_i(\bar{x})$ are asymptotically equivalent it follows that $\mathsf{C}(\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}))$ and $\mathsf{C}(\psi_1(\bar{x}), \ldots, \psi_k(\bar{x}))$ are asymptotically equivalent. Since the formula $\mathsf{C}(\psi_1(\bar{x}), \ldots, \psi_k(\bar{x}))$ is aggregation-free it follows, by part (1) of Assumption 5.5, that it is equivalent to an L_0-basic formula $\psi'(\bar{x})$. It now follows that $\mathsf{C}(\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}))$ and $\psi'(\bar{x})$ are asymptotically equivalent. $\square$

Definition 5.7 (Frequency parameters of an L_0-basic formula) Let L_0 and L_1 be sublogics of PLA^* that satisfy Assumption 5.5. Let $\psi(\bar{x}, \bar{y})$ be an L_0-basic formula, so $\psi(\bar{x}, \bar{y})$ has the form $\bigwedge_{i=1}^{s} \big(\psi_i(\bar{x}, \bar{y}) \to c_i\big)$ where $\psi_i \in L_0$ and $c_i \in [0, 1]$ for $i = 1, \ldots, s$, and $\lim_{n\to\infty} \mathbb{P}_n\big(\forall\bar{x}, \bar{y} \bigvee_{i=1}^{s} \psi_i(\bar{x}, \bar{y})\big) = 1$. Suppose that $\chi(\bar{x}, \bar{y}) \in \bigcap_{i=1}^{s} L_{\psi_i(\bar{x},\bar{y})}$.

It follows from Assumption 5.5 that there are $\theta_1(\bar{x}), \ldots, \theta_t(\bar{x}) \in L_0$ and $\alpha_{i,j} \in [0, 1]$, for $i = 1, \ldots, t$ and $j = 1, \ldots, s$, such that for every $\varepsilon > 0$ there is $\mathbf{Y}_n^{\varepsilon} \subseteq \mathbf{W}_n$ such that $\lim_{n\to\infty} \mathbb{P}_n\big(\mathbf{Y}_n^{\delta}\big) = 1$ and for all $\mathcal{A} \in \mathbf{Y}_n^{\varepsilon}$

$$\mathcal{A} \models \forall\bar{x} \bigvee_{i=1}^{t} \theta_i(\bar{x}),$$

$$\text{if } i \neq j \text{ then } \mathcal{A} \models \forall\bar{x}\neg(\theta_i(\bar{x}) \wedge \theta_j(\bar{x})),$$

and, for all $i = 1, \ldots, t$ and all $j = 1, \ldots, s$, if $\bar{a} \in (D_n)^{|\bar{x}|}$, $\mathcal{A} \models \theta_i(\bar{a})$ and $\chi(\bar{a}, \mathcal{A}) \neq \emptyset$, then

$$\alpha_{i,j} - \varepsilon \leq \frac{|\psi_j(\bar{a}, \mathcal{A}) \cap \chi(\bar{a}, \mathcal{A})|}{|\chi(\bar{a}, \mathcal{A})|} \leq \alpha_{i,j} + \varepsilon.$$

Fix some $i \in \{1, \ldots, t\}$. Let $c \in \{c_1, \ldots, c_s\}$ and let $j_1, \ldots, j_r$ enumerate, without repetition, all indices i such that $c_i = c$. Let $\beta_c = \alpha_{i,j_1} + \ldots + \alpha_{i,j_r}$. Let $d_1, \ldots, d_{s'}$ be an enumeration of $\{c_1, \ldots, c_s\}$ without repetition. Then we call $(d_1, \ldots, d_{s'}, \beta_{d_1}, \ldots, \beta_{d_{s'}})$ the *sequence of $\bar{y}$-frequency parameters of ψ relative to χ and θ_i* (and it is unique up to the order of the d_i and the corresponding β_{d_i}). A *sequence of $\bar{y}$-frequency parameters of ψ relative to χ* is, by definition, a sequence of $\bar{y}$-frequency parameters of ψ relative to χ and θ_i, for some i.

The above definition can be generalized, for any $r > 1$, to r L_0-basic formulas $\psi_1(\bar{x}, \bar{y}), \ldots, \psi_r(\bar{x}, \bar{y})$. Suppose that, for each $i = 1, \ldots, r$, ψ_i has the form

$$\bigwedge_{k=1}^{s_i} \big(\psi_{i,k}(\bar{x}, \bar{y}) \to c_{i,k}\big),$$

so each $\psi_{i,k}$ belongs to L_0. Suppose that, for $i = 1, \ldots, r$, $\chi_i(\bar{x}, \bar{y}) \in \bigcap_{k=1}^{s_i} L_{\psi_{i,k}(\bar{x},\bar{y})}$. By Assumption 5.5 there are $\theta_1(\bar{x}), \ldots, \theta_t(\bar{x}) \in L_0$ and $\alpha_{i,j,k} \in [0, 1]$ for $i = 1, \ldots, t$, $j = 1, \ldots, r$ and $k = 1, \ldots, s_j$, such that for every $\varepsilon > 0$ there is $\mathbf{Y}_n^{\varepsilon} \subseteq \mathbf{W}_n$ such that $\lim_{n\to\infty} \mathbb{P}_n\big(\mathbf{Y}_n^{\delta}\big) = 1$ and for all $\mathcal{A} \in \mathbf{Y}_n^{\varepsilon}$

$$\mathcal{A} \models \forall\bar{x} \bigvee_{i=1}^{t} \theta_i(\bar{x}),$$

$$\text{if } i \neq j \text{ then } \mathcal{A} \models \forall\bar{x}\neg(\theta_i(\bar{x}) \wedge \theta_j(\bar{x})),$$

and, for all $i = 1, \ldots, t$ and all $j = 1, \ldots, r$, if $\bar{a} \in (D_n)^{|\bar{x}|}$, $\mathcal{A} \models \theta_i(\bar{a})$ and $\chi_j(\bar{a}, \mathcal{A}) \neq \emptyset$, then

$$\alpha_{i,j,k} - \varepsilon \leq \frac{|\psi_{j,k}(\bar{a}, \mathcal{A}) \cap \chi_j(\bar{a}, \mathcal{A})|}{|\chi_j(\bar{a}, \mathcal{A})|} \leq \alpha_{i,j,k} + \varepsilon.$$

The r-tuple of sequences of $\bar{y}$-frequency parameters of $(\psi_1, \ldots, \psi_r)$ relative to $(\chi_1, \ldots, \chi_r)$ and θ_i is, by definition, $(\bar{p}_1, \ldots, \bar{p}_r)$, where $\bar{p}_j$ is the sequence of $\bar{y}$-frequency parameters of ψ_j relative to χ_j and θ_i. An r-tuple is an *r-tuple of $\bar{y}$-frequency parameters of $(\psi_1, \ldots, \psi_r)$ relative to $(\chi_1, \ldots, \chi_r)$ if, for some i, it is the sequence of $\bar{y}$-frequency parameters of $(\psi_1, \ldots, \psi_r)$ relative to $(\chi_1, \ldots, \chi_r)$ and θ_i.*

The next lemma follows more or less directly from Definition 5.7 and states the properties of frequency parameters that will be relevant later.

Lemma 5.8 *Let $\psi(\bar{x}, \bar{y})$ denote the L_0-basic formula $\bigwedge_{i=1}^{s} \big(\psi_i(\bar{x}, \bar{y}) \rightarrow c_i\big)$ and suppose that $\chi(\bar{x}, \bar{y}) \in \bigcap_{i=1}^{s} L_{\psi_i(\bar{x},\bar{y})}$. Also let $\theta_1(\bar{x}), \ldots, \theta_t(\bar{x}) \in L_0$ be formulas as in Definition 5.7, and let $(d_{i,1}, \ldots, d_{i,s_i}, \beta_{i,1}, \ldots, \beta_{i,s_i})$ be the sequence of $\bar{y}$-frequency parameters of ψ relative to χ and θ_i. Then:*

(1) For each $i = 1, \ldots, t$, $d_{i,1}, \ldots, d_{i,s_i}$ is an enumeration of $\{c_1, \ldots, c_s\}$ without repetition.

(2) Suppose that $\mathbf{Y}_n^{\varepsilon}$ is like in Definition 5.7 and $\mathcal{A} \in \mathbf{Y}_n^{\varepsilon}$. Let $i \in \{1, \ldots, t\}$ and suppose that $\bar{a} \in (D_n)^{|\bar{x}|}$, $\chi(\bar{a}, \mathcal{A}) \neq \emptyset$, $\mathcal{A} \models \theta_i(\bar{a})$, and let

$$\bar{p} = \big(\mathcal{A}\big(\psi(\bar{a}, \bar{b})\big) : \bar{b} \in (D_n)^{|\bar{x}|} \text{ and } \mathcal{A} \models \chi(\bar{a}, \bar{b})\big).$$

Then

(a) $rng(\bar{p}) \subseteq \{c_1, \ldots, c_s\} = \{d_{i,1}, \ldots, d_{i,s_i}\}$.

(b) For all $j = 1, \ldots, s_i$, then number of coordinates in $\bar{p}$ which are equal to $d_{i,j}$ is between $(\beta_{i,j} - s\varepsilon)|\chi(\bar{a}, \mathcal{A})|$ and $(\beta_{i,j} + s\varepsilon)|\chi(\bar{a}, \mathcal{A})|$.

Theorem 5.9 *Let L_0 and L_1 be sublogics of $PLA^*(\sigma)$ that satisfy Assumption 5.5. Let $F : \big([0,1]^{<\omega}\big)^m \rightarrow [0,1]$ be an m-ary aggregation function, let $\varphi_i(\bar{x}, \bar{y}) \in PLA^*(\sigma)$, for $i = 1, \ldots, m$, and suppose that each $\varphi_i(\bar{x}, \bar{y})$ is asymptotically equivalent to an L_0-basic formula $\psi_i(\bar{x}, \bar{y})$ where ψ_i has the form $\bigwedge_{k=1}^{s_i} (\psi_{i,k}(\bar{x}, \bar{y}) \rightarrow c_{i,k})$. Suppose that for $i = 1, \ldots, m$, $\chi_i(\bar{x}, \bar{y}) \in \bigcap_{k=1}^{s_i} L_{\psi_{i,k}(\bar{x},\bar{y})}$. Let $\varphi(\bar{x})$ denote the $PLA^*(\sigma)$-formula*

$$F\big(\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_m(\bar{x}, \bar{y}) : \bar{y} : \chi_1(\bar{x}, \bar{y}), \ldots, \chi_m(\bar{x}, \bar{y})\big).$$

(i) If F is up-continuous with respect to every m-tuple of $\bar{y}$-frequency parameters of $(\psi_1, \ldots, \psi_m)$ relative to $(\chi_1, \ldots, \chi_m)$, then $\varphi(\bar{x})$ is asymptotically equivalent to an L_0-basic formula.

(ii) Suppose that there is $\delta > 0$ such that F is ct-continuous with respect to every m-tuple of (sequences of) parameters $(\bar{p}_1, \ldots, \bar{p}_m)$ such that the following holds:

For each $i = 1, \dots, m$, *if* $\bar{p}_i = (c_{i,1}, \dots, c_{i,k_i}, \alpha_{i,1}, \dots, \alpha_{i,k_i})$, *then there are* $\beta_{i,j} \in [0, 1]$ *such that* $\alpha_{i,j} \in (\beta_{i,j} - \delta, \beta_{i,j} + \delta)$ *for* $j = 1, \dots, k_i$, *and if*

$$\bar{q}_i = (c_{i,1}, \dots, c_{i,k_i}, \beta_{i,1}, \dots, \beta_{i,k_i}),$$

then $(\bar{q}_1, \dots, \bar{q}_m)$ *is an* m*-tuple of* $\bar{y}$*-frequency parameters of* $(\psi_1, \dots, \psi_m)$ *relative to* $(\chi_1, \dots, \chi_m)$.

Then $\varphi(\bar{x}, \bar{y})$ *is asymptotically equivalent to an* L_0*-basic formula.*

Example 5.10 As a concrete example of an application of Theorem 5.9 let us consider the following context. Let σ be a finite relational signature and let $\mathbf{W}_n$ be the set of all σ-structures with domain $\{1, \dots, n\}$. Suppose that $\mathbb{P}_n$ is the uniform probability distribution on $\mathbf{W}_n$ for all n. Let L_0 be the set of all conjunctions of first-order literals (over σ) and let $L_1 = L_0$. In this setting, an L_0-basic formula is one of the form $\bigwedge_{i=1}^{s}(\theta_i(\bar{x}) \rightarrow c_i)$ where $c_i \in [0, 1]$ and $\theta_i(\bar{x}) \in L_0$ for all $i = 1, \dots, s$, and $\forall \bar{x} \bigvee_{i=1}^{s} \theta_i(\bar{x})$ is a valid sentence.

One can easily prove, as in [17], that every aggregation-free formula is equivalent to an L_0-basic formula so part (1) Assumption 5.5 holds. From the proofs of the classical 0-1 law (in particular the arguments in [7]) it follows that also part (2) of Assumption 5.5 holds and we can, with the notation of that assumption, let $L_{\varphi_j(\bar{x}, \bar{y})} = L_1$ for all $\varphi_j(\bar{x}, \bar{y}) \in L_0$. It now follows from Example 4.12 and Theorem 5.9 that if $\varphi(\bar{x}, \bar{y})$ is an L_0-basic formula and $\chi(\bar{x}, \bar{y}) \in L_1$, then the formulas

$$\mathrm{am}\big(\varphi(\bar{x}, \bar{y}) : \bar{y} : \chi(\bar{x}, \bar{y})\big) \quad \text{and} \quad \mathrm{gm}\big(\varphi(\bar{x}, \bar{y}) : \bar{y} : \chi(\bar{x}, \bar{y})\big)$$

(where am and gm are the arithmetic, respectively gemetric, means) are asymptotically equivalent to L_0-basic formulas. If we require that the formula called $\chi(\bar{x}, \bar{y})$ above is a conjuction of formulas of the form $x = y$ or $x \neq y$, then, by the results in [17, 18], we get the same conclusion for more probability distributions (than the uniform one) and for more aggregation functions (than am and gm). In [15, 16], which use Theorem 5.9, other choices of L_0 and L_1 turn out to be appropriate for the results aimed at there.[2]

Remark 5.11 (Applications of Theorem 5.9) (i) Suppose that Assumption 5.5 holds and $\varphi(\bar{x}) \in PLA^*(\sigma)$. Part (1) of Assumption 5.5, Lemma 5.6, and Theorem 5.9, tell that $\varphi(\bar{x})$ can be "reduced", step by step, to an asymptotically equivalent L_0-basic formula as long as, in a step when an aggregation function F is to be "asymptotically eliminated" we do not encounter parameters with respect to which F is not up-continuous. The proofs of Lemma 5.6 and Theorem 5.9 indicate how the steps in such a reduction are carried out.

(ii) Now suppose that when trying to asymptotically eliminate an aggregation function F as in Theorem 5.9 this F is *not* up-continuous with respect to the parameters $c_{i,j}, \alpha_{i,j}$ that arise in this step. Then there may be another aggregation function G which behaves "almost" like F but is up-continuous with respect to the parameters

[2] The articles [15, 16] were written after this article but published before.

$c_{i,j}, \alpha_{i,j}$. If F is replaced by G in the formula then the change of meaning of the new formula may be negligible in a particular context, but now the aggregation function G can be eliminated. For example, if F is as in Example 4.13 and we encounter parameters c_j, α_j with respect to which F is not up-continuous, or equivalently not ct-continuous, then we can consider an aggregation function G which is defined just as F except that we replace β with some $\beta' \neq \beta$ which is very close to β.

(iii) As a special case of Theorem 5.9 we have the following: Suppose that Assumption 5.5 holds for some $L_0, L_1 \subseteq PLA^*(\sigma)$ where $L_1 = \{\top\}$ and $L_{\varphi(\bar{x},\bar{y})} = L_1$ for all $\varphi(\bar{x}, \bar{y}) \in L_0$. Also let $coPLA^*(\sigma, L_0, L_1)$ be the set of all formulas $\varphi(\bar{x}) \in PLA^*(\sigma)$ such that every subformula of $\varphi(\bar{x})$ of the form

$$F\big(\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_m(\bar{x}, \bar{y}) : \bar{y} : \chi_1(\bar{x}, \bar{y}), \ldots, \chi_m(\bar{x}, \bar{y})\big).$$

is such that $\chi_1, \ldots, \chi_m \in L_1$ (i.e. all χ_i are $\top$) and F is up-continuous with respect to all choices of parameters, or equivalently (by Proposition 4.10), ct-continuous with respect to all choices of parameters. Examples of such aggregation functions are given by Proposition 4.4 and Example 4.9. Then Lemma 5.6 and Theorem 5.9 imply that *every formula in $coPLA^*(\sigma, L_0, L_1)$ is asymptotically equivalent to an L_0-basic formula.*

Remark 5.12 **(Necessity of continuity)** In Theorem 5.9 we cannot remove the assumption that F is up-continuous, or ct-continuous, with respect to certain parameters. This can be seen in different ways. One way is the following, roughly explained. The work in [13] coveres the context where $\mathbf{W}_n$ is the set of all σ-structures with domain $D_n = \{1, \ldots, n\}$ and $\mathbb{P}_n$ is the uniform probability distribution on $\mathbf{W}_n$. The work in [13] shows that Assumption 5.5 is satisfied if L_0 is the set of (consistent) conjunctions of first-order literals, $L_1 = \{\top\}$, and $L_{\varphi(\bar{x},\bar{y})} = L_1$ for all $\varphi(\bar{x}, \bar{y}) \in L_0$. Let $PLA^*(\sigma, L_0, L_1)$ be the set of formulas $\varphi(\bar{x}) \in PLA^*(\sigma)$ such that every subformula of $\varphi(\bar{x})$ of the form

$$F\big(\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_m(\bar{x}, \bar{y}) : \bar{y} : \chi_1(\bar{x}, \bar{y}), \ldots, \chi_m(\bar{x}, \bar{y})\big).$$

is such that $\chi_1, \ldots, \chi_m \in L_1$ (i.e. all χ_i are $\top$). As was explained in Example 4.13 every formula of the (0/1-valued) "probability logic" L considered in [13] is equivalent to a $PLA^*(\sigma, L_0, L_1)$-formula (if σ contains all relation symbols used by L). If Theorem 5.9 would hold without the continuity assumption, then every $PLA^*(\sigma, L_0, L_1)$-formula would be asymptotically equivalent to an L_0-basic formula. It would follow (as L_0 contains only quantifier-free formulas) that every formula of the logic L considered in [13] is asymptotically equivalent to a quantifier-free first-order formula, and from this it would follow that for every $\varphi \in L$, $\lim_{n\to\infty} \mathbb{P}_n(\varphi)$ exists (see [13, Corollary 4.10]). But this is not the case, because as shown in [13, Proposition 3.1] there is $\varphi \in L$ such that the limit does not exist.

5.1 Proof of Theorem 5.9

Part (ii) of Theorem 5.9 is a consequence of part (i) and Proposition 4.10. For if the assumptions in part (ii) hold, then F is up-continuous with respect to every m-tuple of $\bar{y}$-frequency parameters of $(\psi_1, \dots, \psi_m)$ relative to $(\chi_1, \dots, \chi_m)$, so by part (i) $\varphi(\bar{x})$ is asymptotically equivalent to an L_0-basic formula.

Hence it remains to prove part (i) of Theorem 5.9 and the rest of this section is devoted to this. In order to make the arguments more clear by avoiding heavy notation we will assume that $m = 1$. The general case is proved in the same way except that we need to keep track of more formulas and parameters.

Throughout the rest of this section we assume that $\psi(\bar{x}, \bar{y})$ denotes the L_0-basic formula $\bigwedge_{i=1}^{s} \big(\psi_i(\bar{x}, \bar{y}) \to c_i\big)$, so $\psi_i \in L_0$ for all i. We also assume that $\chi(\bar{x}, \bar{y}) \in \bigcap_{i=1}^{s} L_{\psi_i(\bar{x}, \bar{y})}$. By Assumption 5.5 there are $\theta_1(\bar{x}), \dots, \theta_t(\bar{x}) \in L_0$, $\chi_1(\bar{x}), \dots, \chi_k(\bar{x}) \in L_0, \alpha_{i,j} \in [0, 1]$ for $i = 1, \dots, t$ and $j = 1, \dots, s$, and $\mathbf{Y}_n^{\delta} \subseteq \mathbf{W}_n$ such that $\lim_{n\to\infty} \mathbb{P}_n(\mathbf{Y}_n^{\delta}) = 1$ and for all $\mathcal{A} \in \mathbf{Y}_n^{\delta}$,

$$(a)\ \mathcal{A} \models \forall \bar{x} \bigvee_{i=1}^{t} \theta_i(\bar{x}) = 1,$$

$$(b)\ \text{if } i \neq j \text{ then } \mathcal{A} \models \forall \bar{x} \neg(\theta_i(\bar{x}) \wedge \theta_j(\bar{x})),$$

$$(c)\ \mathcal{A} \models \forall \bar{x} \Big(\neg \exists \bar{y} \chi(\bar{x}, \bar{y}) \leftrightarrow \Big(\bigvee_{i=1}^{k} \chi_i(\bar{x})\Big)\Big), \text{ and}$$

(d) for all $i = 1, \dots, t,$ if $\bar{a} \in (D_n)^{|\bar{x}|}$, and $\mathcal{A} \models \theta_i(\bar{a})$,
then $(\alpha_{i,j} - \varepsilon)|\chi(\bar{a}, \mathcal{A})| \leq |\psi_j(\bar{a}, \mathcal{A}) \cap \chi(\bar{a}, \mathcal{A})| \leq (\alpha_{i,j} + \varepsilon)|\chi(\bar{a}, \mathcal{A})|$.

Furthermore, suppose that $F : [0, 1]^{<\omega} \to [0, 1]$ is an aggregation function that is up-continuous with respect to every sequence of $\bar{y}$-frequency parameters of ψ relative to χ.

Our first goal, achieved by Lemma 5.14 below, is to prove that $F\big(\psi(\bar{x}, \bar{y}) : \bar{y} : \chi(\bar{x}, \bar{y})\big)$ is asymptotically equivalent to an L_0-basic formula. Then we prove, as stated by Proposition 5.15 below, that if $\varphi(\bar{x}, \bar{y}) \in PLA^*(\sigma)$ and $\varphi(\bar{x}, \bar{y})$ is asymptotically equivalent to $\psi(\bar{x}, \bar{y})$, then $F\big(\varphi(\bar{x}, \bar{y}) : \bar{y} : \chi(\bar{x}, \bar{y})\big)$ is asymptotically equivalent to an L_0-basic formula. This completes the proof of Theorem 5.9.

Lemma 5.13 *Fix an arbitrary index $1 \leq i \leq t$. There is $d_i \in [0, 1]$, depending only on ψ, χ, θ_i, F, and $(\mathbb{P}_n : n \in \mathbb{N}^+)$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all sufficiently large n, all $\mathcal{A} \in \mathbf{Y}_n^{\delta}$, and all $\bar{a} \in (D_n)^{|\bar{x}|}$, if $\chi(\bar{a}, \mathcal{A}) \neq \emptyset$, and $\mathcal{A} \models \theta_i(\bar{a})$, then*

$$\big|\mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - d_i\big| < \varepsilon.$$

Proof. Fix $i \in \{1, \dots, t\}$ and let $\varepsilon > 0$. By the semantics of PLA^* it suffices to show that there is $\delta > 0$ such that if $\mathcal{A}_1 \in \mathbf{Y}_{n_1}^{\delta}$, $\mathcal{A}_2 \in \mathbf{Y}_{n_2}^{\delta}$, $\bar{a}_1 \in [n_1]^{|\bar{x}|}$, $\bar{a}_2 \in [n_2]^{|\bar{x}|}$, $\chi(\bar{a}_1, \mathcal{A}_1) \neq \emptyset$, $\chi(\bar{a}_2, \mathcal{A}_2) \neq \emptyset$, $\mathcal{A}_1 \models \theta_i(\bar{a}_1)$, $\mathcal{A}_2 \models \theta_i(\bar{a}_2)$, and, for $k = 1, 2$,

$$\bar{p}_k = \big(\mathcal{A}_k(\psi(\bar{a}_k, \bar{b})) : \bar{b} \in [n_k]^{|\bar{y}|} \text{ and } \mathcal{A} \models \chi(\bar{a}, \bar{b})\big),$$

then $|F(\bar{p}_1) - F(\bar{p}_2)| < \varepsilon$.

Let, according to Definition 5.7, $(d_{i,1}, \dots, d_{i,s_i}, \beta_{i,1}, \dots, \beta_{i,s_i})$ be the sequence of $\bar{y}$-frequency parameters of ψ relative to χ and θ_i. By Lemma 5.8, $d_{i,1}, \dots, d_{i,s_i}$ is an enumeration of $\{c_1, \dots, c_s\}$ without repetition, for $k = 1, 2$, $\text{rng}(\bar{p}_k) \subseteq \{c_1, \dots, c_s\}$, and for all $j = 1, \dots, s_i$ the number of coordinates in $\bar{p}_k$ which are equal to $d_{i,j}$ is between $(\beta_{i,j} - s\delta)|\chi(\bar{a}, \mathcal{A}_k)|$ and $(\beta_{i,j} + s\delta)|\chi(\bar{a}, \mathcal{A}_k)|$. It follows that $\mu_1^u(\bar{p}_1, \bar{p}_2) < C\delta$ where the constant C depends only on s_i.

We assume that F is up-continuous with respect to every sequence of $\bar{y}$-frequency parameters of ψ relative to χ, so in particular F is up-continuous with respect to $(d_{i,1}, \dots, d_{i,s_i}, \beta_{i,1}, \dots, \beta_{i,s_i})$. It follows from part (1) of the definition of up-continuity that if $\delta > 0$ is chosen small enough then $|F(\bar{p}_1) - F(\bar{p}_2)| < \varepsilon$. $\square$

Lemma 5.14 *There is an L_0-basic formula $\psi'(\bar{x})$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all sufficiently large n, all $\mathcal{A} \in \mathbf{Y}_n^\delta$, and all $\bar{a} \in (D_n)^{|\bar{x}|}$,*

$$\big|\mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - \mathcal{A}\big(\psi'(\bar{a})\big)\big| < \varepsilon.$$

Proof. Let $d_1, \dots, d_t \in [0, 1]$ be as in Lemma 5.13. By (c) above, for all $\mathcal{A} \in \mathbf{Y}_n^\delta$,

$$\mathcal{A} \models \forall \bar{x}\Big(\neg \exists \bar{y} \chi(\bar{x}, \bar{y}) \leftrightarrow \big(\bigvee_{i=1}^{k} \chi_i(\bar{x})\big)\Big).$$

Let $\psi'(\bar{x})$ be the formula $\bigwedge_{i=1}^{s}(\theta_i(\bar{x}) \to d_i) \wedge \bigwedge_{j=1}^{k}(\chi_j(\bar{x}) \to 0)$, so ψ' is an L_0-basic formula. We now verify that the claim of the lemma holds with this choice of ψ'. First note that if $\bar{a} \in (D_n)^{|\bar{x}|}$, $\mathcal{A} \in \mathbf{Y}_n^\delta$ and $\chi(\bar{a}, \mathcal{A}) = \emptyset$, then, for some j, $\mathcal{A}(\chi_j(\bar{a})) = 1$, so $\mathcal{A}(\chi_j(\bar{a}) \to 0) = 0$ for some j and hence

$$\mathcal{A}(\psi'(\bar{a})) = 0 = \mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big).$$

Now suppose that $\bar{a} \in (D_n)^{|\bar{x}|}$, $\mathcal{A} \in \mathbf{Y}_n^\delta$ and $\chi(\bar{a}, \mathcal{A}) \neq \emptyset$. Hence $\mathcal{A} \not\models \chi_i(\bar{a})$ for all $i = 1, \dots, k$ and thus $\mathcal{A}\big(\bigwedge_{j=1}^{k}(\chi_j(\bar{x}) \to 0)\big) = 1$. By (a) and (b) above there is a unique i such that $\mathcal{A} \models \theta_i(\bar{a})$. Hence $\mathcal{A}\big(\theta_i(\bar{a}) \to d_i\big) = d_i$ and $\mathcal{A}\big(\theta_{i'}(\bar{a}) \to d_{i'}\big) = 1$ for all $i' \neq i$. It follows that $\mathcal{A}(\psi'(\bar{a})) = d_i$. By Lemma 5.13, if $\delta > 0$ is sufficiently small we get

$$\big|\mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - d_i\big| < \varepsilon.$$

and the lemma now follows. $\square$

Proposition 5.15 *Suppose that $\varphi(\bar{x}, \bar{y}) \in PLA^*(\sigma)$ and that $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are asymptotically equivalent. Then $F(\varphi(\bar{x}, \bar{y}) : \bar{y} : \chi(\bar{x}, \bar{y}))$ is asymptotically equivalent to an L_0-basic formula.*

Proof. Suppose that $\varphi(\bar{x}, \bar{y}) \in PLA^*(\sigma)$ and that $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are asymptotically equivalent. By Lemma 5.14, there is an L_0-basic formula $\psi'(\bar{x})$ such that, for all $\varepsilon > 0$, if $\delta > 0$ is small enough then for all sufficiently large n, all $\mathcal{A} \in \mathbf{Y}_n^\delta$, and all $\bar{a} \in (D_n)^{|\bar{x}|}$,

$$\big|\mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - \mathcal{A}\big(\psi'(\bar{a})\big)\big| < \varepsilon/2. \tag{5.1}$$

For any $\delta > 0$ define

$$\mathbf{X}_n^\delta = \big\{\mathcal{A} \in \mathbf{W}_n : \text{ for all } \bar{a} \in (D_n)^{|\bar{x}|} \text{ and all } \bar{b} \in (D_n)^{|\bar{y}|}, \big|\mathcal{A}(\varphi(\bar{a}, \bar{b})) - \mathcal{A}(\psi(\bar{a}, \bar{b}))\big| < \delta\big\}.$$

Since $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are asymptotically equivalent we have $\lim_{n\to\infty} \mathbb{P}_n(\mathbf{X}_n^\delta) = 1$. Hence $\lim_{n\to\infty} \mathbb{P}_n(\mathbf{X}_n^\delta \cap \mathbf{Y}_n^\delta) = 1$. Therefore it suffices to show that if $\delta > 0$ is small enough, $\mathcal{A} \in \mathbf{X}_n^\delta \cap \mathbf{Y}_n^\delta$ and $\bar{a} \in (D_n)^{|\bar{x}|}$, then

$$\big|\mathcal{A}\big(F\big(\varphi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - \mathcal{A}\big(\psi'(\bar{a})\big)\big| < \varepsilon. \tag{5.2}$$

The inequality (5.2) is a consequence of (5.1) and the following inequality:

$$\big|\mathcal{A}\big(F\big(\varphi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) - \mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) < \varepsilon/2. \tag{5.3}$$

So we need to prove that if $\delta > 0$ is sufficiently small and n sufficiently large, then (5.3) holds for all $\mathcal{A} \in \mathbf{X}_n^\delta \cap \mathbf{Y}_n^\delta$ and all $\bar{a} \in (D_n)^{|\bar{x}|}$.

Let $\mathcal{A} \in \mathbf{X}_n^\delta \cap \mathbf{Y}_n^\delta$ and $\bar{a} \in (D_n)^{|\bar{x}|}$. First suppose that $\chi(\bar{a}, \mathcal{A}) = \emptyset$. Then

$$\mathcal{A}\big(F\big(\varphi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big) \;=\; 0 \;=\; \mathcal{A}\big(F\big(\psi(\bar{a}, \bar{y}) : \bar{y} : \chi(\bar{a}, \bar{y})\big)\big).$$

Now suppose that $\chi(\bar{a}, \mathcal{A}) \neq \emptyset$. There is (by (a) and (b) above) a unique i such that $\mathcal{A} \models \theta_i(\bar{a})$ and the following sequences are nonempty:

$$\bar{p} = \big(\mathcal{A}\big(\varphi(\bar{a}, \bar{b})\big) : \bar{b} \in (D_n)^{|\bar{y}|} \text{ and } \mathcal{A} \models \chi(\bar{a}, \bar{b})\big),$$
$$\bar{q} = \big(\mathcal{A}\big(\psi(\bar{a}, \bar{b})\big) : \bar{b} \in (D_n)^{|\bar{y}|} \text{ and } \mathcal{A} \models \chi(\bar{a}, \bar{b})\big).$$

From Lemma 5.8 it follows that if $(c'_1, \ldots, c'_m, \alpha_1, \ldots, \alpha_m)$ is the sequence of $\bar{y}$-frequency parameters of $\psi(\bar{x}, \bar{y})$ relative to $\chi(\bar{x}, \bar{y})$ and $\theta_i(\bar{x})$, then $\mathrm{rng}(\bar{q}) \subseteq \{c'_1, \ldots, c'_m\}$ and the number of coordinates in $\bar{q}$ which are equal to c'_j is between $(\alpha_j - s\delta)|\chi(\bar{a}, \mathcal{A})|$ and $(\alpha_j + s\delta)|\chi(\bar{a}, \mathcal{A})|$. Since $\mathcal{A} \in \mathbf{X}_n^\delta$ it follows from the definition of μ_∞^o that $\mu_\infty^o(\bar{p}, \bar{q}) < \delta$.

Since F is up-continuous with respect to every sequence of $\bar{y}$-frequency parameters of $\psi(\bar{x}, \bar{y})$ relative to $\chi(\bar{x}, \bar{y})$, it is, in particular, up-continuous with respect to the sequence $(c'_1, \ldots, c'_m, \alpha_1, \ldots, \alpha_m)$. It now follows from condition (2) of the definition of up-continuity with respect to the sequence of parameters $(c'_1, \ldots, c'_m, \alpha_1, \ldots, \alpha_m)$ that if $\delta > 0$ is sufficiently small, then $|F(\bar{p}) - F(\bar{q})| < \varepsilon/2$. This together with the definitions of $\bar{p}$ and $\bar{q}$ and the semantics of PLA^* imply that (5.3) holds, so the proof is completed. □

6 Conclusion

We have considered the quite expressive logic PLA^* with truth values in the unit interval and which uses aggregation functions instead of (generalized) quantifiers. We have shown that generalized quantifiers in the sense of Mostowski [21, 22] can be expressed by appropriate aggregation functions, so PLA^* subsumes (on finite structures) first-order logic extendend by such generalized quantifiers.

We have also studied continuity properties of aggregation functions and how the continuity properties of an aggregation function F influence whether a PLA^*-formula that uses F is asymptotically equivalent to a simpler formula without occurences of F, where 'asymptotical equivalence' generalizes 'almost sure equivalence' to the context of continuous-valued logics. The same approach has been used earlier in [17, 18], but in more specific contexts. This study has identified conditions which make sense in wider contexts than that of [17, 18] and the main result (Theorem 5.9) may therefore be useful in a variety of situations, as already exemplified by [15, 16] which use the main result of this study.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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