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Divided by A Common Language: A Response to White Brahmia and Thompson

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Abstract
This summary provides commentary on the paper by White Brahmia and Thompson (this issue). We respond to issues raised by the paper and amplify several of them. In particular we highlight elements of practice in physics instruction that are not well aligned with what students encounter in their calculus courses and respond to the value, and limitations, of the proposed framework using the Fundamental Theory of Calculus as a unifying theme. Finally, we repeat the call for increased attention to interdisciplinary work.

Introduction

This issue is an inspired and inspiring collection of work that seeks to cross the disciplinary boundaries that so often confine our work as instructors and researchers. This brief commentary seeks to respond primarily to the work by White Brahmia and Thompson (this issue, WB&T hereinafter) on the Fundamental Theorem of Calculus as a unifying framework across calculus and introductory physics, with some comments at the end as to some of the other disciplines addressed by other papers.

As a disclaimer, I offer first of all a brief statement of positionality. First, I have collaborated with White Brahmia and her team on several projects. The subset of physics education research (henceforth PER) that focuses on mathematics is not large, a subfield of a subfield, and some overlap is to be expected. While every effort is made to be objective, readers might consider whether there is a blind spot there. Second, for roughly two decades my primary scholarly focus has been on intermediate-level courses for physics majors in the United States, including thermal and statistical physics. More recently this has included the mathematical methods course that is

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offered to prepare physics majors for the mathematical challenges of their upper-level theory courses. This course can foreground many aspects of students' reasoning with calculus that have impacts on their later experiences in physics courses, and that will be part of the lens applied here. Third, what I find most compelling might certainly be idiosyncratic and this piece is too short to respond to all of the issues brought up by WB&T.

WB&T build a case for a framework that positions the Fundamental Theorem of Calculus as a central element in a common approach to the definite integral in physics and mathematics. They articulate the challenges that this framework can address and illustrate some theoretical and empirical work that supports this approach. This commentary highlights areas of agreement with their assessment of the problems and seek to offer some amplification of several points. Later sections briefly describe some challenges to their proposal and some additional perspectives.

The Problems Addressed

It should be the easiest and least controversial connection in academia: learning physics requires calculus and calculus classes should prepare students for better understanding of physics. These two fields are deeply intertwined historically. And yet, the evidence suggests that things are not so rosy. Biza and colleagues (2022) described the roles of calculus as a *filter* or *scaffold*: in the former case, calculus is used to determine who is qualified for later coursework; in the latter, it is useful as supporting knowledge to enhance success in later STEM coursework. Ideally, calculus would serve as a scaffold: learning physics would build upon knowledge of calculus, and conversely, it would as well deepen and broaden a students' understanding of calculus. In practice, calculus too often serves instead as a barrier.

For practitioners, mathematics in physics instruction is seemingly driven by a pragmatic rather than theoretical take. That is to say, the instructional practices of many physics instructors and choices made by textbook authors reflect an uneasy tension between intellectual rigor, prerequisite structures, and the reality of student and instructor experiences. From a high-level view they can appear to be a piecemeal and highly localized set of strategies that change from chapter to chapter even within a single textbook. Instructional approaches can reflect instructors' own beliefs about the nature of learning and their students' backgrounds. As a discipline, physics shows little sign of having reflected upon the structure and practice of mathematics—and calculus in particular—in the teaching of introductory physics, or the ways in which the mathematics encountered in physics is complemented or contradicted by prior instruction in mathematics departments.

Moreover, the mathematics that is needed in physics courses is not necessarily closely matched to the presence or sequencing of topics in mathematics courses. Prerequisite structures are constrained because the courses offered in mathematics departments are not always a good or timely match for physics, particularly when university requirements are considered. As a result, there is a broad and open question of when and where students are supposed to learn the mathematical practices and quantitative reasoning skills that they will use in physics courses. This tension is visible in nearly every physics course; many textbooks and instructors have taken it upon themselves to have a full introductory chapter of the necessary mathematical tools without regard to context, so as to lay the groundwork for the use of these tools in the chapters to come.I In introductory texts this includes a chapter dedicated to vectors (Knight, 2016; Resnick et al., 2014). For more advanced texts this might include a full chapter dedicated to the most relevant math, such as vector calculus in electricity and magnetism (Griffiths, 2013) or eigentheory and linear algebra in quantum mechanics (Zettili, 2001).

Added to this are the sociopolitical factors that go hand-in-hand with access to higher education, as illustrated by recent work (White Brahmia & Cochran, 2025). The mathematical prerequisites of physics instruction impose a disproportionate burden on students whose high schools do not offer calculus.

Mathematics in physics, and in particular calculus in physics, has been the subject of an ever-growing body of literature. Studies have characterized how students approach physics problems, how experts perceive mathematics in physics, and everything in between. WB&T motivate their proposal by a thorough investigation of the literature on students' learning and bridge this with an experts'view of the mathematical underpinnings of key elements of the physics curriculum. For the sake of this commentary, there are a sequence of key insights to highlight that together paint a picture:

> *Experts perceive differences in physics and mathematics, even as they are intertwined.*

There are multiple points of view that point to the distinctions in mathematical practice between mathematicians and physicists. Physicists tend to load meaning on symbols and equations, rather than considering decontextualized mathematical relationships (Redish & Kuo, 2015). An alternative approach is to highlight the distinction between calculus and its focus on functions and physics with its emphasis on quantities. Formal studies using frameworks such as the Anthropological Theory of the Didactic (ATD; Bosch et al., 2019) have highlighted differences in practice among instructors (e.g., Hitier & González-Martín, 2022).

However, even as experts perceive differences, it is important to note that when experts or even students are reasoning with mathematics in physics, it is difficult to separate the two. Several reports have highlighted the close connections between physics and mathematics in practice, noting their 'inextricability' and choosing to describe blended reasoning that is no longer purely physics nor mathematics, but rather a blend (Schermerhorn & Thompson, 2023; Serbin & Wawro, 2022; van den Eynde et al., 2022; Zimmerman et al., 2020).

Students perceive differences in the mathematics they encounter in physics and what they have learned in calculus.

In addition to distinctions that are visible from an expert's perspective, there have been studies that highlight students' perceptions of mathematics in physics. At the first (2023) and second (2025) conferences on *The Learning and Teaching of Calculus Across Disciplines*, several reports have addressed this important question. Hitier and González-Martín (2023) reported that "in mechanics, ready-to-use formulas are provided, so students can solve these problems without using knowledge from derivatives nor thinking in terms of covariation" (p. 116). Noah-Sella et al. (2023) described several differences in the ways that students perceive related problems in physics and calculus, including the perception that physics problems require 'more thought' than calculus and associated rate of change with physics more than calculus (p. 139). Taylor and Loverude (2023) described students' perceiving the mathematical expectations as so different that one math major stated that "there's not really math to it…just plug in your numbers, get your answer, go home" (p. 171). Another student stated, "I forget about math when I go to physics, I forget about physics when I go to math" (p. 171).

The building of skills in reasoning with quantities and making meaning is often absent from calculus instruction.

There is an extensive and growing body of research that shows the ways in which students find it challenging to use calculus to model physical systems. This case is ably made by WB&T, but is worth reinforcing. As the situation currently stands, a physics instructor may feel that she has limited tools to address the situation. Most physics instructors are not familiar with instructional trends in calculus and some would have difficulty identifying which content is covered in pre-requisite courses. From the point of view of physics, elements in particular need of attention include units and the use of mathematics in modeling. While calculus instructors might agree, these aspects of mathematical meaning might well be absent from their own training. Conversations across disciplines are necessary.

Prerequisite structures for physics are not well-aligned.

Analyses of instructional practices (Robles et al., 2025) have suggested that there are numerous examples in the introductory physics sequence of misalignment with the co- and prerequisites at many institutions. In many US universities the first physics course has Calculus I as a co-requisite. Derivatives and integrals might appear in the study of kinematics in the first week of class, while students in the co-requisite Calculus I course might not have encountered derivatives (Hitier & González-Martín, 2023). Work is encountered in the middle of the mechanics portion of the course. Work is more generally described as a path integral, or at least as a dot product, and some instructors and textbooks will emphasize this, despite the fact that these ideas will be encountered much later in the calculus curriculum.

In the second course, electricity and magnetism, students might encounter integrals of vector quantities in three-dimensional situations by the second week, while students in a co-requisite Calculus II are likely studying single-variable integration. Soon after in physics, students could encounter integrals that include dot and cross products (in electric potential and magnetic fields, respectively), and possibly even partial derivatives and the gradient (in electric potential). An early chapter of introductory calculus-based electricity and magnetism covers Gauss' law, essentially the divergence theorem. Stoke's theorem appears later, in the form of Ampere's law. The third course of physics generally covers some combination of waves, optics, and modern physics, but is not necessarily taken by all students. For the students that do take this course, there are very few instances of differentiation or integration.

Some authors have suggested changing course sequences, providing additional supporting structures including co-requisite courses (White Brahmia & Cochran, 2025), or even removing the calculus prerequisite entirely (Capece & Richards, 2024). While each has its merits, these reforms have yet to take hold on a broader scale. Admittedly, many students are ahead of the required co- or pre-requisites, often due to having calculus in high school. As noted above, that means that the burden falls disproportionately on students who are already disadvantaged.

Calculus is more absent from introductory physics than many recognize.

The final statement is perhaps the natural consequence of all of the above. In a project on mathematics in courses for physics majors, physics majors were surveyed as to their experiences with mathematics in general, and calculus in particular, in introductory physics coursework. Many physics majors described an introductory physics experience that was nearly devoid of calculus (Loverude, 2017). One described the course sequence as 'just barely calculus based.' Others indicated that the calculus was used briefly to illustrate concepts or derive equations but for little else:

> *"The calculus that we used was mentioned once, and then the students were expected to just memorize the equations of motion instead of deriving them …"* (p. 17).

A follow-up study is ongoing to determine the extent to which this is broadly true of introductory physics and better characterize instructional practices relating to calculus and quantitative reasoning. One feature of this has been an analysis of leading US introductory physics textbooks. Following related work by other authors, the textbooks were examined and skills and concepts from the Calculus Concept Framework (Sofronas et al., 2011) were coded.

In the preliminary analysis (Robles et al., 2025) including two textbooks, the skills coded most frequently concerned algebra and trigonometry, each of which were far more common than purely calculus codes like integration and derivatives. Even in the most calculus intensive parts of the course, Manipulating Algebraic Expressions was the single most common code, over 30% of all codes. Trigonometric Manipulations was second most frequent, representing approximately 14% of all codes, though it was more present in some chapters than others.

Whereas derivatives were more prevalent than integrals in the mechanics portion of the class, this trend was reversed in electricity and magnetism, with integrals being coded approximately twice as frequently as derivatives. In addition, for integrals and derivatives, the codes for concepts were an order of magnitude more common than the codes for corresponding skills. For example, the concept of derivative was coded more frequently than the skill of computing a derivative. At least in the introductory physics sequence, students are generally not evaluating complicated derivatives or using sophisticated techniques of integration.

At least with respect to introductory mechanics, these data are supported by results to interviews with instructors (to be published). One instructor stated:

> *"But up to now they don't … do the calculus. I do the calculus to derive things for them. In [chapter on work], they're going to start doing that. Yeah, but still not excessive."*

Other instructors estimated the frequency of students encountering calculus:

> *"in a semester, teaching twice a week, ... that's 30 times that we meet together, maybe calculus rears its face in five of those or six of those 30 lectures. I think that's fair."*

> *"I have, like, maybe some, like, 15 weeks of slides . . . there [are] derivatives in at least 5 of them. 5, 6 of them, so it's not…quite up to half, but it's close."*

Again, these results point toward a missed opportunity. While calculus could serve as a valuable scaffold for student learning in physics, in practice the factors above mean that it often does not. It should also be noted that there is a tension between 'calculus' and its underlying conceptual foundations. While there might be no instances of derivatives in a lecture or section of a textbook, that does not mean that the concept of rate of change is absent. The ideas of rate of change and accumulation are sufficiently fundamental that they are even frequently encountered in the algebra-based version of these courses that do not have a calculus prerequisite. For many students, the formalism of calculus, which is heavily emphasized in calculus courses, is likely to be less important than fundamental ideas of rate of change and accumulation, which often are not as heavily emphasized.

Responses To the Fundamental Theorem of Calculus as a Unifying Framework

Given the challenges noted above, WB&T propose a number of aspects of increased coherence between calculus instruction and physics. The unifying framework from the title is most prominent among these but there are several other elements of the argument that are worthy of further highlighting and amplifying.

First, this is lovely and elegant and intellectually pleasing to an expert. While many physics instructors likely will not be surprised by this characterization, it is nevertheless an exceedingly clear and clean framing of the role of integration that crosses disciplinary boundaries. It has intellectual coherence, both within the context of the discipline of physics, and from the point of view of those who study the teaching and learning of both disciplines.

a. The value of quantities and covariational reasoning

As much as PER has turned its attention to mathematical reasoning, it has often done so without effective frameworks to describe reasoning processes. Mathematics education has provided work that can fill in these gaps for PER. Reasoning with quantities and covariational reasoning are two streams of work that are extremely relevant to physics instruction, but there has been too little uptake in either physics instruction or PER.

The notion of quantity is hardly a new one in physics or physics teaching. In 1917, Tolman published 'The measurable quantities in physics,' which placed the notion of quantity, including extensive and intensive quantities, on a firm intellectual footing. And yet, until recently this work had not percolated into conversations on instruction or student learning (White Brahmia, 2019).

Covariational reasoning (Carlson et al., 2002) and later multivariational reasoning (Jones, 2022) are frameworks from mathematics education that are relevant to physics instruction and PER alike. Covariational reasoning describes the coordination of two quantities and the ways that variations in one coincide with variations in the other; multivariation extends this to additional variables. Despite the obvious relevance of these constructs in physics classrooms, there are only a handful of papers in PER that have taken up these constructs (e.g., Akinyemi et al., 2025).

b. The value of a conceptual unifying theme

There is considerable value, for instructors, for researchers, and particularly for students, of a unifying framework that can provide a larger structure to the topics of introductory physics. PER studies have demonstrated that students perceive introductory physics to be a disconnected collection of equations, problem-solving algorithms, and topics (Redish et al., 1998). Instructors and the authors of textbooks might see the underlying narrative through-line as the mechanics portion of the course progresses from kinematics to dynamics, work and energy, momentum, and then to rotational motion. For students, however, the course is perceived as a disparate set of topics, quantities, and problem-solving approaches. This tendency is only enhanced when students reach electricity and magnetism, with a similarly large list of new ideas but the added complications of increased abstraction and mathematical complexity.

c. Additional ideas for amplification

WB&T raise a great many ideas. In this section several supporting points are offered to amplify points raised in their article, particularly the recommendations in Sect. 6, that are particularly resonant for the cross-disciplinary conversations on physics and calculus.

Expressions in physics courses are complex with many symbols, but in practice students often are only asked to reason using simpler functions.

WB&T highlight the fact that the actual functions that students encounter in physics are simpler than calculus practice might support. Also, students encounter many complex integrals in calculus, particularly as they are studying the various techniques of integration, but the functions that students actually need to integrate in most of their physics courses are far simpler than their calculus courses might assume.

As a supporting example, consider the many contexts in electricity and magnetism in which students encounter integrals of vector quantities in multiple dimensions. While the expressions that students encounter can be algebraically complex, as a practical matter the integrals that are evaluated are often quite simple. For example, Resnick et al. (2014) derive the electric field of a charged ring (see Fig. 1). Despite the complexity of the expression for the electric field, the integral that is ultimately evaluated is a definite integral of a single variable: $\int_0^{2\pi R} ds$. All the complexity

Integrating. Because we must sum a huge number of these components, each small, we set up an integral that moves along the ring, from element to element, from a starting point (call it $s = 0$) through the full circumference ($s = 2\pi R$). Only the quantity s varies as we go through the elements; the other symbols in Eq. 22-14 remain the same, so we move them outside the integral. We find

$$E = \int dE \cos\theta = \frac{z\lambda}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}} \int_0^{2\pi R} ds$$
$$= \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}}. \qquad (22\text{-}15)$$

Fig. 1 The integral begins with a difficult sum of vector components in three dimensions, but the integral evaluated is of a single variable (Resnick et al., 2014)

of the situation is in modeling the system, reasoning with the physics quantities of charge and electric field, and then setting up and simplifying the integral. The integral evaluation is, in principle, trivial (though this may not be so for students).

Differentials are omnipresent in physics instruction

Another point of departure, noted by WB&T and others, is the use of, and interpretation of, differentials. There has been a great deal written about the role of differentials in recent years. This includes excellent documentation of the extent to which their usage is different in different disciplines (Artigue et al., 1990) as well as calls to rethink their absence in calculus to better scaffold student learning (Dray & Manogue, 2010; Ely, 2021).

It should be noted that very few physics instructors are even aware of this debate. Differentials are part of undergraduate physics instructional practice from very early on, and their use is particularly pronounced in some upper-level courses including thermal physics. Throughout physics instruction, students will see textbooks and instructors going through the process of 'setting up' integrals. In many cases these will begin with the explicit statement of a small something that is to be summed (which might even be a vector): dx, dv, $d\vec{E}$. It should be noted, however, that differentials can also be encountered in the derivation of various expressions and physical laws, without any integration intended. For example, in the derivation of acceleration in circular motion in one textbook (Knight, 2016, p.97) students see a figure that includes four different differential quantities: ds, dt, $d\vec{v}$, and $d\theta$ (Fig. 2).

In another example, differentials are used to determine the current in a single loop circuit (Fig. 3):

It is possible that mathematicians will be troubled by these examples as well as the lack of distinction between differential and infinitesimal in these discussions, but physics frequently uses such reasoning and makes little distinction between the two terms (see also, in this special issue, Faulkner et al., this issue). And yet, the question of 'how small' that is inherent to writing a quantity as d[something] is not merely an abstract discussion in physics. In many physics applications the mathematical notion of approaching zero clashes with the physical reality that is being modeled.

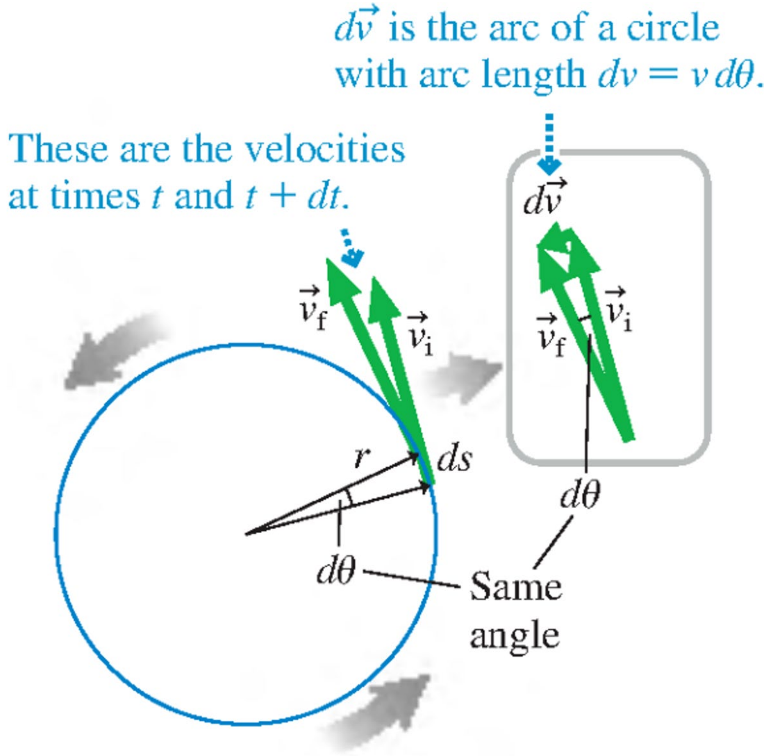
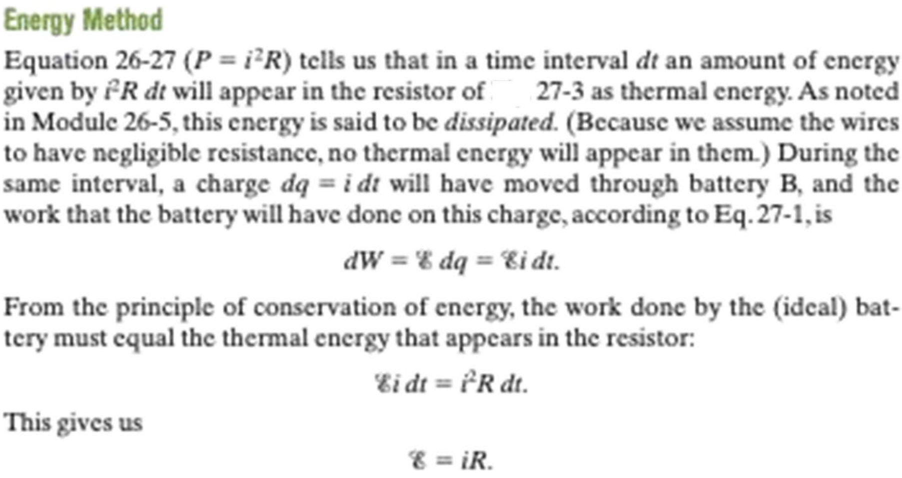


Fig. 2 A figure from a textbook presenting derivation with multiple differentials (Knight, 2016)



Energy Method

Equation 26-27 ($P = i^2R$) tells us that in a time interval dt an amount of energy given by $i^2R\,dt$ will appear in the resistor of 27-3 as thermal energy. As noted in Module 26-5, this energy is said to be *dissipated*. (Because we assume the wires to have negligible resistance, no thermal energy will appear in them.) During the same interval, a charge $dq = i\,dt$ will have moved through battery B, and the work that the battery will have done on this charge, according to Eq. 27-1, is

$$dW = \mathscr{E}\,dq = \mathscr{E}i\,dt.$$

From the principle of conservation of energy, the work done by the (ideal) battery must equal the thermal energy that appears in the resistor:

$$\mathscr{E}i\,dt = i^2R\,dt.$$

This gives us

$$\mathscr{E} = iR.$$

Fig. 3 Determining current in a single loop circuit using differentials (Resnick et al., 2014)

To illustrate these tensions, consider this excerpt from a textbook in thermal physics (Zemansky & Dittman, 1997, p. 33):

> *If the change of, say, V is very small in comparison to V and very large in comparison with the space occupied by a few molecules, then this change of V may be written as a differential dV.*
> ...
> *Every infinitesimal in thermodynamics must satisfy the requirement that it represents a change quantity which is small with respect to the quantity itself and large in comparison with the effect produced by the behavior of a few molecules."*

Imagining a limit as a quantity that goes to zero is simply not an option in some fields of physics.

Function notation can be a valuable scaffold but is underused in physics

As WB&T highlight, there are potential scaffolds that might support student reasoning with quantities in physics. One that is underutilized in physics is function notation. While it might seem trivial to experts, notation can be more of an obstacle to students than instructors might expect. In one study, many students struggled to reconcile the notation in physics with that in calculus while interpreting kinematics graphs (Taylor & Loverude, 2023). One extended passage from an interview illustrates this challenge; the student (a math major) wrote $f(t)=x$ then scribbled it out (p. 171):

> S6: So slope, I don't know … let's call this $f(x)$. So $f'(0)$ would be 3, right? If x is t, I don't know. If x is t. I don't know what I'm doing.
> ...
> S6: Well, I was originally going to try to make this look mathematical with a whole function and everything, but if I had $f(x)$ I guess I could do that. No that doesn't make sense, $f(t)$ is equal to x, I guess. Right, cuz at some time you get your value, so it's really $f'(t)$.
> ...
> S6: In the math books, it's always in terms of x and y, so it's either gonna be x or x' or x'' … and then in physics, there's like, well, no, this isn't f of x', it's velocity … No, it's f of x'! So, I guess it's the same thing, it's just written differently.

Results of the textbook analysis showed use of function notation in the first sections of kinematics, but that the presence of function notation later in the course ranges from sporadic to nonexistent. Even in the context of kinematics in one dimension, function notation is often omitted unless as a shorthand reference to a graph (e.g., an $a(t)$ graph displays acceleration as a function of time). But the scaffolding of communicating which of the many symbols in an expression are parameters and which might be considered an independent variable is largely absent. Early on, the much-discussed 'kinematics equations' are derived; in many treatments there are five and four of the five are implicitly functions of time. The remaining equation is often written as $(v_f)^2 = (v_i)^2 + 2a(x_f - x_i)$; the absence of explicit time dependence can make this is useful in problem solving (and in later reasoning with work and kinetic energy). However, without the scaffold, a student might well wonder whether this is to be considered a function, and, if so, what should be considered the independent variable.

Later in the course, there are even more complex expressions, including functions with multiple independent variables, and almost no use of function notation to signal key dependences. When students do encounter function notation in physics texts, it can be as a means of calling attention to otherwise invisible functional dependence. For example, in one text (Resnick et al., 2014) the energy of a magnetic dipole in an external magnetic field is given as a function of theta, though no theta is explicitly present in the problem: $U(\theta) = -\vec{\mu} \bullet \vec{B}$.

Physics students would benefit from attention to units in calculus classes.

As WB&T note, attendant with the importance that physics attaches to quantities is the idea of units. This is a recurring tension between stripped down expressions that contain only the functional dependence vs. the symbol-rich expressions encountered in physics, and one element of that tension is the units of quantities. For example, in one physics text (Knight, 2016), an example gives a particle's position as $x(t) = \left(-t^3 + 3t\right) m$. Many physics instructors would be troubled because the coefficients of the t and t^3 terms cannot have the same units. The units are an important sensemaking tool for physics students.

While an expert might look at the expression above and implicitly append the appropriate units such that the quantity in parentheses ends up dimensionless, it is possible for students to move on to upper-level physics courses and not have considered the issue. In a study of physics majors in a math methods course (Loverude, 2015), student success in determining the units associated with parameters and constants in a physics problem varied from over 90% in the simplest cases of position and velocity to under 20% in tasks on the course final involving differential equations. Student responses to tasks with derivatives or integrals suggested that many students did not know what the effects of, say, a derivative *d/dx* would be on the units of a quantity, assuming that *x* represents a position. The lack of focus on quantities with units can also impact students in topics they encounter in later physics courses, from Taylor series (Loverude, 2024) to linear algebra (Her & Loverude, 2020). In each of these cases, students accustomed to reasoning with pure numbers struggled with tasks in physics applications where the units of the quantity might matter.

Challenges, and Alternative Perspectives

The previous section highlighted some aspects of the paper by WB&T, and amplified several of its claims. This section will offer some counterarguments and cautions, or at least alternative perspectives to consider. These should not be interpreted as attempts to contradict or invalidate the work by WB&T, but to probe and explore potential applications of the model.

a. Applications beyond classical mechanics.

The framework as it is given is articulated in the context of classical mechanics, to which it is extremely well matched. A natural question is whether, and to what extent, the framework can be useful in later parts of introductory physics as well as in subsequent courses.

The example of Gauss' law is illustrative. Three aspects of Gauss' law show the value of a calculus sequence that emphasizes meaning and modeling through quantities but also illustrate the gaps between practice in physics and mathematics.

First, the most general form of Gauss' law, as an application of the divergence theorem, involves not only the flux integral described in the text, but also a second integral for the charge enclosed in a region. The latter is most generally a volume integral for the enclosed region but is usually simplified in introductory physics to be the enclosed charge. Students who go on in physics, or who have a particularly ambitious introductory course, may be asked to compute or at least consider this integral as well. Many instructors and textbooks, however, acknowledge that these integrals are both extremely challenging for students and beyond the scope of the mathematics that most students have encountered in the calculus sequence. If an actual flux integral is encountered, it is often reduced to the form $\int dA$ and replaced with the area of the appropriate surface. While the flux here is in principle a vector calculus idea involving three dimensions, in practice the symmetry of the situation is used to reduce the calculation to multiplication.

Second, the conceptual emphasis in instruction on Gauss' law at the introductory level is not well matched to calculus instruction. PER studies of student reasoning with Gauss' law have focused on the symmetry arguments, confusion between flux and electric field, the dot product and scalar nature of flux, and the relationship between charge and field, including the principle of superposition (Campos et al., 2023; Li & Singh, 2018). They seldom considered either of the integrals or the notion of accumulation. This in part reflects prerequisite structures; at many institutions students may be co-enrolled in the second calculus course at this time and might not have encountered multivariable calculus or the vector calculus theorems in any detail. However, it may also reflect a disconnect between PER and other fields. If students do actually consider integration in Gauss' law, and apply the dominant conception of an integral as area under the curve rather than accumulation, this reasoning will badly fail them (Jones, 2015). It is neither productive nor customary to interpret either of the integrals in Gauss' law as the area under a curve, even though area is central to the concept of flux. It is not surprising that this can be confusing. A former research student, when analyzing data on this task, said that he now understood why he had never 'got' Gauss' law, he was stuck trying to figure out which area and which curve.

Third, since Gauss' law is an application of the divergence theorem, it can be considered an extension of the Fundamental Theorem of Calculus to additional dimensions. And yet in the context of instruction, this aspect of the relationship is at most secondary to the many physics aspects of the problem. Assigning the elements of the framework to Gauss' law is a challenge.

b. Integrals that do not fit.

While many of the critical concepts in classical mechanics lend themselves to interpretation through the lens proposed by WB&T, there will be other integrals that students encounter that are not easily mapped to this paradigm. That can be because they are not generally interpreted in terms of a rate times a *d*[something], or because the total change is not easily interpreted.

In introductory mechanics, for example, instructors or students might integrate to determine the center of mass or moment of inertia of a rigid body. The form of the center of mass integral in one dimension, for instance for the center of mass of a rod, is $\frac{1}{M}\int\ x\,dm$ and the mass term is generally then converted to $\rho\,dx$ in terms of the (possibly variable) mass density of the rod, giving $\frac{1}{M}\int\ \rho\,x\,dx$. The moment of inertia is of the form $I=\int\ x^2\,dm$ or $\int\ \rho\,x^2\,dx$. Many introductory courses omit these integrals, but they are found in introductory textbooks. Interpreting these as a sum is productive, but the interpretation is not generally framed as a change. It would be extremely unusual for a physics instructor to speak of, say, the position as the rate of change of center of mass with respect to mass, or the square of the position as $\frac{dI}{dm}$.

Other topics will challenge this interpretation in other ways. In thermal physics, students encounter work integrals that are path-dependent (see section below for more). The change in internal energy in a system is not wholly attributed to the work (heat transfer is also present), and integrals that start and end at the same location (e.g., in a thermodynamic cycle for a heat engine) will have nonzero work. In electricity and magnetism, integrals include many examples of accumulation (electric field, electric potential, charge, magnetic field) but these might not be easily interpreted as an extension of the Fundamental Theorem. Faraday's law includes the poorly named electromotive force, which is the path integral of the electric field. In the presence of a changing magnetic field, this path integral around a closed loop is not zero.

c. The merits of flexibility.

In a textbook analysis, several introductory physics textbooks were coded for all instances of integration (Pina & Loverude, 2019). The coding was guided by Jones' (2015a) work on students' conceptions of integration; codes included adding up pieces, area / perimeter, and antiderivatives as well as the analytical derivation lens (Bajracharya & Thompson, 2016). The data show that each of these interpretations appear with varying frequencies across the many chapters of the various textbooks, with adding up pieces making up 56% of the integrals coded across all textbooks. This contrasts with the common outcome of calculus instruction that focuses on area under the curve.

The prevalence of the adding-up-pieces conceptualization supports the suggestion by WB&T. However, there is another possible interpretation: one might argue instead that the data portray an expectation that the students can call upon multiple interpretations of integration and fluidly move between the interpretations as needed. Indeed, students might be asked to do so within a single problem, constructing an integral by considering adding up small pieces but then recognizing the utility of a geometric rather than analytical computation. Students are seemingly expected to master several interpretations and perceive which is most salient and productive for each new context. As students move through physics, they are increasingly likely to encounter integrals that require that they extend their understanding. As another scholar wrote (Wagner, 2016):

> *"thinking of the educational task at hand as supporting students in developing a variety of concept projections for the integral can be helpful in developing and measuring the success of these efforts."* (p. 1381)

Concluding thoughts, and a Plea for Interdisciplinary Conversations

The proposed framework by WB&T is, I believe, a valuable contribution but also part of a broader conversation. Changing instructional practice is challenging; the existing system seems to have near-infinite inertia. That said, any hope of change cannot begin without cogent arguments and productive frameworks for discussion. Ultimately the proof will be in the value that instructors and students find, or do not find in the framework, but this is clearly a valuable contribution to the discussion.

The cross-disciplinary spirit of this issue and the conference series that birthed it are admirable. Too often disciplines get caught in narrow silos and there is plenty of published work in all our fields that would benefit from a look across disciplinary lines. The work in the chemistry (Towns et al., this issue) and engineering papers (Faulkner et al., this issue) includes topics (*PV* work, torque, hydrostatic pressure) that are often covered in introductory physics courses and for which there are PER studies that are relevant. This special issue and the conference series that birthed it have provided opportunities for conversations between mathematics and other disciplines, but this work in only beginning. Studies in the education research subfields should make extra efforts to learn from scholarship in their sister fields. It is notably difficult to work across disciplinary boundaries, but with that difficulty should come great rewards of insight and reduction of duplicated effort. In particular, disciplines should take advantage of this opportunity to increase conversation not only between mathematics and the other disciplines, but across all disciplines and communities.

For example, the concept of work in thermodynamic processes arises in chemistry and engineering, but has also been studied in PER. This work goes back as far as initial studies of student reasoning with the first law of thermodynamics (Loverude et al., 2002; Meltzer, 2004). Work is taught as the area under a curve describing a process on a *PV* diagram, but student responses suggested difficulties in reasoning with the diagram and in recognizing that work depends on the path on the PV diagram and not just the endpoints—in physics language, students had difficulties in interpreting work as a state function. Meltzer (2004) developed a task that has been widely adapted, and reported that 15–20% of students stated incorrectly that work is independent of path. A series of subsequent articles have investigated student use of integration to determine work in *PV* diagrams, drawing upon the Meltzer task (Christensen & Thompson, 2010; Pollock et al., 2007). This work led to a series of paired questions in which a concept was addressed in a physics context and as a purely mathematical task. While the research was carried out in physics courses, this content is also part of chemistry and engineering courses but it is not clear that it is widely known in those fields.

The converse is also true; PER has often failed to attend to relevant scholarship in related fields. Despite the recent growth of work in mathematical and quantitative reasoning in PER, there is surprisingly little uptake of the results of mathematics education research. There is a clear benefit for physics in considering scholarship in other fields, because of the overlapping content areas but also the value of frameworks and approaches drawn from other education researchers. As another example drawn directly from this special issue, PER has paid too little attention to scholarship in engineering education, which has findings to offer physics. The priorities of engineering faculty with respect to mathematical practice overlap greatly with physics; one report highlighted using units and dimensions, evaluating solutions to problems, and creation and interpretation of graphs (Tague et al., 2013), all of which are highly important in physics instruction. The work of Faulkner et al. (this issue) and prior studies (e.g., Faulkner et al., 2019) is relevant to physics educators at all levels.

Interdisciplinary conversations are not easy, and often these will uncover cultural differences including notation, terminology, and conventions (Rabin et al., 2021). The authors of the various articles in this special issue and participants at the associated conferences are at the forefront of these conversations. Within this group one sees different perspectives, but it is also pleasing to see a convergence of viewpoints in the value of working across disciplines and continents. The far greater task that remains is to communicate with scholars and practitioners who are not only not part of these conversations but also content with current practices. Thanks to the organizers and editors for helping us to take this first set of steps.

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