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Editorial Introduction

1. Some of the Contributions of Constantine Tsinakis

Constantine (Costas) Tsinakis' contributions to mathematics are wide-ranging and difficult to summarise concisely. Perhaps the most fitting keywords are those found in the title of a conference series he helped to promote: "Order, Algebra, and Logic". Indeed, one of the *leitmotifs* in his scientific work has been the pursuit of a unified framework that brings together research on ordered structures, universal algebra, and logical calculi.

The scientific and intellectual environments in which Costas spent his formative years played a decisive role in shaping his holistic approach to these disciplines. At the University of Houston, where he earned his M.Sc. in Mathematics in 1974, he was profoundly influenced by the work and teaching of Jürgen Schmidt — an early pioneer in exploring the interplay between order theory, general algebra, and topology, and how their intersections could yield fresh insights. While at Berkeley, where he received his Ph.D. in Mathematics in 1979, he immersed himself in the unique hotbed of genius provided by Alfred Tarski's school, a cradle for many of the past century's major mathematical developments, including universal algebra, algebraic logic, model theory, the study of structures based on Boolean algebras, formal semantics, and more. Two of Tarski's former students — Ralph McKenzie and Bjarni Jónsson — would later become his colleagues at Vanderbilt University, exerting a profound and lasting influence on his work. Ralph McKenzie served as Costas' PhD supervisor, while of Bjarni Jónsson he wrote:

> Bjarni Jónsson was a remarkable mathematician who made field-defining and path-breaking contributions in universal algebra, lattice theory and algebraic logic. Anyone who had the fortune to know him admired his integrity, kindness and immense respect for colleagues and

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friends. His influence on my personal and mathematical life has been enormous, and it is a great privilege that I have had the opportunity to work with and learn from him.

During the first two decades of his scientific career, Costas focused intensively on several core areas: pseudocomplemented structures, one of his mentor Schmidt's pet topics [43,45], lattice theory [16,44], and, most notably, lattice-ordered groups (ℓ-groups), a field to which he made decisive contributions, often in collaboration with W.B. Powell. In particular:

- He conducted an in-depth study of free products within the variety of ℓ-groups [31–34,37–39].
- He developed an interest in the radical class of projectable ℓ-groups, namely, ℓ-groups (including, for example, conditionally σ-complete ℓ-groups) in which every principal polar is a cardinal summand. He showed that an ℓ-group is projectable if and only if every interval from the unit to a positive element is a Stone lattice, thereby offering a purely order-theoretic characterisation of projectability. This result foreshadowed a central theme that would become prominent in his later work [46].
- He studied the amalgamation property (AP) in various classes of ℓ-groups, providing in particular an elegant proof that Abelian ℓ-groups have the AP (first established by Pierce) and a counterexample to the AP for representable ℓ-groups [35,40,41]
- He investigated the lattice of subvarieties of ℓ-groups [36], answering in the affirmative (and independently of Kopytov [21]) the question as to whether there are representable non-solvable covers of the Abelian variety [42].

From 1993 to 2004, Costas held several key administrative roles at Vanderbilt University, serving as Chair of the Department of Mathematics, Associate Provost for Faculty Affairs, and Executive Dean of the College of Arts and Science. Far from the stereotype of the ivory tower mathematician detached from the world beyond pure science, Costas always embraced active engagement in university administration. He viewed such involvement as both demanding and rewarding, offering a broader perspective on the inner workings of academic institutions.

After this "administrative interlude", Costas' research began to take a new direction. He came to believe that many fundamental properties of ℓ-groups are not intrinsic to that specific variety, but rather stem from the fact that ℓ-groups are key examples of *residuated structures*. Although the

study of residuated lattices — originally motivated by the investigation of lattices of ideals in rings — had been introduced over sixty years earlier [47,48], it had not yet gained traction as a mainstream topic in either algebra or order theory. Interestingly, however, commutative, integral, and bounded residuated lattices had drawn the attention of logicians working on the algebraic semantics of substructural logics [14,29]. Costas recognised that a more general notion, much closer to Ward and Dilworth's original concept, could encompass both ℓ-groups and the structures of interest to logicians (including MV-algebras, Heyting algebras, Boolean algebras, and the algebras of linear and relevance logics) under the same umbrella. Before that, in [18], Costas and Bjarni Jónsson had already recognised that relation algebras can be viewed as residuated Boolean algebras.

Together with a number of students and collaborators, Costas began a sustained investigation into the structure theory of residuated lattices [6,15,17], extending to this broader context several classical results from the theory of ℓ-groups. For instance, the theorem stating that, in any residuated lattice, the lattices of congruences and convex normal subalgebras are isomorphic generalises the well-known result that in any ℓ-group the lattice of congruences is isomorphic to the lattice of convex normal ℓ-subgroups. He also sought to isolate the consequences of specific properties of ℓ-groups, such as cancellativity and divisibility, by studying the cancellative and divisible varieties of residuated lattices [3,11]. At the same time, he built or strengthened ties with the community of logicians centred around figures like Antonio Di Nola, Petr Hájek, Franco Montagna, Daniele Mundici, or Hiroakira Ono, who regarded certain classes of residuated lattices as central to the algebraic study of many-valued and substructural logics. The aforementioned conference series on "Order, Algebra, and Logic", along with the "ASubL — Algebra and Substructural Logics" series, signalled the convergence of these two research communities into a unified scientific network that has continued to collaborate fruitfully ever since. This confluence is vividly epitomised by the 2007 monograph [10], coauthored by researchers from both communities.

The interest in residuated lattices, however, did not overshadow Tsinakis' other scientific impulses driven by universal algebra and abstract algebraic logic. Indeed, another of Bjarni Jónsson's insights opened up a new avenue of research. In collaboration with Wim Blok, Jónsson developed in 2006 an innovative theory concerning the equivalence of consequence operators [4]. Contrary to the received view that consequence relations in propositional logic hold only between sets of formulas of a propositional language, Blok and Jónsson considered more general relations among syntactic units of

an unspecified type — not only formulas, but also, for instance, sequents or equations. Since substitutions are nothing but endomorphisms of the formula algebra, the crucial property of *substitution invariance* had to be redesigned in terms of actions of a monoid on a set, an abstract counterpart for the action of the substitution monoid on the set of formulas. In the paper [12], this theory is further refined through the application of order-theoretic and categorical methods. Around the same time, Costas explored another connection between algebra and logic: the relationship between the interpolation property (IP) in logic and the amalgamation property (AP) in algebra. Although the AP and the IP had been studied for many specific classes of algebras, papers adopting a general, universal algebraic perspective were rather thin on the ground [2,8,30]. The paper [25] examines several distinct versions of both the IP and AP, often conflated in the literature, and clarifies their properties and mutual relations, with applications to varieties of residuated lattices.

In recent years, Costas has devoted his efforts to reviving an approach to ℓ-groups that originated with the work of Paul Conrad [7], and which also influenced other scholars such as Jorge Martinez [24]. According to Conrad, the order-theoretic structure of the lattice of convex ℓ-subgroups (as opposed to the lattice of convex *normal* ℓ-subgroups, which, as previously noted, correspond bijectively to congruences) is a fundamental source of insight into the properties of an ℓ-group. More generally, Conrad maintained that many of the key results on the structure theory of ℓ-groups are order-theoretic in nature, as they can be derived from general properties of the various lattices associated with each ℓ-group. Costas has extended this *Conrad's programme* to the more general setting of e-cyclic residuated lattices, conducting an in-depth study of their lattices of convex subalgebras, and giving lattice-theoretic characterisations of projectable and Archimedean residuated lattices [5,13,23].

Over the past quarter-century, intensive research — driven by Costas and other scholars — has brought residuated lattices into the spotlight, spurring rapid growth in the field and a surge in the related literature. Today, residuated lattices constitute a well-established area of research at the intersection of order theory, algebra, and logic. Unlike in 2007, when the monograph [10] was still able to provide a comprehensive overview of residuated structures and substructural logics, any attempt at an exhaustive survey today would be futile. Consequently, the recent volume [26] adopts a "selected topics" approach. In addition to presenting the fundamentals of the theory and its historical development, the book focuses on those classes of residuated lattices, and algebraic properties, that have repeatedly

demonstrated their importance in terms of both mathematical depth and scope of application.

We cannot conclude this introductory note without mentioning two further aspects of Costas' personality that are just as important as his mathematical accomplishments: his belief in science as a collaborative enterprise, and his unwavering support for young scholars. Only a small fraction of Costas' extensive body of work consists of single-authored papers — a telling indication of how profoundly he values collaboration. He has always understood that while mathematical ideas can be refined in solitude, they are most fruitfully shaped through passionate, shoulder-to-shoulder discussion — whether in front of a whiteboard or, more recently, over a video call. Equally remarkable is his attitude toward early-career researchers. Costas has consistently demonstrated an ability to recognise talent, coupled with a genuine concern for the challenges that prevent many promising mathematicians from pursuing academic careers. Whether or not they were his own students, he has always gone to great lengths to support and encourage them, doing whatever he could to ensure they had both the motivation and the means to thrive in the scientific community. For all these reasons, to us, Costas is not only a colleague and a friend, but a benchmark, both as a scientist and as a person. We are looking forward to many more collaborations with him.

2. Outline of the Special Issue

This special issue features papers that build upon Costas' pioneering research in various ways: by advancing his foundational work on classes of residuated lattices, addressing open problems he posed, or employing methods and techniques that frequently appear in his contributions.

Several articles are dedicated to *MV-algebras* and their generalisations — an area that has consistently held a prominent place in Costas' research interests. This is in large part due to their deep connection with Abelian ℓ-groups with a strong unit, as revealed by Mundici's categorical equivalence theorem [27], later extended to the non-commutative setting by Dvurečenskij [9]. One of Costas' notable contributions was to offer a fresh perspective on this result, by broadening its scope and uncovering its roots in the theory of nuclei on residuated lattices. In fact, together with one of the present writers, he proved that integral generalised MV-algebras are categorically equivalent to negative cones of ℓ-groups with a dense nucleus [11]. The paper "Topological classes of MV-algebras", by Giuseppina Barbieri, Antonio

Di Nola, and Giacomo Lenzi, studies the topological space of prime ideals of an MV-algebra, endowed with the Zariski topology. The authors explore properties of MV-algebras that are determined solely by this prime spectrum — such as simplicity and strong simplicity — as opposed to properties like being perfect, which are not. The article "Ulam-Rényi games, MV-algebras, Specker ℓ-groups", by Daniele Mundici, expands on the well-known link between the Ulam-Rényi game, Łukasiewicz logic and MV-algebras, first noted in [28]. In the original version of the game, the "secret" to be guessed lies in a finite search space, and the respondent is allowed a fixed, though unspecified, number of incorrect answers. This paper extends the analysis to games with infinite search spaces and variable numbers of incorrect answers, and investigates the MV-algebras and the attendant ℓ-groups that arise from these games. The article "Some results on quasi MV-algebras and perfect quasi MV-algebras" by Anatolij Dvurečenskij and Omad Zahiri, focuses on quasi MV-algebras, a generalisation of MV-algebras where the semigroup reduct need not have a unit [22]. These algebras are motivated by the study of algebras of density operators in tensor products of $\mathbb{C}^2$, used as models of quantum computation. The authors investigate various classes of congruences and ideals in quasi MV-algebras and establish a categorical equivalence between perfect quasi MV-algebras and certain "quasi ℓ-groups".

As we have seen, Costas has consistently regarded the amalgamation and interpolation properties as central themes of investigation, dedicating numerous studies to them — initially within the framework of ℓ-groups, and later in the broader setting of residuated lattices [25,35,40,41]. The monograph [26] enumerates no fewer than seven open problems concerning the AP and the IP in various classes of residuated lattices. Some of these problems are solved in the paper "Amalgamation in semilinear residuated lattices" by Wesley Fussner and Simon Santschi. In particular, the authors show that the variety of commutative semilinear residuated lattices does not have the AP. They further investigate the AP and IP in idempotent semilinear varieties of residuated lattices and then extend their analysis to the more intricate case in which idempotency is relaxed to knotted inequalities.

Another recurring theme in Costas' body of work is the extension of constructions that are especially effective for ℓ-groups, such as semidirect products or wreath products, to broader classes of residuated lattices [19]. The paper "A new representation of finite hoops", by Michal Botur, follows a similar line of inquiry. He defines a new type of product of hoops which, in the case of finite hoops, can decompose an arbitrary hoop into a filter and the corresponding homomorphic image. The main result of the paper

establishes that every finite hoop can be represented as a product of finite MV-chains.

Finally, two contributions revisit themes that Costas explored at the very outset of his career, when he was deeply influenced by Jürgen Schmidt — namely, pseudocomplemented lattices and Stone lattices. The article "Orthomodular and unsharp orthomodular lattices: A categorical equivalence", by Antonio Ledda and Gandolfo Vergottini, generalises to the non-distributive case the categorical equivalence between regular double Stone algebras and Boolean algebras with a designated filter [20]. They show that a certain category that properly includes the algebraic category of regular double Stone algebras, whose objects can be viewed as unions of their double Stone blocks, is equivalent to the category of orthomodular lattices with a designated congruence filter. The paper "Quasivarieties of p-algebras: Some new results", by Tomasz Kowalski and Katarzyna Słomczyńska, investigates the lattice of quasivarieties of distributive pseudocomplemented lattices. Building on previous work by Wroński [49] and Adams [1], they show that certain intervals in the lattice contain continuum many quasivarieties, and identify a quasivariety containing all the quasivarieties generated by free distributive pseudocomplemented lattices.

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