



An Upper Bound for the Nevanlinna Matrix of an Indeterminate Moment Sequence

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Abstract

The solutions of an indeterminate Hamburger moment problem can be parameterised using the Nevanlinna matrix of the problem. The entries of this matrix are entire functions of minimal exponential type, and any growth less than that can occur. An indeterminate moment problem can be considered as a canonical system in limit circle case by rewriting the three-term recurrence of the problem into a discrete vector-valued differential equation. We give a bound for the growth of the Nevanlinna matrix in terms of the parameters of this canonical system. In most situations this bound can be evaluated explicitly. It is sharp in the sense that for well-behaved parameters it coincides with the actual growth of the Nevanlinna matrix up to multiplicative constants.

Keywords Indeterminate moment problem · Nevanlinna matrix · Order of entire function · Canonical system · Distribution of eigenvalues

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1 Introduction

Let $(s_n)_{n=0}^\infty$ be a sequence of real numbers. The *Hamburger moment problem* is the task of describing the set

$$\mathcal{M}((s_n)_{n=0}^\infty) := \left\{ \mu \mid \begin{array}{l} \mu \text{ positive Borel measure on } \mathbb{R} \\ s_n = \int_{\mathbb{R}} t^n d\mu(t) \text{ for } n = 0, 1, 2, \dots \end{array} \right\}.$$

This is a classical problem of analysis and was treated extensively in work of H. Hamburger, M. Riesz, R. Nevanlinna, and many others. Standard references are, e.g., [1, 22, 24]. It is known that one of the following alternatives takes place for the set $\mathcal{M}((s_n)_{n=0}^\infty)$: it is either empty, or contains exactly one element, or contains infinitely many elements. In the last case $(s_n)_{n=0}^\infty$ is called an *indeterminate moment sequence*, and this is the case we are concerned with in the present paper.

For an indeterminate moment sequence $(s_n)_{n=0}^\infty$ the set $\mathcal{M}((s_n)_{n=0}^\infty)$ can be parameterised: There exist four entire functions A, B, C, D , such that the formula

$$\int_{\mathbb{R}} \frac{1}{t - z} d\mu(t) = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)} \quad (1.1)$$

establishes a bijection between $\mathcal{M}((s_n)_{n=0}^\infty)$ and the set of all functions τ that are analytic in the open upper half-plane $\mathbb{C}^+$ and have nonnegative imaginary part (formally including the function constant equal to ∞). The matrix

$$W(z) := \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is called the *Nevanlinna matrix* of the sequence $(s_n)_{n=0}^\infty$.

The Nevanlinna matrix admits an operator theoretic interpretation (and this viewpoint could be used to prove (1.1)). To the moment sequence there is an associated sequence $(p_n)_{n=0}^\infty$ of orthogonal polynomials that satisfies a three-term recurrence of the form

$$zp_n(z) = b_n p_{n+1}(z) + a_n p_n(z) + b_{n-1} p_{n-1}(z), \quad n = 0, 1, 2, \dots \quad (1.2)$$

with certain parameters $a_n \in \mathbb{R}$ and $b_n > 0$ for $n = 0, 1, 2, \dots$ (and formally $b_{-1} = 0$). The *Jacobi operator* is the closure of the operator

$$(Ju)_n = \begin{cases} b_0 u_1 + a_0 u_0 & \text{if } n = 0, \\ b_n u_{n+1} + a_n u_n + b_{n-1} u_{n-1} & \text{if } n \geq 1, \end{cases}$$

which is defined on $\mathcal{D} := \{u \in \ell^2 \mid u_n = 0 \text{ for almost all } n\}$, and (1.2) is the formal eigenvalue equation for J . The Jacobi operator is symmetric, and has deficiency index $(1, 1)$ if $(s_n)_{n=0}^\infty$ is indeterminate. In this case the self-adjoint extensions of the Jacobi operator (we write again J for simplicity) are described by M. G. Krein's resolvent

formula. The Nevanlinna matrix of $(s_n)_{n=0}^\infty$ is precisely the u -resolvent matrix of J for a certain generating element u .

So far, we talked about two different (equivalent) objects, i.e., the moment sequence and the Jacobi operator. Our method actually relies on a third object – a *canonical system* of differential equations. It is obtained from rewriting the three-term recurrence (1.2) as a first order vector difference equation and interpreting this as a discrete differential equation. In the indeterminate case the *monodromy matrix* of the system coincides with the Nevanlinna matrix W (up to multiplication with constant invertible factors).

The canonical systems occurring in the context of moment problems are represented by a *Hamiltonian* which reflects the discrete nature of the difference equation. It is determined by two sequences, its *lengths* $l_j > 0$ and *angles* $\phi_j \in \mathbb{R}$, which we call the *Hamiltonian parameters* (details are given below, cf. Definition 1.1). This model for an indeterminate moment sequence is well suited for the study of various properties. For example, the moment sequence is indeterminate if and only if $(l_j)_{j=1}^\infty$ is summable.

The entries A, B, C, D of the Nevanlinna matrix all have the same growth [5], and a classical theorem of M. Riesz in [18] states that they are of minimal exponential type. Moreover, it is known that any growth up to “order 1, minimal type” may occur, e.g. [8]. Revealing more refined information about the growth of the Nevanlinna matrix is an intricate problem. It is of great interest also for spectral theoretic reasons: due to the above interpretation of W as a resolvent matrix, the set of zeros of C coincides with the spectrum of a particular self-adjoint extension of J . Hence, if the growth of W is known, information about the distribution of eigenvalues can be obtained using tools from complex analysis.

Let us briefly review a couple of earlier results on the growth of Nevanlinna matrices. The history of the subject after Riesz’s paper starts probably with a theorem of M. S. Livšic in [12] that gives a lower bound for

$$M(r) := \max_{|z|=r} \|W(z)\|.$$

This bound is formulated in terms of the moment sequence itself. It is easy to handle, but will usually not give the correct size. In recent work of C. Berg and R. Szwarc [6] it is shown that the order and type of W

$$\rho := \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}, \quad \tau := \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho}$$

coincide with those of a certain entire function built in a complicated way from the coefficients of the orthonormal polynomials. In theory this is a precise formula, but in practice it is almost never computable. A theorem which takes the Jacobi parameters as input is due to Yu. M. Berezanskii [4] and was generalised in [6]. It asserts that ρ equals the convergence exponent of the sequence $(b_n)_{n=0}^\infty$, but the assumptions on the given data are very restrictive, involving regularity of b_n and smallness of a_n . Bounds for the order of W in terms of the Hamiltonian parameters, which under certain conditions give the correct value, are obtained in [15].

In our present paper we work almost exclusively in the canonical systems model. We take the Hamiltonian parameters $(l_j)_{j=1}^\infty$ and $(\phi_j)_{j=1}^\infty$ as input and deduce knowledge about $\log M(r)$. The philosophy is: if lengths and increments of angles are dominated by (or comparable to) “well-behaved” functions, then one can give an expression in terms of these functions such that $\log M(r)$ is dominated by (or comparable to) this expression. Here “domination” should be understood in a pointwise and averaged sense. The reason to work with dominating functions instead of lengths and angles themselves is that in this way one can treat also very irregularly behaving data.

There are three major results.

- ▷ A very general, albeit complicated, upper bound which assumes only weak “well-behavedness” of the dominating functions. Namely, we assume measurability and monotonicity. This is Theorem 2.2; we recommend the reader to first visit Corollary 2.5 where estimates for order and type are given.
- ▷ Under a more restrictive assumption on the dominating functions (namely, regular variation in Karamata sense), the general bound can be evaluated and becomes a rather explicit expression. This is done in Theorem 4.6 which in turn is based on Theorem 4.1.
- ▷ We consider the situation where the data is not only dominated, but comparable to regularly varying comparison functions. For the case of fast decay (corresponding to order less than $\frac{1}{2}$), we prove a simple and explicit formula that fully determines the growth of $\log M(r)$. This is Theorem 5.3.

The proof of our foundational Theorem 2.2 builds upon an idea going back to R. Romanov [20]. The trick is to cut the Hamiltonian, use an explicit estimate for the beginning section and Romanov’s method (in its refined variant from [16]) for the end section. The optimal cutting point is then found by balancing the contributions. Theorems 4.1 and 4.6 are established by an involved analysis of functions, and Theorem 5.3 by combining the so obtained upper bounds with a lower bound from [16].

We close the paper by showing an application to the moment problem itself – in the form of Corollaries 5.10 and 5.11 which are formulated in terms of the Jacobi parameters.

Canonical Systems with Hamburger Hamiltonian

A two-dimensional canonical system is an equation of the form

$$y'(t) = zJH(t)y(t), \quad t \in (a, b) \text{ a.e.},$$

where

- ▷ $-\infty \leq a < b \leq \infty$,
- ▷ $H \in L^1_{\text{loc}}((a, b), \mathbb{R}^{2 \times 2})$, and $H(t) \geq 0$ and $H(t) \neq 0$ for $t \in (a, b)$ a.e.,
- ▷ $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $z \in \mathbb{C}$,

and the solution $y: (a, b) \rightarrow \mathbb{C}^2$ is required to be locally absolutely continuous. The function H is called the *Hamiltonian* of the system. Some general references about canonical systems are [3, 17, 19]. There the reader can find background information about solutions, the operator model, or Weyl's limit circle/limit point distinction. The connection with the moment problem was around for many years and is made explicit in [9].

Canonical systems that occur from indeterminate moment sequences are those whose Hamiltonian has the following particular – discrete – form. Here we denote

$$\xi_\phi := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \phi \in \mathbb{R}.$$

1.1 Definition Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers and $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Set $x_0 := 0$, $x_j := \sum_{i=1}^j l_i$ for $j \geq 1$, and $L := \sum_{j=1}^\infty l_j < \infty$. The *Hamburger Hamiltonian* with lengths l_j and angles ϕ_j is the function H defined on the interval $(0, L)$ as

$$H(t) := \xi_{\phi_j} \xi_{\phi_j}^T \quad \text{for } j \in \mathbb{N} \text{ and } x_{j-1} \leq t < x_j.$$

A Hamburger Hamiltonian H thus can be pictured as

$$H: \quad \begin{array}{c} \xleftarrow{\xi_{\phi_1} \xi_{\phi_1}^T} \quad \xleftarrow{\xi_{\phi_2} \xi_{\phi_2}^T} \quad \xleftarrow{\xi_{\phi_3} \xi_{\phi_3}^T} \quad \dots \\ \hline x_0 \quad \xrightarrow{l_1} \quad x_1 \quad \xrightarrow{l_2} \quad x_2 \quad \xrightarrow{l_3} \quad x_3 \quad \dots \quad L \end{array}$$

Since a Hamburger Hamiltonian in the sense of the above definition is even integrable on the whole interval $(0, L)$, there exists a unique 2×2 matrix-valued solution $W: [0, L] \times \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$ of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(t; z) J = z W(t; z) H(t), & t \in (0, L) \text{ a.e.}, \\ W(0; z) = I. \end{cases}$$

We refer to W as the *fundamental solution* of H , and to the matrix $W_H(z) := W(L; z)$ as its *monodromy matrix*.

A Notational Convention

We frequently compare functions up to multiplicative constants or asymptotically, and throughout the paper use the following notation.

1.2 Notation Let X be a set and $f, g: X \rightarrow (0, \infty)$.

- (i) We write “ $f \lesssim g$ ” (or “ $f(x) \lesssim g(x)$ ”) to say that there exists a constant $C > 0$ such that $f(x) \leq C \cdot g(x)$ for all $x \in X$. We write “ $f \gtrsim g$ ” if $g \lesssim f$, and “ $f \asymp g$ ” if $f \lesssim g$ and $f \gtrsim g$.
- (ii) Assume that X is directed. We say that “ $f \lesssim g$ for sufficiently large x ”, if there exists $x_0 \in X$ such that $f|_Y \lesssim g|_Y$ where $Y := \{x \in X \mid x \geq x_0\}$.

- (iii) Assume again that X is directed. We write “ $f \sim g$ ” if $\lim_{x \in X} \frac{f(x)}{g(x)} = 1$, and “ $f \ll g$ ” if $\lim_{x \in X} \frac{f(x)}{g(x)} = 0$, and “ $f \approx g$ ” if $\lim_{x \in X} \frac{f(x)}{g(x)}$ exists in $(0, \infty)$.
- (iv) Assume that X is a subset of a topological space. We say that “ $f \lesssim g$ locally”, if $f|_K \lesssim g|_K$ for every compact subset K of X . Analogous wording applies to “ $\asymp$ ”.

For the convenience of the reader, we include an appendix where the definition and some basic results about regularly varying functions in Karamata sense are recalled.

2 An Upper Bound for the Monodromy Matrix

In this section we give a generic upper bound for $\log \|W_H(z)\|$. The functions d_l, d_ϕ used below play the role of well-behaved comparison functions for the lengths and for the differences of consecutive angles of the Hamiltonian.

2.1 Definition Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally. Then we denote

$$\begin{aligned} \kappa(R) &:= \sup \left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{2}{R(d_l d_\phi)(s)} \leq 1 \right\}, \\ \hbar(R) &:= \sup \left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{d_\phi(s)}{R d_l(s)} \leq 1 \right\}, \end{aligned} \quad (2.1)$$

where $R \in [\frac{2}{(d_l d_\phi)(1)}, \infty)$ and $R \in [\frac{d_\phi(1)}{d_l(1)}, \infty)$, respectively. Further, we set

$$g(t, R) := \begin{cases} \log(R(d_l d_\phi)(t)) & \text{if } 1 \leq t < \kappa(R), \\ R^{\frac{1}{2}}(d_l d_\phi)^{\frac{1}{2}}(t) & \text{if } \kappa(R) \leq t < \hbar(R), \\ R d_l(t) & \text{if } \hbar(R) \leq t, \end{cases}$$

where $(t, R) \in [1, \infty) \times [\frac{2}{(d_l d_\phi)(1)}, \infty)$, and

$$\underline{g}(t, R) = \int_1^t g(s, R) \, ds. \quad (2.2)$$

Note here that $d_\phi(t) \leq 1$, and hence $\kappa(R) \leq \hbar(R)$ for all $R \geq \frac{2}{(d_l d_\phi)(1)}$.

2.2 Theorem Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.

Let $\psi \in \mathbb{R}$ and let $d_l, d_\phi, c_l, c_\phi: [1, \infty) \rightarrow (0, \infty)$ be measurable and nonincreasing, with $d_\phi \leq 1$ and $c_\phi \leq c_l$. Assume that

$$\forall j \in \mathbb{N}: l_j \leq d_l(j) \wedge |\sin(\phi_{j+1} - \phi_j)| \leq d_\phi(j), \quad (2.3)$$

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \leq c_l(N) \wedge \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \leq c_\phi(N). \quad (2.4)$$

Let $q(t, R)$ be as in (2.2), and set

$$\begin{aligned} L(t, R) := & 1 + \log^+ R + \log^+ \frac{c_l(\lceil t \rceil)}{c_\phi(\lceil t \rceil)} + \log^+ \frac{d_l(1)}{d_\phi(1)} \\ & + \log^+ \frac{d_l(\min\{\lceil t \rceil, \lfloor h(R) \rfloor\})}{d_\phi(\min\{\lceil t \rceil, \lfloor h(R) \rfloor\})} + \sum_{j=1}^{\min\{\lceil t \rceil, \lfloor h(R) \rfloor\}-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \bigg/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right|. \end{aligned} \quad (2.5)$$

Then we have, for all $R \geq \frac{2}{(d_l d_\phi)(1)}$,

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \leq 9 \cdot \inf_{t \geq 1} \left(\max \left\{ q(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\} + L(t, R) \right). \quad (2.6)$$

2.3 Remark The terms appearing in the upper bound in (2.6) have the following meaning:

- (i) $q(t, R)$ estimates the contribution of the first $\lceil t \rceil$ intervals of H ;
- (ii) $R(c_l c_\phi)^{\frac{1}{2}}(t)$ estimates the contribution of the remaining intervals of H ;
- (iii) $L(t, R)$ is usually a remainder term.

We mention two possible scenarios. If lengths and angle differences are nonincreasing but decay slowly, choosing $d_l(j), d_\phi(j)$ such that equality holds in (2.3) makes $q(t, R)$ a rather precise bound for the contribution of the first $\lceil t \rceil$ intervals. On the other hand, if lengths and angle differences behave irregularly, then $q(t, R)$ likely overestimates the contribution of the first $\lceil t \rceil$ intervals. However, choosing c_l, c_ϕ such that equality holds in (2.4), the bound (2.6) is still a good estimate for $\log \|W_H(z)\|$ as long as the decay is sufficiently fast.

If $\lim_{t \rightarrow \infty} (c_l c_\phi)(t) > 0$, the bound (2.6) is $\gtrsim R$ and thus trivial: we know that W_H is of minimal exponential type by the classical Krein-de Branges formula. Hence we may safely assume that $\lim_{t \rightarrow \infty} (c_l c_\phi)(t) = 0$ whenever this is convenient.

The following lemma hints at a way to evaluate the upper bound in Theorem 2.2.

2.4 Lemma Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally, and let c_l, c_ϕ be continuous and nonincreasing with $\lim_{t \rightarrow \infty} (c_l c_\phi)(t) = 0$. Then the equation

$$q(t, R) = R(c_l c_\phi)^{\frac{1}{2}}(t), \quad (2.7)$$

has a unique solution $T(R)$, and

$$\min_{t \geq 1} \max \left\{ q(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\} = q(T(R), R) = R(c_l c_\phi)^{\frac{1}{2}}(T(R)).$$

In addition, $\lim_{R \rightarrow \infty} T(R) = \infty$.

Proof For each fixed R the function $t \mapsto g(t, R)$ is continuous and increasing while $t \mapsto R(c_l c_\phi)^{\frac{1}{2}}(t)$ is continuous and nonincreasing. Moreover, $g(1, R) = 0 < R(c_l c_\phi)^{\frac{1}{2}}(1)$ and $\lim_{t \rightarrow \infty} R(c_l c_\phi)^{\frac{1}{2}}(t) = 0$. By the intermediate value theorem, (2.7) has a unique solution $T(R)$.

If $\liminf_{R \rightarrow \infty} T(R) < \infty$, choose a sequence $(R_n)_{n \in \mathbb{N}}$ such that $(T(R_n))_{n \in \mathbb{N}}$ is bounded. Using the crude estimate

$$g(t, R) \leq t \log(R(d_l d_\phi)(1)) + \sqrt{2}t + t \lesssim t \log R,$$

this leads to the contradiction

$$R(c_l c_\phi)^{\frac{1}{2}}(T(R_n)) = g(T(R_n), R) \lesssim T(R_n) \cdot \log R \ll R \asymp R(c_l c_\phi)^{\frac{1}{2}}(T(R_n)).$$

□

In many situations, we are going to first determine $T(R)$, and then show that $L(T(R), R)$ is small. The bound (2.6) is then asymptotically equal to $R(c_l c_\phi)^{\frac{1}{2}}(T(R))$.

Before we go into the proof of Theorem 2.2, let us discuss a particular situation.

2.5 Corollary Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.

Assume that we have $\Psi \in R$ and $\delta_l, \delta_\phi, \gamma_l, \gamma_\phi \geq 0$, such that

$$l_j \lesssim j^{-\delta_l} \wedge |\sin(\phi_{j+1} - \phi_j)| \lesssim j^{-\delta_\phi},$$

$$\sum_{j=N+1}^\infty l_j \lesssim N^{-\gamma_l} \wedge \sum_{j=N+1}^\infty l_j \sin^2(\phi_j - \psi) \lesssim N^{-\gamma_\phi},$$

i.e., we are in the situation of Theorem 2.2 with

$$d_l(t) = ct^{-\delta_l}, \quad d_\phi(t) = ct^{-\delta_\phi}, \quad c_l(t) = ct^{-\gamma_l}, \quad c_\phi(t) = ct^{-\gamma_\phi},$$

where c is some positive constant. Set $\delta := \delta_l + \delta_\phi$, $\gamma := \frac{1}{2}(\gamma_l + \gamma_\phi)$, and assume that $\delta > 0$ and $\gamma > 0$. Then we have, for sufficiently large R , the following bounds for $T(R)$ (cf. Lemma 2.4), $\log \max_{|z|=R} \|W_H(z)\|$ and the order ρ_H of W_H .

<i>Data satisfies</i>	$T(R) \asymp$	$\log \max_{ z =R} \ W_H(z)\ \lesssim$	$\rho_H \leq$
$\delta < 1 + \gamma$	$\left(\frac{R}{\log R}\right)^{\frac{1}{1+\gamma}}$	$R^{\frac{1}{1+\gamma}} (\log R)^{\frac{\gamma}{1+\gamma}}$	$\frac{1}{1+\gamma}$
$\delta = 1 + \gamma$	$R^{\frac{1}{\delta}}$	$R^{\frac{1}{\delta}}$	$\frac{1}{\delta}$
$1 + \gamma < \delta, \delta > 2,$	$R^{\frac{\delta-1}{\gamma\delta}}$	$R^{\frac{1}{\delta}}$	$\frac{1}{\delta}$
$\delta = 2,$	$\left(\frac{R^{\frac{1}{2}}}{\log R}\right)^{\frac{1}{\gamma}}$	$R^{\frac{1}{2}} \log R$	$\frac{1}{2}$
$\delta < 2, \delta_l \leq 1 + \gamma$	$R^{\frac{1}{2-\delta+2\gamma}}$	$R^{\frac{2-\delta+\gamma}{2-\delta+2\gamma}}$	$\frac{2-\delta+\gamma}{2-\delta+2\gamma}$
$1 + \gamma < \delta_l$	$R^{\frac{\delta_l-1}{\gamma(\delta_l-\delta_\phi)}}$	$R^{\frac{1-\delta_\phi}{\delta_l-\delta_\phi}}$	$\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$

All these formulae are checked by elementary calculations which we do not elaborate here; the necessary computations can be found in the preprint version of this article [14].

Corollary 2.5 already shows that our present results may improve drastically upon our previous work [15]. The main feature that enables this is the use of $\mathcal{C}_\phi$: the present results exploit convergence of angles more efficiently.

2.6 Example Let $\alpha > 1$ and $\beta \geq 0$, and let H lengths and angles

$$l_j := j^{-\alpha}, \quad \phi_j := (-1)^j j^{-\beta}.$$

Considering the expressions (2.3) and (2.4), we see that Corollary 2.5 can be applied with

$$\delta_l := \alpha, \quad \delta_\phi := \beta, \quad \gamma_l := \alpha - 1, \quad \gamma_\phi := \alpha + 2\beta - 1.$$

Since $\delta = \alpha + \beta$ and $\gamma = \alpha + \beta - 1$, this yields

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim R^{\frac{1}{\alpha+\beta}},$$

and hence $\rho_H \leq \frac{1}{\alpha+\beta}$. Now recall [15, Example 2.23], where we saw that the order ρ_H is at least $\frac{1}{\alpha+\beta}$. Thus, $\rho_H = \frac{1}{\alpha+\beta}$.

Let us compare this with what we can obtain from [15]. For $\alpha + \beta \geq 2$, the formula $\rho_H = \frac{1}{\alpha+\beta}$ was already established in [15, Example 2.23]. For $\alpha + \beta < 2$, we had obtained the upper bound $\frac{1-\beta}{\alpha-\beta}$ in that place. However, this did not take into account that angles converge. If we want to take convergence into account, we should use [15, Corollary 2.9]. The quantities used there identify as

$$\Delta_l^+ = \alpha, \quad \Delta_\phi^* = \Lambda^* = \beta,$$

and the bound for order so obtained thus is $\frac{1-\beta}{\alpha}$. If $\beta > 0$, this lies strictly between $\frac{1}{\alpha+\beta}$ and $\frac{1-\beta}{\alpha-\beta}$, since

$$\begin{aligned} \frac{1-\beta}{\alpha-\beta} - \frac{1-\frac{\beta}{2}}{\alpha} &= \frac{\frac{\beta}{2}(2-\alpha-\beta)}{(\alpha-\beta)\alpha} > 0, \\ \frac{1-\frac{\beta}{2}}{\alpha} - \frac{1}{\alpha+\beta} &= \frac{\frac{\beta}{2}(2-\alpha-\beta)}{\alpha(\alpha+\beta)} > 0. \end{aligned}$$

2.1 Proof of Theorem 2.2

Denote by $W(s, t; z)$ the unique solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} W(s, t; z) J = z W(s, t; z) H(t), & t \in (0, L) \text{ a.e.}, \\ W(s, s; z) = I. \end{cases}$$

and observe the multiplicativity property

$$W(s, t; z) W(t, u; z) = W(s, u; z).$$

Our method for the proof of Theorem 2.2 is to fix $t \geq 1$ and estimate $\|W(0, x_{[t]}; z)\|$ and $\|W(x_{[t]}, L; z)\|$ separately. The contribution of the first $[t]$ intervals is estimated using the explicit form of the fundamental solution on indivisible intervals. On the remaining part, we use Grönwall's Lemma. This method is a refined version of an idea from [20] and its improved variant from [16].

Set $W_j(z) := W(x_{j-1}, x_j; z)$, then by a direct computation

$$W_j(z) = I + z l_j \xi_{\phi_j} \xi_{\phi_j}^T J.$$

Moreover, multiplicativity yields

$$W(0, x_N; z) = W_1(z) \cdot \dots \cdot W_N(z), \quad N \in \mathbb{N}. \quad (2.8)$$

We often use the matrices introduced in [16, Definition 4.4]: for $a > 0$ and $\psi \in \mathbb{R}$, denote

$$\Omega(a, \psi) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \exp(-\psi J) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix}.$$

Recall the following properties from [16, Lemma 4.6].

2.7 Lemma *Let $a, b > 0$ and $\phi, \psi \in \mathbb{R}$.*

- (i) $\|\Omega(a, \psi)\| = \|\Omega(a, \psi)^{-1}\| = \max\{a, a^{-1}\}$;
- (ii) $\|\Omega(a, \psi) \xi_{\phi} \xi_{\phi}^T J \Omega(a, \psi)^{-1}\| = a^2 \cos^2(\phi - \psi) + \frac{1}{a^2} \sin^2(\phi - \psi)$;

$$(iii) \quad \|\Omega(a, \psi)\Omega(b, \phi)^{-1}\| \leq \max\left\{\frac{a}{b}, \frac{b}{a}\right\} |\cos(\phi - \psi)| + \max\left\{ab, \frac{1}{ab}\right\} |\sin(\phi - \psi)|.$$

We use $\Omega(a_j, \psi_j)$ to rotate and dilate $W_j(z)$ such that the norm becomes small. In the following lemma, we make the obvious choice $\psi_j = \phi_j$, but retain the freedom of choosing suitable parameters a_j .

2.8 Lemma *Let $N \in \mathbb{N}$ and let numbers $a_j \in (0, 1]$, $j = 1, \dots, N$, be given. Then, for each $z \in \mathbb{C}$, we have the estimate*

$$\begin{aligned} \|W(0, x_N; z)\| &\leq \frac{1}{a_1 a_N} \cdot \prod_{j=1}^N \left(1 + |z| l_j a_j^2\right) \\ &\cdot \prod_{j=1}^{N-1} \left(\max\left\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\right\} \cdot |\cos(\phi_j - \phi_{j+1})| + \frac{|\sin(\phi_j - \phi_{j+1})|}{a_j a_{j+1}}\right). \end{aligned} \quad (2.9)$$

Proof With $\Omega_j := \Omega(a_j, \phi_j)$ we have

$$\Omega_j W_j(z) \Omega_j^{-1} = I + z l_j \Omega_j \xi_{\phi_j} \xi_{\phi_j}^T J \Omega_j^{-1},$$

and we can use Lemma 2.7, (ii), to estimate

$$\|\Omega_j W_j(z) \Omega_j^{-1}\| \leq 1 + |z| l_j a_j^2.$$

Expanding on (2.8),

$$W(0, x_N; z) = \Omega_1^{-1} \cdot (\Omega_1 W_1(z) \Omega_1^{-1}) \cdot \Omega_1 \Omega_2^{-1} \cdot \dots \cdot (\Omega_N W_N(z) \Omega_N^{-1}) \cdot \Omega_N.$$

Using submultiplicativity of the norm and Lemma 2.7, (i), (iii), we arrive at the desired estimate. $\square$

The next lemma is the ingredient of Theorem 2.2 that accounts for the term $g(t, R)$ in (2.6).

2.9 Lemma *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Let H be the Hamburger Hamiltonian with these lengths and angles. Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable and nonincreasing, and assume that*

$$\forall j \in \mathbb{N}: l_j \leq d_l(j) \wedge |\sin(\phi_{j+1} - \phi_j)| \leq d_\phi(j).$$

Then, for every $(N, R) \in \mathbb{N} \times \left[\frac{2}{(d_l d_\phi)(1)}, \infty\right)$,

$$\log \left(\max_{|z|=R} \|W(0, x_N; z)\| \right) \leq \frac{7}{2} \cdot \left[g(N, R) + \lambda(\min\{N, \hbar(R)\}, R) \right]$$

where

$$\lambda(t, R) := 1 + \log^+ R + \log^+ \frac{d_l(1)}{d_\phi(1)} + \log^+ \frac{d_l(\lfloor t \rfloor)}{d_\phi(\lfloor t \rfloor)} + \sum_{j=1}^{\lfloor t \rfloor - 1} \left| \log \frac{d_\phi(j) d_l(j+1)}{d_l(j) d_\phi(j+1)} \right|.$$

Proof Fix z with $|z| = R$. In the first step we estimate $\log \|W(0, x_N; z)\|$ when $N \leq \hbar(R)$. This is done by an application of Lemma 2.8 with

$$a_j := \left(\frac{d_\phi(j)}{R d_l(j)} \right)^{\frac{1}{4}}, \quad j = 1, \dots, N.$$

Since $N \leq \hbar(R)$ we have $a_j \in (0, 1]$. The factors in (2.9) are treated separately:

$$\frac{1}{a_1 a_N} = R^{\frac{1}{2}} \cdot \left(\frac{d_l(1)}{d_\phi(1)} \right)^{\frac{1}{4}} \cdot \left(\frac{d_l(N)}{d_\phi(N)} \right)^{\frac{1}{4}},$$

$$1 + |z| l_j a_j^2 \leq 1 + R d_l(j) a_j^2 = 1 + (R(d_l d_\phi)(j))^{\frac{1}{2}},$$

$$\begin{aligned} & \max \left\{ \frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j} \right\} \left| \cos(\psi_j - \psi_{j+1}) \right| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}} \\ &= \max \left\{ \frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j} \right\} \cdot \left(|\cos(\phi_{j+1} - \phi_j)| + \frac{|\sin(\phi_{j+1} - \phi_j)|}{\max\{a_j^2, a_{j+1}^2\}} \right) \\ &\leq \max \left\{ \frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j} \right\} \cdot \left(1 + \frac{d_\phi(j)}{a_j^2} \right) \\ &= \left[\max \left\{ \frac{d_\phi(j) d_l(j+1)}{d_l(j) d_\phi(j+1)}, \frac{d_l(j) d_\phi(j+1)}{d_\phi(j) d_l(j+1)} \right\} \right]^{\frac{1}{4}} \cdot \left(1 + (R(d_l d_\phi)(j))^{\frac{1}{2}} \right). \end{aligned}$$

Now (2.9) yields

$$\begin{aligned} \log \|W(0, x_N; z)\| &\leq \frac{1}{2} \log^+ R + \frac{1}{4} \log^+ \frac{d_l(1)}{d_\phi(1)} + \frac{1}{4} \log^+ \frac{d_l(N)}{d_\phi(N)} \\ &\quad + \sum_{j=1}^N \log \left(1 + (R(d_l d_\phi)(j))^{\frac{1}{2}} \right) \\ &\quad + \frac{1}{4} \sum_{j=1}^{N-1} \left| \log \frac{d_\phi(j) d_l(j+1)}{d_l(j) d_\phi(j+1)} \right| + \sum_{j=1}^{N-1} \log \left(1 + (R(d_l d_\phi)(j))^{\frac{1}{2}} \right). \end{aligned}$$

Since d_l, d_ϕ are nonincreasing, we can estimate sums by integrals as

$$\begin{aligned} \sum_{j=2}^N \log \left(1 + (R(d_l d_\phi)(j))^{\frac{1}{2}} \right) &\leq \int_1^N \log \left(1 + (R(d_l d_\phi)(s))^{\frac{1}{2}} \right) ds \\ &\leq \frac{3}{2} \cdot \int_1^{\min\{N, \hat{k}(R)\}} \log (R(d_l d_\phi)(s)) ds + \int_{\min\{N, \hat{k}(R)\}}^N (R(d_l d_\phi)(s))^{\frac{1}{2}} ds. \end{aligned}$$

Noting that

$$\begin{aligned} \log \left(1 + (R(d_l d_\phi)(1))^{\frac{1}{2}} \right) &\leq \frac{3}{2} \left(\log^+ R + \log^+ (d_l d_\phi)(1) \right) \\ &\leq \frac{3}{2} \left(\log^+ R + \log^+ \frac{d_l(1)}{d_\phi(1)} \right), \end{aligned}$$

leads to

$$\begin{aligned} \log \|W(0, x_N; z)\| &\leq \frac{7}{2} \log^+ R + \frac{13}{4} \log^+ \frac{d_l(1)}{d_\phi(1)} + \frac{1}{4} \log^+ \frac{d_l(N)}{d_\phi(N)} \\ &\quad + \frac{1}{4} \sum_{j=1}^{N-1} \left| \log \frac{d_\phi(j) d_l(j+1)}{d_l(j) d_\phi(j+1)} \right| \\ &\quad + 3 \cdot \int_1^{\min\{N, \hat{k}(R)\}} \log (R(d_l d_\phi)(s)) ds + 2 \cdot \int_{\min\{N, \hat{k}(R)\}}^N (R(d_l d_\phi)(s))^{\frac{1}{2}} ds. \end{aligned} \quad (2.10)$$

We see that

$$\log \|W(0, x_N; z)\| \leq \frac{7}{2} \cdot [\mathcal{Q}(N, R) + \lambda(N, R)].$$

Consider now the case that $N > \hat{k}(R)$. Since $\|\xi_{\phi_j} \xi_{\phi_j}^T J\| = 1$, we see that

$$\begin{aligned} \|W(x_{\lfloor \hat{k}(R) \rfloor}, x_N; z)\| &= \left\| \prod_{j=\lfloor \hat{k}(R) \rfloor+1}^N W(x_{j-1}, x_j; z) \right\| \\ &\leq \prod_{j=\lfloor \hat{k}(R) \rfloor+1}^N \left\| I + z l_j \xi_{\phi_j} \xi_{\phi_j}^T J \right\| \leq \prod_{j=\lfloor \hat{k}(R) \rfloor+1}^N (1 + R d_l(j)). \end{aligned}$$

Noting that

$$R d_l(\lfloor \hat{k}(R) \rfloor + 1) < 1,$$

and estimating the sum by an integral we arrive at

$$\log \|W(x_{\lfloor \hat{k}(R) \rfloor}, x_N; z)\| \leq \sum_{j=\lfloor \hat{k}(R) \rfloor+1}^N \log (1 + R d_l(j))$$

$$< \log 2 + \int_{[\hbar(R)]+1}^N R d_l(s) ds.$$

We combine this with (2.10) to obtain

$$\begin{aligned} \log \|W(0, x_N; z)\| &\leq \log \|W(0, x_{[\hbar(R)]}; z)\| + \log \|W(x_{[\hbar(R)]}, x_N; z)\| \\ &\leq \frac{7}{2} \cdot [\mathcal{Q}(N, R) + \lambda(\hbar(R), R)]. \end{aligned}$$

□

The second lemma accounts for the contribution of the remaining intervals.

2.10 Lemma *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^\infty$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Let $c_l, c_\phi: [1, \infty) \rightarrow (0, \infty)$ be two functions with $c_\phi \leq c_l$. Choose $\psi \in \mathbb{R}$ and assume that

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^\infty l_j \leq c_l(N) \wedge \sum_{j=N+1}^\infty l_j \sin^2(\phi_j - \psi) \leq c_\phi(N).$$

Then, for any $N \in \mathbb{N}$ and $z \in \mathbb{C}$,

$$\|W(x_N, L; z)\| \leq \left(\frac{c_l(N)}{c_\phi(N)} \right)^{\frac{1}{2}} \exp \left(2|z| (c_l c_\phi)^{\frac{1}{2}}(N) \right).$$

Proof Let $\Omega \in \text{GL}(2, \mathbb{R})$ and consider the function $\tilde{W}(t) := \Omega W(x_N, t; z) \Omega^{-1}$ defined on $[x_N, L]$. This function satisfies $\tilde{W}(x_N) = I$ and

$$\tilde{W}'(t) = -z \tilde{W}(t) \Omega H(t) J \Omega^{-1}, \quad t \in [x_N, L].$$

By Grönwall's Lemma,

$$\|\tilde{W}(t)\| \leq \exp \left(|z| \int_{x_N}^t \|\Omega H(s) J \Omega^{-1}\| ds \right).$$

Set $a := \left(\frac{c_\phi(N)}{c_l(N)} \right)^{\frac{1}{4}}$ and $\Omega = \Omega(a, \psi)$. Using Lemma 2.7, (i), (ii), we obtain

$$\begin{aligned} \|W(x_N, L; z)\| &\leq \|\Omega^{-1}\| \|\tilde{W}(L)\| \|\Omega\| \leq \frac{1}{a^2} \exp \left(|z| \int_{x_N}^L \|\Omega H(s) J \Omega^{-1}\| ds \right) \\ &= \left(\frac{c_l(N)}{c_\phi(N)} \right)^{\frac{1}{2}} \exp \left(|z| \sum_{j=N+1}^\infty l_j \cdot \|\Omega \xi_{\phi_j} \xi_{\phi_j}^T J \Omega^{-1}\| \right) \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{c_l(N)}{c_\phi(N)} \right)^{\frac{1}{2}} \exp \left(|z| \sum_{j=N+1}^{\infty} l_j \left[a^2 \cos^2(\phi_j - \psi) + \frac{1}{a^2} \sin^2(\phi_j - \psi) \right] \right) \\
 &\leq \left(\frac{c_l(N)}{c_\phi(N)} \right)^{\frac{1}{2}} \exp \left(|z| \left[a^2 \sum_{j=N+1}^{\infty} l_j + \frac{1}{a^2} \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \right] \right) \\
 &\leq \left(\frac{c_l(N)}{c_\phi(N)} \right)^{\frac{1}{2}} \exp \left(2|z| (c_l c_\phi)^{\frac{1}{2}}(N) \right)
 \end{aligned}$$

□

The proof of Theorem 2.2 is now easily completed.

Proof of Theorem 2.2 Let $(t, R) \in [1, \infty) \times \left[\frac{2}{(d_l d_\phi)(1)} \right)$ and set $N := \lceil t \rceil$. We use Lemma 2.9 and Lemma 2.10 to estimate, for $|z| = R$,

$$\begin{aligned}
 \log \|W_H(z)\| &\leq \log \|W(0, x_N; z)\| + \log \|W(x_N, L; z)\| \\
 &\leq \frac{7}{2} \cdot \left[g(N, R) + \lambda(\min\{N, \ell(R)\}, R) \right] \\
 &\quad + \left[\frac{1}{2} \log \frac{c_l(N)}{c_\phi(N)} + 2R (c_l c_\phi)^{\frac{1}{2}}(N) \right] \\
 &\leq \frac{7}{2} \cdot \left[g(t, R) + R (c_l c_\phi)^{\frac{1}{2}}(t) \right] + \frac{7}{2} \cdot |g(N, R) - g(t, R)| \\
 &\quad + \frac{7}{2} \cdot \lambda(\min\{N, \ell(R)\}, R) + \frac{1}{2} \log \frac{c_l(N)}{c_\phi(N)}.
 \end{aligned}$$

Since $0 \leq N - t \leq 1$ and

$$\begin{aligned}
 g(s, R) &\leq \begin{cases} \frac{3}{2} (\log^+ R + \log^+ \frac{d_l(1)}{d_\phi(1)}) & \text{if } 1 \leq s < \ell(R), \\ \sqrt{2} & \text{if } \ell(R) \leq s < \ell(R), \\ 1 & \text{if } \ell(R) \leq s, \end{cases} \\
 &\leq \frac{3}{2} \cdot \lambda(\min\{N, \ell(R)\}, R),
 \end{aligned}$$

we have

$$|g(N, R) - g(t, R)| \leq \frac{3}{2} \cdot \lambda(\min\{N, \ell(R)\}, R),$$

and obtain

$$\log \|W_H(z)\| \leq 9 \cdot \left[\max \left\{ g(t, R), R (c_l c_\phi)^{\frac{1}{2}}(t) \right\} + L(t, R) \right].$$

Since t was arbitrary, we can pass to the infimum on the right side.

□

3 Properties of the Upper Bound

In this section we study the expression on the right side of (2.6) independently of its meaning in the context of Theorem 2.2.

3.1 Definition Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally. Then we denote

$$B(R) := \inf_{t \geq 1} \left(\max \left\{ g(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\} + L(t, R) \right). \quad (3.1)$$

We consider the questions

- ▷ Is $L(t, R)$ small compared to $\max \left\{ g(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\}$?
- ▷ Does $\inf_{t \geq 1} \max \left\{ g(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\}$ depend monotonically on the data d_l, d_ϕ, c_l, c_ϕ ?

It turns out that already very weak assumptions on the data, concerning monotonicity, continuity, variation, and power boundedness, are sufficient to ensure that the answers are affirmative. This tells us that usually $L(t, R)$ can be dropped from (3.1), and hence $B(R)$ can be evaluated much more easily, and that usually the natural intuition which stems from the context of Theorem 2.2, that faster decay means smaller growth, is indeed reflected in $B(R)$.

3.1 About Monotonicity in the Data

The essence is the following result.

3.2 Proposition Let $d_l, \widehat{d}_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi, \widehat{d}_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp d_\phi \asymp \widehat{d}_l \asymp \widehat{d}_\phi \asymp 1$ locally, and assume that $\widehat{d}_l \widehat{d}_\phi$ is $\asymp$ to some nonincreasing function. If $\widehat{d}_l \lesssim d_l$ and $\widehat{d}_\phi \lesssim d_\phi$ on $[1, \infty)$, then (quantities $\widehat{g}(t, R)$ etc. are defined correspondingly for $\widehat{d}_l, \widehat{d}_\phi$)

$$\widehat{g}(t, R) \lesssim g(t, R) \quad (3.2)$$

for $R \geq \frac{2}{(d_l d_\phi)(1)}$ and a.e. (t, R) with $t < \widehat{\kappa}(R)$ or $t > \kappa(R)$. If additionally $\frac{\widehat{d}_\phi}{\widehat{d}_l}$ is bounded or $\asymp$ on $[1, \infty)$ to a nondecreasing function, then (3.2) holds for R sufficiently large and a.e. $t \in [1, \infty)$.

The constant that is implicit in the relation (3.2) depends on the constants implicit in the assumptions, but not on the actual functions $d_l, d_\phi, \widehat{d}_l, \widehat{d}_\phi$.

We start with a preparatory lemma.

3.3 Lemma Let $\Phi, \Psi: [1, \infty) \rightarrow (0, \infty)$ with Φ nondecreasing, and assume that $\alpha, \alpha' > 0$ are such that

$$\forall t \in [1, \infty): \alpha \cdot \Phi(t) \leq \Psi(t) \leq \alpha' \cdot \Phi(t).$$

Set

$$t_0 := \sup \left(\{t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \Psi(s) \leq 1\} \cup \{1\} \right) \in [1, \infty].$$

Then, if $t_0 < \infty$,

$$\forall t \in (t_0, \infty): \Psi(t) > \frac{\alpha}{\alpha'}.$$

Proof Since Φ is nondecreasing, we have

$$\begin{aligned} \{t \in [1, \infty) \mid \alpha' \Phi(t) \leq 1\} \cup \{1\} &= \{t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \alpha' \Phi(s) \leq 1\} \cup \{1\} \\ &\subseteq \{t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \Psi(s) \leq 1\} \cup \{1\}. \end{aligned}$$

Set

$$t_1 := \sup \left(\{t \in [1, \infty) \mid \alpha' \Phi(t) \leq 1\} \cup \{1\} \right) \in [1, \infty],$$

then $t_1 \leq t_0$. Again by monotonicity, $\alpha' \Phi(t) > 1$ for all $t > t_1$. and this yields that for $t > t_1$

$$\Psi(t) \geq \alpha \Phi(t) > \frac{\alpha}{\alpha'}.$$

□

Proof of Proposition 3.2 We have to trace constants, and thus make the constants from the assumptions explicit: choose a nondecreasing function u and $\alpha, \alpha' > 0$, such that

$$\alpha u(t) \leq \frac{2}{(\widehat{d}_l \widehat{d}_\phi)(t)} \leq \alpha' u(t), \quad t \in [1, \infty),$$

and choose $\kappa_l, \kappa_\phi > 0$, such that

$$\widehat{d}_l(t) \leq \kappa_l d_l(t), \quad \widehat{d}_\phi(t) \leq \kappa_\phi d_\phi(t), \quad t \in [1, \infty).$$

Moreover, set

$$\lambda := \max_{t \geq 2} \frac{\log t}{\sqrt{t}},$$

and note that

$$x \leq \gamma y \Rightarrow \log x \leq \left(1 + \frac{\log^+ \gamma}{\log 2}\right) \cdot \log y \quad \text{for } x > 0, y \geq 2, \gamma > 0.$$

Next, observe that

$$[R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} < \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}}, \quad t > \widehat{k}(R). \quad (3.3)$$

In order to see this, let R be fixed and apply Lemma 3.3 with $\Phi(t) = \frac{u(t)}{R}$ and $\Psi(t) = \frac{2}{R(\widehat{d}_l \widehat{d}_\phi)(t)}$. Then $t_0 = \widehat{k}(R)$, and hence for $t > \widehat{k}(R)$ we have $\Psi(t) > \frac{\alpha}{\alpha'}$, which implies (3.3).

In order to establish that $\widehat{g}(t, R) \lesssim g(t, R)$ we first assume $t < \widehat{h}(R)$ and distinguish the following six cases according to the definitions of $\widehat{g}(t, R)$ and $g(t, R)$.

▷▷ $1 \leq t < \widehat{k}(R) \wedge 1 \leq t < k(R)$:

$$\log [R(\widehat{d}_l \widehat{d}_\phi)(t)] \leq \left(1 + \frac{\log^+(\kappa_l \kappa_\phi)}{\log 2}\right) \log [R(d_l d_\phi)(t)].$$

▷▷ $1 \leq t < \widehat{k}(R) \wedge k(R) < t < h(R)$:

$$\log [R(\widehat{d}_l \widehat{d}_\phi)(t)] \leq \lambda [R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} \leq \lambda (\kappa_l \kappa_\phi)^{\frac{1}{2}} [R(d_l d_\phi)(t)]^{\frac{1}{2}}.$$

▷▷ $1 \leq t < \widehat{k}(R) \wedge h(R) < t$:

$$\begin{aligned} \log [R(\widehat{d}_l \widehat{d}_\phi)(t)] &\leq \lambda [R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} \leq \lambda [R(\widehat{d}_l \widehat{d}_\phi)(t)] \\ &\leq \lambda [R\widehat{d}_l(t)] \leq \lambda \kappa_l [Rd_l(t)]. \end{aligned}$$

▷▷ $\widehat{k}(R) < t < \widehat{h}(R) \wedge 1 \leq t < k(R)$. By (3.3),

$$[R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} < \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}} \leq \frac{1}{\log 2} \left(\frac{2\alpha'}{\alpha}\right)^{\frac{1}{2}} \log [R(d_l d_\phi)(t)].$$

▷▷ $\widehat{k}(R) < t < \widehat{h}(R) \wedge k(R) < t < h(R)$:

$$[R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} \leq (\kappa_l \kappa_\phi)^{\frac{1}{2}} [R(d_l d_\phi)(t)]^{\frac{1}{2}}.$$

▷▷ $\widehat{k}(R) < t < \widehat{h}(R) \wedge h(R) < t$:

$$[R(\widehat{d}_l \widehat{d}_\phi)(t)]^{\frac{1}{2}} \leq R\widehat{d}_l(t) \leq \kappa_l Rd_l(t).$$

Since (t, R) is assumed to satisfy $t < \widehat{h}(R) \vee h(R) < t$, and $t < \widehat{k}(R)$ was already covered, the only case left to treat is

▷▷ $\widehat{h}(R) < t \wedge h(R) < t$:

$$R\widehat{d}_l(t) \leq \kappa_l Rd_l(t).$$

Now we assume the additional condition on $\frac{\widehat{d}_\phi}{\widehat{d}_l}$. If this quotient is bounded, then $\widehat{h}(R) = \infty$ for sufficiently large R , and there is nothing to show. Otherwise, choose

a nondecreasing function v and $\beta, \beta' > 0$ such that

$$\beta v(t) \leq \frac{\widehat{d}_\phi}{d_l} \leq \beta' v(t), \quad t \in [1, \infty).$$

We claim that

$$R\widehat{d}_l(t) \leq \frac{\beta'}{\beta} \widehat{d}_\phi(t), \quad t > \widehat{h}(R). \quad (3.4)$$

In fact, if we fix R and apply Lemma 3.3 with $\Phi(t) = \frac{v(t)}{R}$ and $\Psi(t) = \frac{\widehat{d}_\phi(t)}{R\widehat{d}_l(t)}$, we find that $t_0 = \widehat{h}(R)$ and $\Psi(t) > \frac{\beta}{\beta'}$ for $t > \widehat{h}(R)$. This is equivalent to (3.4).

With these preparations, the assertion follows from the following simple inequalities:

$$\triangleright \triangleright \widehat{h}(R) < t \wedge 1 \leq t < \widehat{k}(R):$$

$$R\widehat{d}_l(t) \leq \frac{\beta'}{\beta} \widehat{d}_\phi(t) \leq \frac{\beta'}{\beta} \leq \frac{\beta'}{\beta \log 2} \log [R(d_l d_\phi)(t)].$$

$$\triangleright \triangleright \widehat{h}(R) < t \wedge \widehat{k}(R) < t < h(R):$$

$$R\widehat{d}_l(t) \leq \frac{\beta'}{\beta} \widehat{d}_\phi(t) \leq \frac{\beta'}{\beta} \kappa_\phi d_\phi(t) \leq \frac{\beta'}{\beta} \kappa_\phi [R(d_l d_\phi)(t)]^{\frac{1}{2}}.$$

□

3.4 Corollary Let $d_l, d_\phi, \widehat{d}_l, \widehat{d}_\phi$ be as in Proposition 3.2 and subject to the additional condition of this proposition. Further, let $c_l, c_\phi, \widehat{c}_l, \widehat{c}_\phi: [1, \infty) \rightarrow (0, \infty)$. If

$$\widehat{d}_l \lesssim d_l, \widehat{d}_\phi \lesssim d_\phi, \widehat{c}_l \lesssim c_l, \widehat{c}_\phi \lesssim c_\phi$$

on $[1, \infty)$, then

$$\inf_{t \geq 1} \max \{ \widehat{g}(t, R), R(\widehat{c}_l \widehat{c}_\phi)^{\frac{1}{2}}(t) \} \lesssim \inf_{t \geq 1} \max \{ g(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \}$$

for sufficiently large R . The constant that is implicit in the assertion depends on the constants implicit in the assumptions, but not on the actual functions $d_l, \widehat{d}_l$ etc.

3.2 About Smallness of $L(t, R)$

To start with, let us observe that $g(t, R)$ must grow at least logarithmically.

3.5 Lemma Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally. Then, for each $\varepsilon > 0$, there exists a positive constant $c > 0$ such that, for $t \geq 1 + \varepsilon$ and $R \geq \sup_{1 \leq s \leq 1+\varepsilon} \frac{2}{(d_l d_\phi)(s)}$ we have

$$g(t, R) \geq c \log R.$$

Proof Let $\eta := \sup_{1 \leq s \leq 1+\varepsilon} \frac{2}{(d_l d_\phi)(s)}$. Then $R \geq \eta$ is equivalent to $1 + \varepsilon \leq \mathfrak{k}(R)$. Hence

$$\begin{aligned} g(t, R) &\geq \int_1^{1+\varepsilon} \log(R(d_l d_\phi)(s)) \, ds \\ &= \left(\varepsilon + \frac{1}{\log R} \int_1^{1+\varepsilon} \log(d_l d_\phi)(s) \, ds \right) \cdot \log R. \end{aligned}$$

The term in the bracket tends to ε for $R \rightarrow \infty$, and hence we find $R_0 \geq \eta$ such that $g(t, R) \geq \frac{\varepsilon}{2} \cdot \log R$ for all $t \geq 1 + \varepsilon$ and $R \geq R_0$. The function g is nonzero, and hence bounded away from zero, on the compact set $\{1 + \varepsilon\} \times [\eta, R_0]$. By monotonicity, it is therefore also bounded away from zero on $[1 + \varepsilon, \infty) \times [\eta, R_0]$. Together, we see that a constant $c > 0$ can be chosen as required. $\square$

The most cumbersome part of $L(t, R)$ is the sum written as the last of the six summands in (2.5). Assuming monotonicity of the quotient $\frac{d_\phi}{d_l}$, this sum turns into a telescoping sum and can be estimated.

3.6 Lemma Let $d_l, d_\phi: [1, \infty) \rightarrow (0, \infty)$ and let $\mathfrak{k}(R)$ be as in (2.1). Assume that the quotient $\frac{d_\phi}{d_l}$ is eventually monotone but not eventually constant. Then

$$\sum_{j=1}^{\min\{\lceil t \rceil, \lfloor \mathfrak{k}(R) \rfloor\}-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \middle/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \asymp 1 + \left| \log \frac{d_\phi(\min\{\lceil t \rceil, \lfloor \mathfrak{k}(R) \rfloor\})}{d_l(\min\{\lceil t \rceil, \lfloor \mathfrak{k}(R) \rfloor\})} \right|$$

for $(t, R) \in [1, \infty) \times [\frac{d_\phi(1)}{d_l(1)}, \infty)$.

Proof We abbreviate $T := \min\{\lceil t \rceil, \lfloor \mathfrak{k}(R) \rfloor\}$. Choose $M \in \mathbb{N}$ such that $\frac{d_\phi}{d_l}$ is monotone on $[M, \infty)$ and $\frac{d_\phi(M)}{d_l(M)} \neq \frac{d_\phi(M+1)}{d_l(M+1)}$. For all (t, R) such that $T \geq M + 1$ we have

$$\begin{aligned} \sum_{j=M}^{T-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \middle/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| &= \left| \sum_{j=M}^{T-1} \log \left(\frac{d_\phi(j)}{d_l(j)} \middle/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \\ &= \left| \log \frac{d_\phi(M)}{d_l(M)} - \log \frac{d_\phi(T)}{d_l(T)} \right|. \end{aligned}$$

Clearly, this value is nondecreasing in T and positive for $T \geq M + 1$. If $\left| \log \frac{d_\phi(T)}{d_l(T)} \right|$ is bounded, it is $\asymp 1$. If $\left| \log \frac{d_\phi(T)}{d_l(T)} \right|$ is unbounded, it is $\asymp \left| \log \frac{d_\phi(T)}{d_l(T)} \right|$.

The beginning part of the sum which we cut off, i.e.,

$$\sum_{j=1}^{M-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \middle/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right|$$

is some nonnegative number independent of t and R . $\square$

3.7 Corollary Let $d_l, d_\phi: [1, \infty) \rightarrow (0, \infty)$ and let $\hbar(R)$ be as in (2.1). Assume that one of the following two assumptions holds.

- (i) The quotient $\frac{d_\phi}{d_l}$ is eventually nondecreasing and $\frac{d_\phi}{d_l}(t) \lesssim \sup_{1 \leq s < t} \frac{d_\phi}{d_l}(s)$ for sufficiently large t .
- (ii) The quotient $\frac{d_\phi}{d_l}$ is eventually nonincreasing and power bounded from below (i.e. there exists $\alpha > 0$ such that $\frac{d_\phi(t)}{d_l(t)} \gtrsim t^{-\alpha}$ for sufficiently large t).

Then

$$\begin{aligned} & \sum_{j=1}^{\min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \middle/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \\ & \lesssim \begin{cases} 1 + \log^+ R & \text{if (i) holds,} \\ 1 + \log t & \text{if (ii) holds,} \end{cases} \end{aligned}$$

for $(t, R) \in [1, \infty) \times [\frac{d_\phi(1)}{d_l(1)}, \infty)$.

Proof Again abbreviate $T := \min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}$, and let $M \in \mathbb{N}$ be such that $\frac{d_\phi}{d_l}$ is monotone on $[M, \infty)$. Under the assumption (i) we have

$$\frac{d_\phi(T)}{d_l(T)} \leq \frac{d_\phi(\hbar(R))}{d_l(\hbar(R))} \lesssim R,$$

and hence

$$\left| \log \frac{d_\phi(T)}{d_l(T)} \right| \lesssim 1 + \log R.$$

Under the assumption (ii) we have

$$t^{-\alpha} \lesssim T^{-\alpha} \lesssim \frac{d_\phi(T)}{d_l(T)} \lesssim 1,$$

and hence

$$\left| \log \frac{d_\phi(T)}{d_l(T)} \right| \lesssim 1 + \log t.$$

$\square$

Another type of condition on the quotient $\frac{d_\phi}{d_l}$ that ensures that the sum can be estimated, is that its variation is not too fast.

3.8 Lemma Let $d_l, d_\phi: [1, \infty) \rightarrow (0, \infty)$ and let $\hbar(R)$ be as in (2.1). Assume that the quotient $\frac{d_\phi}{d_l}$ can be represented as

$$\frac{d_\phi(t)}{d_l(t)} = c(t) \cdot \exp \left(\int_1^t \epsilon(u) \frac{du}{u} \right), \quad t \in [1, \infty),$$

where $\epsilon: [1, \infty) \rightarrow \mathbb{R}$ is locally integrable and eventually bounded, and where $c: [1, \infty) \rightarrow (0, \infty)$ is eventually constant. Then

$$\sum_{j=1}^{\min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \bigg/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| \lesssim 1 + \log t$$

for $(t, R) \in [1, \infty) \times [\frac{d_\phi(1)}{d_l(1)}, \infty)$.

Proof We again abbreviate $T := \min\{\lceil t \rceil, \lfloor \hbar(R) \rfloor\}$. For $M \in \mathbb{N}$ such that $\epsilon(t)$ is bounded and $c(t)$ is constant on $[M, \infty)$, we can estimate, for all (t, R) such that $T > M$,

$$\begin{aligned} \sum_{j=M}^{T-1} \left| \log \left(\frac{d_\phi(j)}{d_l(j)} \bigg/ \frac{d_\phi(j+1)}{d_l(j+1)} \right) \right| &= \sum_{j=M}^{T-1} \left| \int_j^{j+1} \epsilon(u) \frac{du}{u} \right| \\ &\leq \sum_{j=M}^{T-1} \int_j^{j+1} |\epsilon(u)| \frac{du}{u} = \int_M^T |\epsilon(u)| \frac{du}{u} \lesssim \log t. \end{aligned}$$

□

Now we give three sets of conditions, each of which ensures that $L(t, R)$ can be neglected.

3.9 Proposition Let $d_l: [1, \infty) \rightarrow (0, \infty)$ and $d_\phi: [1, \infty) \rightarrow (0, 1]$ be measurable with $d_l \asymp 1 \asymp d_\phi$ locally, let c_l, c_ϕ be continuous and nonincreasing with $\lim_{t \rightarrow \infty} (c_l c_\phi)(t) = 0$, and let $T(R)$ be the unique solution of (2.7). Assume that one of the following three sets of assumptions holds.

- (i) The quotient $\frac{d_\phi}{d_l}$ is eventually nondecreasing and $\frac{d_\phi}{d_l}(t) \lesssim \sup_{1 \leq s < t} \frac{d_\phi}{d_l}(s)$ for sufficiently large t . We have $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t .
- (ii) The quotient $\frac{d_\phi}{d_l}$ is eventually nonincreasing and power bounded from below. There exists $\alpha > 0$ such that $(c_l c_\phi)(t) \lesssim t^{-\alpha}$ for sufficiently large t . The quotient $\frac{c_l}{c_\phi}$ is power bounded from above or $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t .
- (iii) The quotient $\frac{d_\phi}{d_l}$ can be represented as

$$\frac{d_\phi(t)}{d_l(t)} = c(t) \cdot \exp \left(\int_1^t \epsilon(u) \frac{du}{u} \right), \quad t \in [1, \infty),$$

where $\epsilon: [1, \infty) \rightarrow \mathbb{R}$ is locally integrable and eventually bounded, and where $c: [1, \infty) \rightarrow (0, \infty)$ is eventually constant. There exists $\alpha > 0$ such that $(c_l c_\phi)(t) \lesssim t^{-\alpha}$ for sufficiently large t . The quotient $\frac{c_l}{c_\phi}$ is power bounded from above or $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t .

Let $B(R)$ be as in (3.1). Then $L(T(R), R) \lesssim 1 + \log^+ R$, and

$$B(R) \asymp g(T(R), R) = R(c_l c_\phi)^{\frac{1}{2}}(T(R)).$$

Proof Assume (i). The first sentence of (i) is nothing but the assumption (i) in Corollary 3.7, and hence the sum in the definition of $L(t, R)$ is $\lesssim 1 + \log^+ R$ independently of t . The last but one summand in (2.5) is $\lesssim 1$ by monotonicity of $\frac{d_\phi}{d_l}$. It remains to estimate the third summand in (2.5), and here we use the second sentence in the present assumption: using $\lim_{R \rightarrow \infty} T(R) = \infty$ and Lemma 3.5,

$$\begin{aligned} \log^+ \frac{c_l(\lceil T(R) \rceil)}{c_\phi(\lceil T(R) \rceil)} &\lesssim 1 + \log^+ \frac{c_l(T(R))}{c_\phi(T(R))} \lesssim 1 + \log^+ \frac{1}{(c_l c_\phi)(T(R))} \\ &= 1 + \log^+ \frac{R^2}{g(T(R), R)^2} \lesssim 1 + \log^+ R. \end{aligned} \quad (3.5)$$

Assume (ii). Let $\alpha > 0$ be as in the second sentence of (ii), then $R(c_l c_\phi)^{\frac{1}{2}}(R^{\frac{2}{\alpha}}) \lesssim 1 \ll g(R^{\frac{2}{\alpha}}, R)$, and hence $T(R) \lesssim R^{\frac{2}{\alpha}}$. The first sentence of (ii) is nothing but assumption (ii) in Corollary 3.7. This yields that the sum in the definition of $L(t, R)$ is $\lesssim 1 + \log^+ t$, and hence the sum in $L(T(R), R)$ is $\lesssim 1 + \log^+ R$. The same holds for the last but one summand in (2.5) by power boundedness of $\frac{d_\phi}{d_l}$, and for the third summand (2.5) in case that $\frac{c_l}{c_\phi}$ is power bounded. If $c_\phi(t) \lesssim c_\phi(t+1)$ for sufficiently large t , then the third summand in (2.5) is $\lesssim 1 + \log^+ R$ by (3.5).

Assume (iii). The second sentence again ensures that $T(R)$ is power bounded from above. The first sentence is the assumption of Lemma 3.8, and it follows that the sum in $L(T(R), R)$ is $\lesssim 1 + \log^+ R$. The form of $\frac{d_\phi}{d_l}$ implies that this quotient is power bounded from below, and hence the last but one summand (2.5) is $\lesssim 1 + \log^+ R$. Concerning the third summand, just argue as above.

To finish the proof it remains to refer to Lemma 3.5. $\square$

4 Regularly Varying Decay

In the preceding section, we studied properties of the function $B(R)$, in particular, we saw that the term $L(t, R)$ is usually neglectable. Our next goal is to explicitly evaluate

$$\min_{t \geq 1} \left(\max \left\{ g(t, R), R(c_l c_\phi)^{\frac{1}{2}}(t) \right\} \right)$$

in the situation that d_l, d_ϕ, c_l, c_ϕ are all regularly varying. Recall here the notion of regular variation in Karamata sense and in particular the index of a regularly varying function from Definition A.1.

The first result in this section is Theorem 4.1. As in the preceding section, this is a pure analysis of functions, and independent of the meaning in the context of Theorem 2.2. However, after having completed the proof of Theorem 4.1, we will return to the study of Hamburger Hamiltonians and combine Theorem 4.1 with Theorem 2.2. This establishes an explicit upper bound for the growth of the monodromy matrix, cf. Theorem 4.6.

The setup for Theorem 4.1 is as follows:

- (i) We are given regularly varying functions $d_l, d_\phi, c_l, c_\phi: [1, \infty) \rightarrow (0, \infty)$ with nonpositive indices, $d_\phi \leq 1$, and $d_l \asymp 1 \asymp d_\phi$ locally.
- (ii) We denote

$$\delta_l := -\text{ind } d_l, \quad \delta_\phi := -\text{ind } d_\phi, \quad \gamma_l := -\text{ind } c_l, \quad \gamma_\phi := -\text{ind } c_\phi,$$

and further

$$\mathcal{D}(t) := \frac{1}{(d_l d_\phi)(t)}, \quad \delta := \text{ind } \mathcal{D} = \delta_l + \delta_\phi,$$

$$\mathcal{C}(t) := \frac{1}{(c_l c_\phi)^{\frac{1}{2}}(t)}, \quad \gamma := \text{ind } \mathcal{C} = \frac{\gamma_l + \gamma_\phi}{2}.$$

- (iii) The quantities of interest are

$$\mathcal{B}(t, R) := \max \left\{ \mathcal{Q}(t, R), \frac{R}{\mathcal{C}(t)} \right\},$$

$$\mathcal{B}(R) := \min_{t \geq 1} \mathcal{B}(t, R).$$

Four fundamentally different cases occur. They depend on the relative size of $\mathcal{D}(t)$ vs. $\mathcal{C}(t)$, and on the absolute size of $\mathcal{D}(t)$. Moreover, there are a few exceptional cases that have to be ruled out.

4.1 Theorem *Assume we are given data as in (i) above, and let notation be as in (ii) and (iii). Assume further that $\delta > 0$, that $\mathcal{C}$ is $\asymp$ on $[1, \infty)$ to a nondecreasing function, and that $\lim_{t \rightarrow \infty} \mathcal{C}(t) = \infty$.*

Then the following estimates for $\mathcal{B}(R)$ hold for sufficiently large R .

A Case $\mathcal{D}(t) \lesssim t \mathcal{C}(t)$:

Choose $\alpha \geq 4 \cdot \sup_{t \geq 1} \frac{\mathcal{D}(t)}{t \mathcal{C}(t)}$, and set $\ell(t) := t \mathcal{C}(t) \log \left[\alpha \frac{t \mathcal{C}(t)}{\mathcal{D}(t)} \right]$. Then an asymptotic inverse of ℓ exists, and with $\tau(R) := \ell^{-1} \left(\frac{1}{\alpha} R \right)$ we have

$$\mathcal{B}(R) \asymp \mathcal{B}(\tau(R), R) \asymp \frac{R}{\mathcal{C}(\ell^{-1}(R))}. \quad (4.1)$$

The function on the right side is regularly varying with index $\frac{1}{1+\gamma}$.

If $\delta < 1 + \gamma$ we have $\ell(t) \asymp t\mathcal{C}(t) \log t$, and $\mathcal{B}(R)$ is (up to $\asymp$) independent of d_l and d_ϕ .

- B** Case $t\mathcal{C}(t) \lesssim \mathcal{D}(t)$, $\int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds < \infty$, and if $(\delta_l, \delta_\phi, \gamma) = (1, 1, 0)$ then $\frac{d_\phi}{d_l}$ is bounded or $\asymp$ to a monotone function:

Then

$$\mathcal{K}(R) \lesssim \mathcal{B}(R) \lesssim R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds. \quad (4.2)$$

Both bounds in (4.2) are regularly varying with index $\frac{1}{\delta}$.

If $\delta > 2$, then $\asymp$ holds throughout (4.2) and $\mathcal{B}(R)$ is (up to $\asymp$) independent of c_l and c_ϕ .

- C** Case $\frac{1}{d_l(t)} \lesssim t\mathcal{C}(t) \lesssim \mathcal{D}(t)$, $\int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds = \infty$, and $(\delta, \gamma) \neq (2, 0)$:

Set $\ell_0(t) := \frac{t^2\mathcal{C}(t)^2}{\mathcal{D}(t)}$ and $\ell_1(t) := \left(\mathcal{C}(t) \int_1^t \mathcal{D}(s)^{-\frac{1}{2}} ds \right)^2$. Then asymptotic inverses ℓ_0^-, ℓ_1^- exist, and with

$$\tau(R) := \min \left\{ \max \left\{ \ell_1^-(R), \mathcal{K}(R) \right\}, \mathcal{H}(R) \right\}$$

we have

$$\frac{R}{\mathcal{C}(\ell_0^-(R))} \lesssim \mathcal{B}(R) \lesssim \mathcal{B}(\tau(R), R) \lesssim \frac{R}{\mathcal{C}(\ell_1^-(R))}. \quad (4.3)$$

Both bounds in (4.3) are regularly varying with index $\frac{2-\delta+\gamma}{2-\delta+2\gamma}$.

If $\delta < 2$, then $\asymp$ holds throughout (4.3).

- D** Case $t\mathcal{C}(t) \lesssim \frac{1}{d_l(t)}$, $\int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds = \infty$, and $\int_1^\infty d_l(s) ds < \infty$:

Then $\frac{d_\phi}{d_l}$ is unbounded and

$$R\mathcal{H}(R)d_l(\mathcal{H}(R)) \lesssim \mathcal{B}(R) \lesssim R^{\frac{1}{2}} \int_1^{\mathcal{H}(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds + R \int_{\mathcal{H}(R)}^\infty d_l(s) ds. \quad (4.4)$$

If $\delta_l > \delta_\phi$, then both bounds in (4.4) are regularly varying with index $\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$.

If $\delta < 2$ and $\delta_l > 1$, then $\asymp$ holds throughout (4.4).

4.1 Proof of Theorem 4.1

We start with two lemmata that determine the function $\mathcal{G}(t, R)$ on the intervals $[1, \mathcal{K}(R)]$ and $[\mathcal{K}(R), \mathcal{H}(R)]$, respectively.

4.2 Lemma Let $\mathcal{D}: [1, \infty) \rightarrow (0, \infty)$ be regularly varying with $\text{ind } \mathcal{D} > 0$ and $\mathcal{D} \asymp 1$ locally. Set

$$\mathcal{K}(R) := \sup \left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{2\mathcal{D}(s)}{R} \leq 1 \right\}, \quad R \geq 2\mathcal{D}(1). \quad (4.5)$$

Then we have

$$\int_1^t \log \frac{R}{\mathcal{D}(s)} ds \asymp t \log \frac{R}{\mathcal{D}(t)} \quad (4.6)$$

for $R \in [2\mathcal{D}(1), \infty)$ and $t \in [2 \log \frac{R}{\mathcal{D}(1)}, \mathcal{K}(R)]$.

Proof In the first part of the proof we establish the assertion under the additional assumption that $\mathcal{D}$ is continuously differentiable with $\mathcal{D}'(t) > 0$ for all $t \geq 1$. Let $R \in [2\mathcal{D}(1), \infty)$ and $t \in [2 \log \frac{R}{\mathcal{D}(1)}, \mathcal{K}(R)]$. Integration by parts yields that

$$\int_1^t \log \frac{R}{\mathcal{D}(s)} ds = t \log \frac{R}{\mathcal{D}(t)} - \log \frac{R}{\mathcal{D}(1)} + \int_1^t s \frac{\mathcal{D}'(s)}{\mathcal{D}(s)} ds. \quad (4.7)$$

The relation “ $\gtrsim$ ” in (4.6) readily follows: The integral on the right side is positive, and by the definition of $\mathcal{K}(R)$ we have $\frac{R}{\mathcal{D}(t)} \geq 2$. For $t \geq 2 \log \frac{R}{\mathcal{D}(1)}$ it follows that

$$\begin{aligned} \int_1^t \log \frac{R}{\mathcal{D}(s)} ds &\geq \left(\frac{1}{2} \cdot t \log \frac{R}{\mathcal{D}(t)} + \frac{1}{2} \cdot 2 \log \frac{R}{\mathcal{D}(1)} \cdot \log 2 \right) - \log \frac{R}{\mathcal{D}(1)} \\ &\geq \frac{1}{2} \cdot t \log \frac{R}{\mathcal{D}(t)}. \end{aligned}$$

To show that “ $\lesssim$ ” holds in (4.6) we make a change of variable and use Karamata’s theorem. The inverse function $\mathcal{D}^{-1}$ exists and is continuously differentiable with positive derivative, and is regularly varying with positive index. We obtain that, for $v \rightarrow \infty$,

$$\int_1^v s \frac{\mathcal{D}'(s)}{\mathcal{D}(s)} ds = \int_{\mathcal{D}(1)}^{\mathcal{D}(v)} \mathcal{D}^{-1}(u) \frac{du}{u} \sim (\text{ind } \mathcal{D}) \cdot v.$$

As a function of $v \in [2 \log 2, \infty)$, both v and the integral above are continuous and nonzero. Hence

$$\int_1^v s \frac{\mathcal{D}'(s)}{\mathcal{D}(s)} ds \asymp v, \quad v \in [2 \log 2, \infty).$$

Referring to (4.7) and again using that $\frac{R}{\mathcal{D}(t)} \geq 2$, we obtain

$$\int_1^t \log \frac{R}{\mathcal{D}(s)} ds \leq t \log \frac{R}{\mathcal{D}(t)} + \int_1^t s \frac{\mathcal{D}'(s)}{\mathcal{D}(s)} ds \lesssim t \log \frac{R}{\mathcal{D}(t)}.$$

The second part of the proof is to reduce to the above settled special case. Hence, assume that $\mathcal{D}$ is given as in the statement of the lemma. Lemma A.9 gives a regularly varying function $\mathcal{D}_1$ which is continuously differentiable with $\mathcal{D}'_1 > 0$, such that

$\mathcal{D} \asymp \mathcal{D}_1$ on $[1, \infty)$. Dividing $\mathcal{D}_1$ by a sufficiently large constant, we obtain a function $\tilde{\mathcal{D}}$ with the properties listed above and

$$\frac{1}{\alpha} \mathcal{D}(t) \leq \tilde{\mathcal{D}}(t) \leq \mathcal{D}(t) \quad \text{for } t \geq 1$$

for some $\alpha \geq 1$.

Clearly, $\mathcal{K}(R) \leq \tilde{\mathcal{K}}(R)$ (where $\tilde{\mathcal{K}}(R)$ is defined by (4.5) with $\tilde{\mathcal{D}}$ in place of $\mathcal{D}$). It follows that for $t < \mathcal{K}(R)$

$$\log \frac{R}{\mathcal{D}(t)} \leq \log \frac{R}{\tilde{\mathcal{D}}(t)} \leq \log \left(\alpha \frac{R}{\mathcal{D}(t)} \right) \leq \log \frac{R}{\mathcal{D}(t)} + \log \alpha \leq \left(1 + \frac{\log \alpha}{\log 2} \right) \log \frac{R}{\mathcal{D}(t)}.$$

We see that

$$\int_1^t \log \frac{R}{\mathcal{D}(s)} \, ds \asymp \int_1^t \log \frac{R}{\tilde{\mathcal{D}}(s)} \, ds \asymp t \log \frac{R}{\tilde{\mathcal{D}}(t)} \asymp t \log \frac{R}{\mathcal{D}(t)}.$$

□

4.3 Lemma *Let $\mathcal{D}: [1, \infty) \rightarrow (0, \infty)$ be regularly varying with $\text{ind } \mathcal{D} > 0$ and $\mathcal{D} \asymp 1$ locally. Then we have*

$$R^{\frac{1}{2}} t \mathcal{D}(t)^{-\frac{1}{2}} \lesssim \mathcal{K}(R) + R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^t \mathcal{D}(s)^{-\frac{1}{2}}(s) \, ds \lesssim R^{\frac{1}{2}} \int_1^t \mathcal{D}(s)^{-\frac{1}{2}} \, ds$$

for $R \in [2\mathcal{D}(1), \infty)$ and $t \in [\mathcal{K}(R), \infty)$.

Proof The asserted estimate from above is easy to see: by Karamata's theorem we have

$$\mathcal{K}(R) \asymp R^{\frac{1}{2}} \mathcal{K}(R) \mathcal{D}(\mathcal{K}(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_1^{\mathcal{K}(R)} \mathcal{D}(s)^{-\frac{1}{2}} \, ds.$$

To obtain an estimate from below, we use Lemma A.9. This gives a regularly varying function $\mathcal{D}_1$ which is continuously differentiable with $\mathcal{D}'_1 > 0$ and satisfies $\mathcal{D} \asymp \mathcal{D}_1$. By multiplying $\mathcal{D}_1$ with a sufficiently large constant, we obtain a function $\tilde{\mathcal{D}}$ with the properties listed above and

$$\mathcal{D}(t) \leq \tilde{\mathcal{D}}(t) \leq \alpha \mathcal{D}(t)$$

with some $\alpha \geq 1$.

Consider the functions

$$\ell_1(t) := R^{\frac{1}{2}} t \tilde{\mathcal{D}}(t)^{-\frac{1}{2}}, \quad \ell_2(t) := \sqrt{2} \mathcal{K}(R) + R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^t \tilde{\mathcal{D}}(s)^{-\frac{1}{2}} \, ds.$$

Then

$$\ell_1(\mathcal{K}(R)) = R^{\frac{1}{2}} \mathcal{K}(R) \tilde{\mathcal{D}}(\mathcal{K}(R))^{-\frac{1}{2}}$$

$$\leq R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} = \sqrt{2} \mathfrak{k}(R) = \mathfrak{f}_2(\mathfrak{k}(R)),$$

and, for $t > \mathfrak{k}(R)$,

$$\mathfrak{f}'_1(t) = R^{\frac{1}{2}} \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} + R^{\frac{1}{2}} t \underbrace{\frac{d}{dt} \left(\tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} \right)}_{<0} \leq R^{\frac{1}{2}} \tilde{\mathfrak{D}}(t)^{-\frac{1}{2}} = \mathfrak{f}'_2(t).$$

We see that

$$\mathfrak{f}_1(t) \leq \mathfrak{f}_2(t) \quad \text{for } t \geq \mathfrak{k}(R).$$

It remains to note that

$$\mathfrak{f}_1(t) \gtrsim R^{\frac{1}{2}} t \mathfrak{D}(t)^{-\frac{1}{2}}, \quad \mathfrak{f}_2(t) \lesssim \mathfrak{k}(R) + R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^t \mathfrak{D}(s)^{-\frac{1}{2}} ds.$$

□

4.4 Lemma Consider regularly varying functions d_l , d_ϕ , $\mathfrak{D}$ as introduced in the discussion preceding Theorem 4.1. Suppose that $\int_1^\infty \mathfrak{D}(s)^{-\frac{1}{2}} ds < \infty$. If $\delta_\phi = \delta_l = 1$, we also assume that $\frac{d_\phi}{d_l}$ is bounded or $\asymp$ to a nondecreasing function. Then for sufficiently large R we have, independently of t , that

$$\mathfrak{g}(t, R) \lesssim R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^\infty \mathfrak{D}(s)^{-\frac{1}{2}} ds.$$

Proof We distinguish cases.

▷ Assume that $\frac{d_\phi}{d_l}$ is bounded, i.e., $\mathfrak{h}(R) = \infty$ for all sufficiently large R . This is surely the case if $\delta_l < \delta_\phi$.

By Karamata's theorem

$$\mathfrak{k}(R) = \sqrt{2} R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} \lesssim R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^\infty \mathfrak{D}(s)^{-\frac{1}{2}} ds.$$

Now it follows that, for all $t \geq \mathfrak{k}(R)$,

$$\mathfrak{g}(t, R) \asymp \mathfrak{k}(R) + R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^t \mathfrak{D}(s)^{-\frac{1}{2}} ds \lesssim R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^\infty \mathfrak{D}(s)^{-\frac{1}{2}} ds.$$

▷ Assume that $\frac{d_\phi}{d_l}$ is $\asymp$ to a nondecreasing function. This is surely the case if $\delta_l > \delta_\phi$. If R is large enough and $t > \mathfrak{k}(R)$,

$$R d_l(t) = \left(\frac{R d_l(t)}{d_\phi(t)} \right)^{\frac{1}{2}} \cdot [R(d_l d_\phi)(t)]^{\frac{1}{2}} \lesssim [R(d_l d_\phi)(t)]^{\frac{1}{2}}.$$

We see that also in this case,

$$\begin{aligned} q(t, R) &\asymp \hbar(R) + R^{\frac{1}{2}} \int_{\hbar(R)}^{\min\{t, \hbar(R)\}} \mathcal{D}(s)^{-\frac{1}{2}} ds + R \int_{\min\{t, \hbar(R)\}}^t d_l(s) ds \\ &\lesssim R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds. \end{aligned}$$

▷ Assume that $\delta_l = \delta_\phi > 1$.

By Karamata's theorem,

$$R \int_{\hbar(R)}^{\infty} d_l(s) ds \asymp R \hbar(R) d_l(\hbar(R)) \asymp \hbar(R) d_\phi(\hbar(R)).$$

Since $\delta_\phi > 1$, the function $t \mapsto t d_\phi(t)$ is eventually decreasing. Since it is locally bounded, it is even bounded on $[1, \infty)$. Again we see that $q(t, R) \lesssim R^{\frac{1}{2}} \int_{\hbar(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds$ independently of t .

□

4.5 Lemma *In the situation of Theorem 4.1, the inequality*

$$\min \left\{ q(s, R), \frac{R}{\mathcal{C}(s)} \right\} \lesssim \mathcal{B}(R) \leq \mathcal{B}(t, R) = \max \left\{ q(t, R), \frac{R}{\mathcal{C}(t)} \right\} \quad (4.8)$$

holds for $(s, t, R) \in [1, \infty)^2 \times [\frac{2}{(d_l d_\phi)(1)}, \infty)$. In particular, if $\tau(R)$ is a function satisfying $q(\tau(R), R) \asymp \frac{R}{\mathcal{C}(\tau(R))}$, then

$$\mathcal{B}(R) \asymp q(\tau(R), R) \asymp \frac{R}{\mathcal{C}(\tau(R))}.$$

Proof The upper bound in (4.8) is obvious from the definition of $\mathcal{B}(t, R)$ and $\mathcal{B}(R)$. For the lower bound, we observe that

$$\begin{aligned} s \leq t &\Rightarrow q(s, R) \leq q(t, R), \\ s > t &\Rightarrow \frac{R}{\mathcal{C}(s)} \lesssim \frac{R}{\mathcal{C}(t)} \end{aligned}$$

and hence

$$\min \left\{ q(s, R), \frac{R}{\mathcal{C}(s)} \right\} \lesssim \mathcal{B}(t, R) \quad (s, t) \in [1, \infty)^2.$$

Taking the infimum over t proves the lower bound in (4.8).

□

Now we are ready for the proof of Theorem 4.1.

Proof of Case A We have $\text{ind } \ell = 1 + \gamma > 0$, and hence ℓ^- exists. Set

$$\tau(R) := \ell^-\left(\frac{1}{\alpha}R\right).$$

Then, for $R \rightarrow \infty$,

$$\frac{1}{\alpha}R \sim \ell(\tau(R)) = \tau(R)\mathcal{C}(\tau(R)) \log \left[\alpha \frac{\tau(R)\mathcal{C}(\tau(R))}{\mathcal{D}(\tau(R))} \right],$$

and hence

$$\frac{R}{\mathcal{D}(\tau(R))} \sim \left(\alpha \frac{\tau(R)\mathcal{C}(\tau(R))}{\mathcal{D}(\tau(R))} \right) \log \left[\alpha \frac{\tau(R)\mathcal{C}(\tau(R))}{\mathcal{D}(\tau(R))} \right].$$

Our choice of α implies that $2\mathcal{D}(\tau(R)) \leq R$ for all sufficiently large values of R , and therefore that $\tau(R) \leq \ell(R)$ for such R .

It holds that

$$\frac{t}{\log t} \circ t \log t \asymp t \quad \text{for } t \geq 4,$$

and we further obtain

$$\frac{\frac{R}{\mathcal{D}(\tau(R))}}{\log\left(\frac{R}{\mathcal{D}(\tau(R))}\right)} \asymp \frac{\tau(R)\mathcal{C}(\tau(R))}{\mathcal{D}(\tau(R))}.$$

Using this and Lemma 4.2, thus

$$\frac{R}{\mathcal{C}(\tau(R))} \asymp \tau(R) \log \left(\frac{R}{\mathcal{D}(\tau(R))} \right) \asymp g(\tau(R), R).$$

In view of Lemma 4.5, this completes the proof of (4.1).

To see the additional statements, note first that

$$\text{ind } \frac{R}{\mathcal{C}(\ell^-(R))} = 1 - \gamma \cdot \frac{1}{1 + \gamma} = \frac{1}{1 + \gamma}.$$

Further $\delta < 1 + \gamma$ implies $\text{ind } \frac{\ell^-\mathcal{C}(t)}{\mathcal{D}(t)} > 0$, and thus $\log \left(\alpha \frac{\ell^-\mathcal{C}(t)}{\mathcal{D}(t)} \right) \asymp \log t$. □

Proof of Case B To obtain the bound from below, we use Lemma 4.2 and our assumption that $\ell^-\mathcal{C}(t) \lesssim \mathcal{D}(t)$. Setting $t = \ell(R) \asymp \mathcal{D}^-(R)$, we find that

$$g(\ell(R), R) \asymp \ell(R), \quad \frac{R}{\mathcal{C}(\ell(R))} \asymp \frac{R}{\mathcal{C}(\mathcal{D}^-(R))} \gtrsim \mathcal{D}^-(R) \asymp \ell(R).$$

Now we prove the upper bound for $\mathcal{B}(R)$. If Lemma 4.4 is applicable, then

$$g(t, R) \lesssim R^{\frac{1}{2}} \int_{\ell(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds.$$

Choosing $\tau(R)$ so large that $\frac{R}{\mathcal{C}(\tau(R))}$ is less than the right hand side of this relation, we see that

$$\mathcal{B}(R) \leq \mathcal{B}(\tau(R), R) \lesssim R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds.$$

If Lemma 4.4 is not applicable, we must have $\delta_l = \delta_\phi = 1$ and $\gamma > 0$. Choose ρ so large that

$$\text{ind } \mathcal{K} = \frac{1}{\delta} > 1 - \gamma\rho = \text{ind} \left(\frac{R}{\mathcal{C}(R^\rho)} \right).$$

Since $\frac{d_\phi}{d_l}$ is slowly varying, we have $R^\rho \lesssim \mathcal{K}(R)$, cf. Lemma A.8. Therefore,

$$q(\mathcal{K}(R), R) \gtrsim \mathcal{K}(R) \gg \frac{R}{\mathcal{C}(R^\rho)} \geq \frac{R}{\mathcal{C}(\mathcal{K}(R))}.$$

We thus obtain

$$\mathcal{B}(R) \lesssim \mathcal{B}(\mathcal{K}(R), R) \asymp q(\mathcal{K}(R), R) \lesssim R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds.$$

To see the additional statements, note first that $\text{ind } \mathcal{K} = \frac{1}{\delta}$ and also

$$\text{ind} \left(R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds \right) = \frac{1}{2} + \frac{1}{\delta} \cdot \left(1 - \frac{\delta}{2} \right) = \frac{1}{\delta}.$$

Further, $\delta > 2$ implies that

$$R^{\frac{1}{2}} \int_{\mathcal{K}(R)}^{\infty} \mathcal{D}(s)^{-\frac{1}{2}} ds \asymp R^{\frac{1}{2}} \mathcal{K}(R) \mathcal{D}(\mathcal{K}(R))^{-\frac{1}{2}} \asymp \mathcal{K}(R).$$

□

Proof of Case $\boxed{C}$ We have

$$\text{ind } \ell_0 = 2 - \delta + 2\gamma, \quad \text{ind } \ell_1 = 2 \left(\gamma + \left(1 - \frac{\delta}{2} \right) \right) = 2 - \delta + 2\gamma.$$

Since $\mathcal{D}^{-\frac{1}{2}}$ is not integrable, we must have $\delta \leq 2$. The case that $\delta = 2$ and $\gamma = 0$ is ruled out by assumption, and it follows that $2 - \delta + 2\gamma > 0$. Thus, asymptotic inverses ℓ_0^- and ℓ_1^- exist.

To prove the bound from below, we use $\ell_0^-(R)$ pushed into the interval $[\mathcal{K}(R), \mathcal{K}(R)]$, namely we set

$$\tau(R) := \min \left\{ \max \left\{ \ell_0^-(R), \mathcal{K}(R) \right\}, \mathcal{K}(R) \right\}.$$

Our assumption that $\frac{1}{d_l(t)} \lesssim t \mathcal{C}(t) \lesssim \mathcal{D}(t)$ gives

$$\frac{d_\phi(t)}{d_l(t)} \lesssim f_0(t) \lesssim \mathcal{D}(t),$$

and hence $k(R) \lesssim f_0^-(R) \lesssim \hbar(R)$. Thus, $\tau(R) \asymp f_0^-(R)$.

Using Lemma 4.3 we obtain

$$g(\tau(R), R) \gtrsim R^{\frac{1}{2}} \tau(R) \mathcal{D}(\tau(R))^{-\frac{1}{2}} \asymp R^{\frac{1}{2}} f_0^-(R) \mathcal{D}(f_0^-(R))^{-\frac{1}{2}}.$$

On the other hand, the definition of f_0 yields for $R \rightarrow \infty$ that

$$R \sim f_0(f_0^-(R)) = \frac{f_0^-(R)^2 \mathcal{C}(f_0^-(R))^2}{\mathcal{D}(f_0^-(R))},$$

and hence

$$\frac{R^{\frac{1}{2}}}{\mathcal{C}(f_0^-(R))} \sim f_0^-(R) \mathcal{D}(f_0^-(R))^{-\frac{1}{2}}.$$

Together, thus,

$$\min \left\{ g(\tau(R), R), \frac{R}{\mathcal{C}(\tau(R))} \right\} \gtrsim R^{\frac{1}{2}} f_0^-(R) \mathcal{D}(f_0^-(R))^{-\frac{1}{2}} \asymp \frac{R}{\mathcal{C}(f_0^-(R))}.$$

For the proof of the upper bound we use

$$\tau(R) := \min \left\{ \max \{ f_1^-(R), \hbar(R) \}, \hbar(R) \right\}.$$

Since $f_0 \lesssim f_1$ we have $f_1^- \lesssim f_0^- \lesssim \hbar$, and therefore $f_1^-(R) \lesssim \tau(R)$. This readily implies that

$$\frac{R}{\mathcal{C}(\tau(R))} \lesssim \frac{R}{\mathcal{C}(f_1^-(R))}.$$

To estimate $g(\tau(R), R)$ we distinguish two cases. First assume that $f_1^-(R) \leq \hbar(R)$. Then $\tau(R) = \hbar(R)$, and hence

$$g(\tau(R), R) \asymp \hbar(R) \lesssim \frac{R}{\mathcal{C}(\hbar(R))} \leq \frac{R}{\mathcal{C}(f_1^-(R))}.$$

Second assume that $f_1^-(R) \geq \hbar(R)$. Then $\tau(R) \leq f_1^-(R)$. We use Lemma 4.3 to obtain

$$g(\tau(R), R) \lesssim R^{\frac{1}{2}} \int_1^{\tau(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds \lesssim R^{\frac{1}{2}} \int_1^{f_1^-(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds.$$

The last integral can be evaluated from the definition of ℓ_1 : for $R \rightarrow \infty$

$$R \sim \ell_1(\ell_1^-(R)) = \left(\mathcal{C}(\ell_1^-(R)) \int_1^{\ell_1^-(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds \right)^2,$$

and hence

$$\int_1^{\ell_1^-(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds \sim \frac{R^{\frac{1}{2}}}{\mathcal{C}(\ell_1^-(R))}.$$

Altogether we see that always

$$\max \left\{ \mathcal{Q}(\tau(R), R), \frac{R}{\mathcal{C}(\tau(R))} \right\} \lesssim \frac{R}{\mathcal{C}(\ell_1^-(R))}.$$

This completes the proof of (4.3).

To see the additional statements, recall that $\text{ind } \ell_0 = \text{ind } \ell_1 = 2 - \delta + 2\gamma$. This implies that

$$\text{ind } \frac{R}{\mathcal{C}(\ell_0^-(R))} = \text{ind } \frac{R}{\mathcal{C}(\ell_1^-(R))} = 1 - \gamma \cdot \frac{1}{2 - \delta + 2\gamma} = \frac{2 - \delta + \gamma}{2 - \delta + 2\gamma}.$$

Further, $\delta < 2$ implies that $\int_1^t \mathcal{D}(s)^{-\frac{1}{2}} ds \asymp t \mathcal{D}(t)^{-\frac{1}{2}}$, and hence $\ell_0 \asymp \ell_1$. □

Proof of Case D The first thing to show is that $\frac{d_\phi}{d_l}$ is unbounded: if we had $d_\phi(t) \lesssim d_l(t)$, it would follow that

$$\mathcal{D}(t)^{-\frac{1}{2}} = \sqrt{d_l(t) d_\phi(t)} \lesssim d_l(t),$$

and this contradicts the fact that $d_l(t)$ is integrable while $\mathcal{D}^{-\frac{1}{2}}(t)$ is not.

To show the bound from below asserted in (4.4), we note that

$$\mathcal{Q}(\hbar(R), R) \gtrsim R^{\frac{1}{2}} \hbar(R) \mathcal{D}(\hbar(R))^{-\frac{1}{2}}$$

by Lemma 4.3, and that

$$\frac{R}{\mathcal{C}(\hbar(R))} \gtrsim R \hbar(R) d_l(\hbar(R))$$

by our assumption that $t \mathcal{C}(t) \lesssim \frac{1}{d_l(t)}$. Since $R \asymp \frac{d_\phi}{d_l}(\hbar(R))$ for $R \rightarrow \infty$, we obtain

$$R \hbar(R) d_l(\hbar(R)) \asymp R^{\frac{1}{2}} \hbar(R) \left(\frac{d_\phi(\hbar(R))}{d_l(\hbar(R))} \right)^{\frac{1}{2}} d_l(\hbar(R)) = R^{\frac{1}{2}} \hbar(R) \mathcal{D}(\hbar(R))^{-\frac{1}{2}},$$

and see that

$$\min \left\{ \mathcal{Q}(\hbar(R), R), \frac{R}{\mathcal{C}(\hbar(R))} \right\} \gtrsim R \hbar(R) d_l(\hbar(R)).$$

For the proof of the bound from above, we show that $g(t, R)$ is bounded independently of $t \geq h(R)$. Using again Lemma 4.3, we find

$$\begin{aligned} g(t, R) &= g(h(R), R) + R \int_{h(R)}^t d_l(s) \, ds \\ &\lesssim R^{\frac{1}{2}} \int_1^{h(R)} \mathcal{D}(s)^{-\frac{1}{2}} \, ds + R \int_{h(R)}^{\infty} d_l(s) \, ds. \end{aligned}$$

It remains to choose $\tau(R)$ sufficiently large, so that $\frac{R}{\mathcal{C}(\tau(R))} \leq g(\tau(R), R)$, which is clearly possible. This completes the proof of (4.4).

To see the additional statements, assume first that $\delta_l > \delta_\phi$. Then $h(R)$ is regularly varying with index $\frac{1}{\delta_l - \delta_\phi}$. We obtain

$$\text{ind} \left(R h(R) d_l(h(R)) \right) = 1 + \frac{1}{\delta_l - \delta_\phi} - \frac{\delta_l}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi}.$$

By Karamata's theorem

$$\begin{aligned} \text{ind} \left(R^{\frac{1}{2}} \int_1^{h(R)} \mathcal{D}(s)^{-\frac{1}{2}} \, ds \right) &= \frac{1}{2} + \left(1 - \frac{\delta}{2}\right) \frac{1}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi}, \\ \text{ind} \left(R \int_{h(R)}^{\infty} d_l(s) \, ds \right) &= 1 + (1 - \delta_l) \frac{1}{\delta_l - \delta_\phi} = \frac{1 - \delta_\phi}{\delta_l - \delta_\phi}. \end{aligned}$$

Further, again referring to Karamata's theorem, $\delta < 2$ implies

$$R^{\frac{1}{2}} \int_1^{h(R)} \mathcal{D}(s)^{-\frac{1}{2}} \, ds \asymp R^{\frac{1}{2}} h(R) \mathcal{D}(h(R))^{-\frac{1}{2}} \asymp R h(R) d_l(h(R)),$$

and $\delta_l > 1$ implies

$$R \int_{h(R)}^{\infty} d_l(s) \, ds \asymp R h(R) d_l(h(R)).$$

□

4.2 Bound for the Monodromy Matrix

We combine Theorem 2.2 with Theorem 4.1 to obtain a bound for the growth of W_H when the lengths and angle differences of H are bounded by regularly varying functions. This yields a far reaching generalisation of Corollary 2.5.

4.6 Theorem *Let $(l_j)_{j=1}^{\infty}$ be a summable sequence of positive numbers, and $(\phi_j)_{j=1}^{\infty}$ a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix.*

Let $\psi \in \mathbb{R}$ and let d_l, d_ϕ, c_l, c_ϕ be regularly varying functions that are $\approx$ to some nonincreasing functions, such that $d_l \asymp 1 \asymp d_\phi$ locally, $c_\phi(t) \lesssim c_l(t)$ for sufficiently large t , and $d_l + \text{ind } d_\phi < 0$, and $\lim_{t \rightarrow \infty} (c_l c_\phi)(t) = 0$. Assume that

$$l_j \lesssim d_l(j), \quad |\sin(\phi_{j+1} - \phi_j)| \lesssim d_\phi(j), \quad j \text{ sufficiently large},$$

$$\sum_{j=N+1}^{\infty} l_j \lesssim c_l(N), \quad \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \lesssim c_\phi(N), \quad N \text{ sufficiently large}.$$

Denote

$$\mathcal{D}(t) := \frac{1}{(d_l d_\phi)(t)}, \quad \delta := \text{ind } \mathcal{D}, \quad \mathcal{C}(t) := \frac{1}{(c_l c_\phi)^{\frac{1}{2}}(t)}, \quad \gamma := \text{ind } \mathcal{C}.$$

Then we have the following bounds for W_H and its order ρ_H .

Data satisfies	$\log(\max_{ z =R} \ W_H(z)\)$ is $\lesssim$	$\rho_H \leq$
$\mathcal{D}(t) \lesssim t^\gamma \mathcal{C}(t)$	$\frac{R}{\mathcal{C}(t_1^{-\gamma}(R))}$ where $f(t) := t^\gamma \mathcal{C}(t) \log \left[\alpha \frac{t^\gamma \mathcal{C}(t)}{\mathcal{D}(t)} \right]$ $\alpha := 4 \sup_{t \geq 1} \frac{\mathcal{D}(t)}{t^\gamma \mathcal{C}(t)}$	$\frac{1}{1+\gamma}$
$t^\gamma \mathcal{C}(t) \lesssim \mathcal{D}(t), \int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds < \infty,$ $(\gamma > 0 \text{ or } \frac{d_\phi}{d_l} \approx \text{to nondecreasing})$	$R^{\frac{1}{2}} \int_{\mathcal{C}(R)}^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds$	$\frac{1}{\delta}$
$\int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds = \infty, \frac{1}{d_l(t)} \lesssim t^\gamma \mathcal{C}(t),$ $(\delta, \gamma) \neq (2, 0)$	$\frac{R}{\mathcal{C}(t_1^{-\gamma}(R))}$ where $f_1(t) := [\mathcal{C}(t) \int_1^t \mathcal{D}(s)^{-\frac{1}{2}} ds]^2$	$\frac{2-\delta+\gamma}{2-\delta+2\gamma}$
$t^\gamma \mathcal{C}(t) \lesssim \frac{1}{d_l(t)},$ $\int_1^\infty d_l(s) ds < \infty,$ $\frac{1}{\delta_l} > \delta_\phi$	$R^{\frac{1}{2}} \int_{\mathcal{C}(R)}^{\mathcal{C}(R)} \mathcal{D}(s)^{-\frac{1}{2}} ds$ $+ R \int_1^\infty d_l(s) ds$	$\frac{1-\delta_\phi}{\delta_l-\delta_\phi}$

In each case the relation $\lesssim$ holds for R sufficiently large.

The main assumptions on the data d_l, d_ϕ, c_l, c_ϕ put in this theorem are that those functions are regularly varying and that $\text{ind } d_l + \text{ind } d_\phi < 0$ (i.e., $\neq 0$). Monotonicity assumptions are only a minor restriction: for example they are automatically fulfilled whenever the function under consideration is not slowly varying. The assumption that $c_\phi \lesssim c_l$ is no loss in generality, since replacing c_ϕ by $\min\{c_\phi, c_l\}$ does not affect validity of any of the other assumptions.

Proof of Theorem 4.6

① Observe that neither the assumptions of the theorem nor the case distinction in the assertion of the theorem depends on the equivalence class modulo $\approx$ of the functions d_l, d_ϕ, c_l, c_ϕ . Further, using Remark A.7, we see that the functions written in the second column of the table change only up to $\approx$ when we pass to other data equivalent modulo $\approx$ to d_l, d_ϕ, c_l, c_ϕ .

Hence, it is enough to prove the theorem for suitable modifications $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$ of d_l, d_ϕ, c_l, c_ϕ which differ only up to $\approx$.

② The task is to define $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$ in such a way that Theorem 2.2, Proposition 3.9, and Theorem 4.1 become applicable. To this end we use Lemma A.10 and the freedom of choice of nonincreasing smoothenings mentioned in Remark A.11.

We choose $\mathcal{S}[c_\phi]$ such that

$$\forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \sin^2(\phi_j - \psi) \leq \mathcal{S}[c_\phi](N),$$

which is clearly possible by first choosing an arbitrary nonincreasing smoothening of c_ϕ and then multiplying it with a sufficiently large positive constant. Next, we choose $\mathcal{S}[c_l]$ such that

$$\mathcal{S}[c_\phi] \leq \mathcal{S}[c_l], \quad \forall N \in \mathbb{N}: \sum_{j=N+1}^{\infty} l_j \leq \mathcal{S}[c_l](N).$$

A suitable modification of d_ϕ is found as follows. If $\liminf_{t \rightarrow \infty} d_\phi(t) > 0$, we choose $\mathcal{S}[d_\phi] := 1$ (which corresponds to the choices $\epsilon[d_\phi] := 0$ and $\kappa[d_\phi] := 1$). If $\lim_{t \rightarrow \infty} d_\phi(t) = 0$, we choose $\mathcal{S}[d_\phi]$ such that

$$\mathcal{S}[d_\phi] \leq 1, \quad \forall j \in \mathbb{N}: |\sin(\phi_{j+1} - \phi_j)| \leq \mathcal{S}[d_\phi](j),$$

which is possible by choosing an arbitrary nonincreasing smoothening of d_ϕ , then multiplying it with a sufficiently large positive constant, and then cutting it off at 1.

It remains to define a modification of d_l . In the generic case that $\gamma > 0$ we proceed just the same as above and choose $\mathcal{S}[c_l]$ such that

$$\forall j \in \mathbb{N}: l_j \leq \mathcal{S}[d_l](j). \quad (4.9)$$

In the boundary case that $\gamma = 0$ we make a further case distinction. If we are in the situation of the 1st or the 3rd row of the table in the theorem, we do just the same as above. If we are in the situation of the 2nd or the 4th row, the additional assumption ensures that $\frac{d_l}{d_\phi}$ is $\approx$ to some nonincreasing function, and we choose

$$\mathcal{S}[d_l] := \mathcal{S}[d_\phi] \cdot \mathcal{S}\left[\frac{d_l}{d_\phi}\right],$$

where $\mathcal{S}\left[\frac{d_l}{d_\phi}\right]$ is sufficiently large to ensure that (4.9) holds.

Now we set

$$\widehat{d}_l := \mathcal{S}[d_l], \quad \widehat{d}_\phi := \mathcal{S}[d_\phi], \quad \widehat{c}_l := \mathcal{S}[c_l], \quad \widehat{c}_\phi := \mathcal{S}[c_\phi].$$

③ We apply our previous results with the data $\widehat{d}_l, \widehat{d}_\phi, \widehat{c}_l, \widehat{c}_\phi$. Here we denote by $\widehat{B}(R), \widehat{L}(t, R), \widehat{\mathcal{B}}(R)$, etc. the correspondingly defined functions.

Theorem 2.2 implies that

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim \widehat{B}(R),$$

and we face the task to control $\widehat{L}(t, R)$. In almost all cases Proposition 3.9 takes care of this:

- ▷ Proposition 3.9 (i) applies if $\delta_\phi > \delta_l$, or if $\gamma = 0$ and we are in the situation of the 2nd row of the table.
- ▷ Proposition 3.9 (ii) applies if $\gamma > 0$ and $\delta_\phi < \delta_l$.
- ▷ Proposition 3.9 (iii) applies if $\gamma > 0$ and $\delta_\phi = \delta_l$.

Thus, in all of these cases, $\widehat{B}(R) \asymp \widehat{\mathcal{B}}(R)$, and the bounds asserted in the second column of the table follows from Theorem 4.1.

It remains to study the situation that $\gamma = 0, \delta_\phi \leq \delta_l$, and we are in the 1st or 3rd row of the table. Applying Corollary 3.7 (ii) if $\delta_\phi < \delta_l$ and Lemma 3.8 if $\delta_\phi = \delta_l$, yields

$$\widehat{L}(t, R) \lesssim 1 + \log t.$$

Let $\tau(R)$ be the power bounded function used in Theorem 4.1 A C to estimate $\widehat{\mathcal{B}}(R)$. Then

$$\widehat{B}(R) \leq \widehat{\mathcal{B}}(\tau(R), R) + \widehat{L}(\tau(R), R) \lesssim \widehat{\mathcal{B}}(\tau(R), R).$$

The bounds asserted in the second column of the table thus follow from Theorem 4.1.

④ The bounds for ρ_H arise simply by taking the indices of the regularly varying functions in the second column.

□

In the following corollary we revisit the setting of Theorem 4.6, except we are not given functions c_l, c_ϕ .

4.7 Corollary *Let H be a Hamburger Hamiltonian and let d_l, d_ϕ be regularly varying. Assume that $d_\phi(t)$ is $\sim$ to a nonincreasing function as $t \rightarrow \infty$ and that $d_l \asymp 1 \asymp d_\phi$ locally.*

Assume that the lengths and angles of H are bounded as

$$l_j \lesssim d_l(j), \quad |\sin(\phi_{j+1} - \phi_j)| \lesssim d_\phi(j) \text{ for sufficiently large } j,$$

and that $d_l \in L^1([1, \infty))$. Then the following statements hold.

- ▷ If $\delta > 2$, then $\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim \mathfrak{k}(R)$.
- ▷ If $0 < \delta < 2$, then $\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim R \int_{\mathfrak{k}(R)}^{\infty} d_l(x) \, dx$.
- ▷ If $0 < \delta \leq 2$, $\delta_l > 1$, and there exists $\psi \in \mathbb{R}$ such that $|\sin(\phi_j - \psi)| \lesssim |\sin(\phi_{j+1} - \phi_j)|$, then again $\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim \mathfrak{k}(R)$.
- ▷ If $\delta = 2$ and $(\delta_l, \delta_\phi) \neq (1, 1)$, then $\rho_H \leq \frac{1}{2}$.

Proof Our goal is to apply Theorem 4.6, and in order to do that we need to construct suitable functions c_l, c_ϕ . Note that the assumption $d_l \in L^1([1, \infty))$ implies $\delta_l \geq 1$.

① Without any a priori assumption, we can set

$$c_l(t) := c_\phi(t) := \int_t^\infty d_l(x) \, dx \gtrsim t d_l(t).$$

Then d_l, d_ϕ together with c_l, c_ϕ satisfy the general assumptions of Theorem 4.6. We have

$$t \mathfrak{C}(t) \lesssim \frac{1}{d_l(t)} \lesssim \mathfrak{D}(t)$$

and $\gamma = \delta_l - 1 \geq 0$. This choice of c_l, c_ϕ is sufficient to prove the asserted bounds in the following cases:

- ▷ $\delta > 2 \wedge \delta_l > 1$. Since $\gamma > 0$, Theorem 4.6 gives the upper bound $R^{\frac{1}{2}} \int_{\mathfrak{k}(R)}^\infty \mathfrak{D}(s)^{-\frac{1}{2}} \, ds \asymp \mathfrak{k}(R)$.
- ▷ $\delta < 2$. Then $\delta_\phi = \delta - \delta_l < 1 \leq \delta_l$, hence $\frac{d_\phi}{d_l}$ has positive index and is eventually nondecreasing. Since

$$R^{\frac{1}{2}} \int_1^{\mathfrak{k}(R)} \mathfrak{D}(s)^{-\frac{1}{2}} \, ds \asymp R^{\frac{1}{2}} \mathfrak{k}(R) \mathfrak{D}(\mathfrak{k}(R))^{-\frac{1}{2}} \lesssim R \int_{\mathfrak{k}(R)}^\infty d_l(s) \, ds,$$

Theorem 4.6 gives the asserted upper bound.

- ▷ $\delta = 2 \wedge \delta_l > 1$. Then we are either in the second or fourth row of Theorem 4.6, but in both cases the upper bound for the order is equal to $\frac{1}{2}$.
- ② Assume $\delta > 2$ and $\delta_\phi > 1$. This implies that $\sum_{j=1}^\infty |\sin(\phi_{j+1} - \phi_j)| < \infty$. We want to choose $\psi := \lim_{j \rightarrow \infty} \phi_j$, so we need to prove that this limit actually exists (at least if the angles ϕ_j are all modified by adding integer multiples of π which leaves H unchanged). Start by choosing n_0 so large that $\sum_{j=n_0}^\infty |\sin(\phi_{j+1} - \phi_j)| \leq \frac{\pi}{8}$. Adding to ϕ_j an integer multiple of π , we may assume $|\phi_j - \phi_{n_0}| \leq \frac{\pi}{2}$ for all $j \in \mathbb{N}$. Since $|x| \leq 2|\sin(x)|$ for $|x| \leq \frac{\pi}{2}$, and $|\sin(x + y)| \leq |\sin(x)| + |\sin(y)|$, we have for $j > k \geq n_0$

$$|\phi_j - \phi_k| \leq |\phi_j - \phi_{n_0}| + |\phi_{n_0} - \phi_k| \leq 2(|\sin(\phi_j - \phi_{n_0})| + |\sin(\phi_{n_0} - \phi_k)|)$$

$$\leq 4 \sum_{n=n_0}^{\infty} |\sin(\phi_{n+1} - \phi_n)| \leq \frac{\pi}{2}.$$

Therefore,

$$|\phi_j - \phi_k| \leq 2 |\sin(\phi_j - \phi_k)| \leq 2 \sum_{n=k}^{j-1} |\sin(\phi_{n+1} - \phi_n)|$$

and thus $(\phi_j)_{j=1}^{\infty}$ is a Cauchy sequence. Let ψ be its limit. We set

$$c_l(t) := \int_t^{\infty} d_l(x) \, dx, \quad c_{\phi}(t) := c_l(t) \cdot \left(\int_t^{\infty} d_{\phi}(x) \, dx \right)^2$$

and observe that d_l, d_{ϕ} together with c_l, c_{ϕ} satisfy the assumptions of Theorem 4.6. A calculation shows that $\gamma = \delta - 2 > 0$. Hence $\text{ind}[t\mathcal{C}(t)] = \delta - 1 < \delta$ and in particular $t\mathcal{C}(t) \ll \mathcal{D}(t)$. Again Theorem 4.6 provides the desired upper bound.

③ Assume that we have $\psi \in \mathbb{R}$ such that $|\phi_j - \psi| \lesssim |\phi_{j+1} - \phi_j|$, and $\delta_l > 1$. Set

$$\begin{aligned} c_l(t) &:= \int_t^{\infty} d_l(x) \, dx \asymp t d_l(t), \\ c_{\phi}(t) &:= \int_t^{\infty} d_l(x) d_{\phi}(x)^2 \, dx \asymp t d_l(t) d_{\phi}(t)^2. \end{aligned}$$

Then $t\mathcal{C}(t) \asymp \mathcal{D}(t)$ and we are in the first row of Theorem 4.6. Now we note that the bound given in the theorem is $\asymp$ to $\ell(R)$.

□

5 Additions and Examples

5.1 Combining with a Bound from Below

In the same way that upper bounds for lengths and angle differences lead to upper bounds for the growth of the monodromy matrix, lower bounds lead to lower bounds. We recall a result obtained in [16, Corollary 2.5].

5.1 Proposition *Let H be a Hamburger Hamiltonian with lengths $(l_j)_{j=1}^{\infty}$ and angles $(\phi_j)_{j=1}^{\infty}$, and let ℓ be regularly varying with positive index. If*

$$l_{j+1} l_j \sin^2(\phi_{j+1} - \phi_j) \gtrsim \frac{1}{\ell(j)}, \quad j \in \mathbb{N}, \quad (5.1)$$

then

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \gtrsim \ell^-(R^2)$$

for sufficiently large R .

Let us translate Proposition 5.1 to the setting where we compare the lengths and the angle differences to regularly varying functions d_l, d_ϕ .

5.2 Corollary *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and let $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles, and let W_H be its monodromy matrix. Assume that d_l, d_ϕ are regularly varying and satisfy*

$$l_j \gtrsim d_l(j) \quad |\sin(\phi_{j+1} - \phi_j)| \gtrsim d_\phi(j).$$

Then

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \gtrsim \left[\frac{1}{d_l d_\phi} \right]^- (R)$$

for sufficiently large R .

In particular, the order of W_H is at least $\frac{1}{\delta}$, where $\delta := -(\text{ind } d_l + \text{ind } d_\phi)$.

Proof Since d_l is regularly varying, we have $d_l(t+1) \sim d_l(t)$. Setting $\mathcal{D}(t) := \frac{1}{(d_l d_\phi)(t)}$, we see that (5.1) is satisfied for $\ell(t) := \mathcal{D}(t)^2$. Since $\ell^-(t) = \mathcal{D}^-(t^{\frac{1}{2}})$, we obtain

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \gtrsim \ell^-(R^2) = \mathcal{D}^-(R) = \left[\frac{1}{d_l d_\phi} \right]^- (R).$$

□

If the lengths and angle differences are well-behaved and summable, the growth of W_H can be determined up to $\asymp$. Note that no functions c_l, c_ϕ appear in the formulation of the following theorem.

5.3 Theorem *Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers, and let $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Denote by H the Hamburger Hamiltonian with these lengths and angles. Consider regularly varying functions d_l, d_ϕ with $d_l \asymp 1 \asymp d_\phi$ locally. If*

- (i) $l_j \asymp d_l(j)$ and $|\sin(\phi_{j+1} - \phi_j)| \asymp d_\phi(j)$ for sufficiently large j ,
- (ii) $\delta := -(\text{ind } d_l + \text{ind } d_\phi) > 2$,

then

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \asymp \left[\frac{1}{d_l d_\phi} \right]^- (R) \quad \text{for sufficiently large } R,$$

and

$$\rho_H = \frac{1}{\delta}.$$

Proof Due to Corollary 5.2, we only need to show that

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim \left[\frac{1}{d_l d_\phi} \right]^- (R).$$

We notice that $\inf d_l \leq -1$ since $(l_j)_{j=1}^\infty$ is summable. Hence d_l is $\sim$ to an eventually monotone function, and thus

$$\int_M^\infty d_l(x) \, dx \lesssim \sum_{j=M}^\infty d_l(j) \lesssim \sum_{j=M}^\infty l_j < \infty.$$

Since $d_l \asymp 1$ locally, this shows that $d_l \in L^1([1, \infty))$. By Corollary 4.7,

$$\log \left(\max_{|z|=R} \|W_H(z)\| \right) \lesssim \mathcal{K}(R) \asymp \left[\frac{1}{d_l d_\phi} \right]^- (R).$$

□

5.2 Power-Log-Majorisations and Exceptional Cases

By considering d_l, d_ϕ, c_l, c_ϕ consisting of a power times a power of a logarithm, we can gain some insight into the exceptional cases of our results. For data of this form all functions occurring in our results can in principle be computed explicitly. We do not aim at giving a complete picture, but rather give a couple of illustrative examples. It should also be added that not all phenomena (counterexamples, exceptional cases, or similar) can be illustrated with functions of this form; one would have to admit an additional double-logarithmic factor.

The facts presented below are shown by straightforward, yet tedious and elaborate, computations. For this reason we defer all proofs to the preprint version of this article [14].

It is practical to use the lexicographic order on $\mathbb{R}^2$, and we denote it by $\preceq$. Explicitly, thus

$$(\alpha, \beta) \preceq (\alpha', \beta') : \Leftrightarrow \alpha < \alpha' \vee (\alpha = \alpha' \wedge \beta \leq \beta')$$

and, as usual, $<$ stands for “ $\preceq$ but not $=$ ”.

5.4 Notation Assume we are given parameters

$$(\delta_l, \alpha_l), (\delta_\phi, \alpha_\phi), (\gamma_l, \beta_l), (\gamma_\phi, \beta_\phi) \in [0, \infty) \times \mathbb{R},$$

and denote

$$\delta := \delta_l + \delta_\phi, \alpha := \alpha_l + \alpha_\phi, \gamma := \frac{1}{2}(\gamma_l + \gamma_\phi), \beta := \frac{1}{2}(\beta_l + \beta_\phi).$$

Assume that these parameters satisfy

- ▷ $(\delta_l, \alpha_l), (\delta_\phi, \alpha_\phi), (\gamma_l, \beta_l), (\gamma_\phi, \beta_\phi) \geq (0, 0)$,
- ▷ $\delta > 0, (\gamma, \beta) \succ (0, 0)$,
- ▷ $(\gamma_l, \beta_l) \preceq (\gamma_\phi, \beta_\phi)$.

Let d_l, d_ϕ, c_l, c_ϕ be continuous and nonincreasing functions, such that $d_\phi \leq 1$, $c_\phi \leq c_l$, and that (for sufficiently large t)

$$\begin{aligned} d_l(t) &= t^{-\delta_l} (\log t)^{-\alpha_l}, & d_\phi(t) &= t^{-\delta_\phi} (\log t)^{-\alpha_\phi}, \\ c_l(t) &= t^{-\gamma_l} (\log t)^{-\beta_l}, & c_\phi(t) &= t^{-\gamma_\phi} (\log t)^{-\beta_\phi}. \end{aligned}$$

We compute some of the basic ingredients. Here, and throughout the following, all formulas are understood to hold for sufficiently large t or R .

5.5 Lemma *We have*

- (i) $\mathcal{D}(t) = t^\delta (\log t)^\alpha$, $\mathcal{C}(t) = t^\gamma (\log t)^\beta$,
 $\frac{d_\phi(t)}{d_l(t)} = t^{\delta_l - \delta_\phi} (\log t)^{\alpha_l - \alpha_\phi}$, $\frac{c_l(t)}{c_\phi(t)} = t^{\gamma_\phi - \gamma_l} (\log t)^{\beta_\phi - \beta_l}$.
- (ii) $\mathcal{K}(R) \approx R^{\frac{1}{\delta}} (\log R)^{-\frac{\alpha}{\delta}}$,
 $\mathcal{H}(R) = \infty \Leftrightarrow (\delta_l, \alpha_l) \preceq (\delta_\phi, \alpha_\phi)$,
 $\mathcal{H}(R) \approx \begin{cases} R^{\frac{1}{\delta_l - \delta_\phi}} (\log R)^{-\frac{\alpha_l - \alpha_\phi}{\delta_l - \delta_\phi}} & \text{if } \delta_l > \delta_\phi, \\ \exp\left(R^{\frac{1}{\alpha_l - \alpha_\phi}}\right) & \text{if } \delta_l = \delta_\phi \wedge \alpha_l > \alpha_\phi. \end{cases}$

5.6 Lemma *We have*

$$\begin{aligned} \int_1^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds &< \infty \Leftrightarrow (\delta, \alpha) \succ (2, 2) \\ \int_t^\infty \mathcal{D}(s)^{-\frac{1}{2}} ds &= \begin{cases} \frac{1}{\frac{\delta}{2}-1} \cdot t^{1-\frac{\delta}{2}} (\log t)^{-\frac{\alpha}{2}} & \text{if } \delta > 2 \\ \frac{1}{\frac{\alpha}{2}-1} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \wedge \alpha > 2 \end{cases} \\ \int_1^t \mathcal{D}(s)^{-\frac{1}{2}} ds &\sim \begin{cases} \frac{1}{1-\frac{\delta}{2}} \cdot t^{1-\frac{\delta}{2}} (\log t)^{-\frac{\alpha}{2}} & \text{if } \delta < 2, \\ \frac{1}{1-\frac{\alpha}{2}} \cdot (\log t)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2 \wedge \alpha < 2, \\ \log \log t & \text{if } \delta = \alpha = 2. \end{cases} \end{aligned}$$

In our first example we discuss the role of the term $L(t, R)$; this fits the context of Proposition 3.9. We show that there are situations where $L(t, R)$ cannot be neglected, but also that the assumptions in Proposition 3.9 are only sufficient for $B(R) \asymp \mathcal{B}(R)$.

5.7 Example Assume that

$$(\delta_l, \alpha_l) \prec (\delta_\phi, \alpha_\phi), \quad \gamma = 0, \quad (\delta, \alpha) \succ (2, 2). \quad (5.2)$$

The first assumption is there to rule out applicability of Proposition 3.9 (i), the second to rule out applicability of Proposition 3.9 (ii), (iii), and the third to reduce computational effort (the facts we want to illustrate occur already under this additional assumption).

For parameters subject to (5.2) it holds that

$$\mathcal{B}(R) \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \delta = 2, \\ R^{\frac{1}{\delta}} (\log R)^{-\frac{\alpha}{\delta}} & \text{if } \delta > 2. \end{cases}$$

$$B(R) \begin{cases} \asymp \mathcal{B}(R) & \text{if } \delta_l = \delta_\phi \vee \beta > 1 \vee (\beta = \delta - 1 > 1 \wedge \alpha < 0), \\ \asymp R^{\frac{1}{1+\beta}} \gg \mathcal{B}(R) & \text{otherwise.} \end{cases}$$

In our second example we discuss the bounds from Theorem 4.1 [C]. We show that they are sharp but need not necessarily be attained.

5.8 Example Assume that

$$(\delta_l, \alpha_l) \leq (1 + \gamma, \beta) \leq (\delta, \alpha), \quad \delta = 2, \alpha \leq 2, \quad \gamma > 0. \quad (5.3)$$

These assumptions ensure that we are in the situation of Theorem 4.1 [C] and that in (4.3) we do not automatically have equality.

For parameters subject to (5.3) it holds that

$$\frac{R}{\mathcal{C}(\ell_0^-(R))} \asymp R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} \asymp \mathcal{R}(R),$$

$$\frac{R}{\mathcal{C}(\ell_1^-(R))} \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log \log R & \text{if } \alpha = 2. \end{cases}$$

$$\mathcal{B}(R) \asymp \begin{cases} R^{\frac{1}{2}} (\log R)^{1-\frac{\alpha}{2}} & \text{if } \gamma < 1, \\ R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} \log \log R & \text{if } \gamma = 1 \wedge \alpha > \beta, \\ R^{\frac{1}{2}} (\log R)^{-\frac{\alpha}{2}} & \text{if } \gamma = 1 \wedge \alpha = \beta. \end{cases}$$

We see that

$$\begin{aligned} &\triangleright \mathcal{B}(R) \asymp \frac{R}{\mathcal{C}(\ell_1^-(R))} \text{ if } \gamma < 1 \wedge \alpha < 2, \\ &\triangleright \frac{R}{\mathcal{C}(\ell_0^-(R))} \ll \mathcal{B}(R) \ll \frac{R}{\mathcal{C}(\ell_1^-(R))} \text{ if } \gamma < 1 \wedge \alpha = 2 \text{ or } \gamma = 1 \wedge \alpha > \beta, \\ &\triangleright \mathcal{B}(R) \asymp \frac{R}{\mathcal{C}(\ell_0^-(R))} \text{ if } \gamma = 1 \wedge \alpha = \beta. \end{aligned}$$

In our third example we discuss the exceptional case “ $(\delta, \gamma) = (2, 0)$ ” in the third row of the table in Theorem 4.6. We show that in some cases (interpreting ℓ_1^- appropriately) the written bound still holds and is even attained by $B(R)$, while in others $B(R)$ is strictly larger due to domination of $L(t, R)$.

5.9 Example Assume that

$$(\delta_l, \alpha_l) \leq (1, 1 + \beta), \quad (\delta, \gamma) = (2, 0), \quad \alpha \leq 2. \quad (5.4)$$

These assumptions ensure that the exceptional case from the third row of the table in Theorem 4.6 (and also of Theorem 4.1 $\square$) takes place. Note that (5.4) implies that $\delta_l \leq \delta_\phi$. Moreover, $\beta > 0$ since $(\gamma, \beta) > (0, 0)$.

For parameters subject to (5.4) it holds that

$$\mathcal{B}(R) \asymp \begin{cases} R^{\frac{2-\alpha+\beta}{2-\alpha+2\beta}} & \text{if } \alpha < 2, \\ R^{\frac{1}{2}} \log R & \text{if } \alpha = 2, \end{cases}$$

and

$$B(R) \begin{cases} \asymp \mathcal{B}(R) & \text{if } \delta_l = \delta_\phi \vee (\delta_l < \delta_\phi \wedge \alpha \leq 1 + \beta), \\ \asymp R^{\frac{1}{1+\beta}} \gg \mathcal{B}(R) & \text{if } \delta_l < \delta_\phi \wedge \alpha > 1 + \beta. \end{cases}$$

Observe, moreover, that the bound for order in the third row of the table in Theorem 4.6 has no continuous extension to $(2, 0)$; its directional limits vary from $\frac{1}{2}$ to 1. The above formula shows that the actual order of the bound $B(R)$ has nothing to do with (δ, γ) (being equal to $(2, 0)$). Yet, it is sometimes given by the same formula, only with the “logarithmic exponents” α, β instead of δ, γ . Also note that the exponent $\frac{1}{1+\beta}$ also occurred in Example 5.7.

5.3 Two Corollaries Given in Terms of Jacobi Parameters

We present two applications in which we return to the regime of power moment problems. The first is a supplement to a result from [13], and in the second we give examples where the Nevanlinna matrix has prescribed growth.

At this point we need the concrete formulae relating Jacobi parameters with Hamiltonian parameters. They read as

$$\begin{aligned} \frac{1}{b_{n-1}} &= \sin(\phi_{n+1} - \phi_n) \sqrt{l_{n+1} l_n}, \\ a_{n-1} &= -\frac{1}{l_n} [\cot(\phi_{n+1} - \phi_n) + \cot(\phi_n - \phi_{n-1})], \end{aligned} \quad (5.5)$$

where the angles are chosen such that $\phi_{n+1} - \phi_n \in [0, \pi)$; these formulae are index-shifted versions of [9, (3.16), (3.17)].

Given the Jacobi parameters, it is in general hard to solve the equations (5.5) for the Hamiltonian parameters. Under the assumption that b_n, a_n have a certain power-like asymptotic behaviour, the approximate size of l_n and $|\sin(\phi_{n+1} - \phi_n)|$ can be determined.

5.10 Corollary *Let $a_n \in \mathbb{R}$ and $b_n > 0$ be sequences which have asymptotics*

$$\begin{aligned} a_n &= n^\beta \left(y_0 + \frac{y_1}{n} + O\left(\frac{1}{n^2}\right) \right), \\ b_n &= n^\beta \left(\frac{|y_0|}{2} + \frac{x_1}{n} + O\left(\frac{1}{n^2}\right) \right), \end{aligned}$$

where $\beta > 2$, $y_0 \neq 0$, $x_1, y_1 \in \mathbb{R}$. Assume that

$$\tau := \beta + 2 \frac{y_1}{y_0} - 4 \frac{x_1}{|y_0|} < 0.$$

Then the moment problem with these Jacobi parameters is indeterminate, and its Nevanlinna matrix $W(z)$ satisfies

$$\log \left(\max_{|z|=R} \|W(z)\| \right) \asymp R^{\frac{1}{\beta}}.$$

Before we come to the proof, let us put this statement into the right context. It deals with the critical situation where diagonal and off-diagonal of the Jacobi matrix are comparable with ratio ± 2 . This setting was considered in [13, Theorem 2]. In that theorem occurrence of limit circle case was characterised in terms of the data of the expansions, and it was shown that $W(z)$ is of order $\frac{1}{\beta}$ with positive type. The significance of Corollary 5.10 is that now we know that $W(z)$ is also of finite type. Furthermore, our conditions on a_n, b_n are slightly relaxed because we use a more recent source for asymptotics of solutions of the Jacobi recurrence [25].

Proof The lengths of the Hamburger Hamiltonian associated with the Jacobi parameters a_n, b_n are given by

$$l_{n+1} = p_n(0)^2 + q_n(0)^2 \quad (5.6)$$

by [9, (3.22)] (and an index shift). We can make this more explicit with the help of an asymptotic formula for a solution of the Jacobi recurrence. By [25, Theorem 2.1] there exists, for each $z \in \mathbb{C}$, a solution $(f_n(z))_{n=0}^\infty$ of (1.2) with asymptotics

$$f_n(z) = (-\operatorname{sgn} y_0)^n n^{-\frac{\beta}{2} + \frac{1}{4}} e^{-2i\sqrt{-\tau n}} (1 + O(n^{-\frac{1}{2}})).$$

For all real x it follows that $(f_n(x))_{n=0}^\infty$ and $(\overline{f_n(x)})_{n=0}^\infty$ form a basis of the solution space of (1.2). Since $(p_n(x))_{n=0}^\infty$ and $(q_n(x))_{n=0}^\infty$ form another basis, we have

$$p_n(x)^2 + q_n(x)^2 \asymp |f_n(x)|^2 + |\overline{f_n(x)}|^2 \asymp |f_n(x)|^2.$$

Taking $x = 0$ and plugging this into (5.6) we obtain $l_n \asymp n^{\frac{1}{2}-\beta}$. Comparing with (5.5), we get $|\sin(\phi_{n+1} - \phi_n)| \asymp n^{-\frac{1}{2}}$. The assumptions of Theorem 5.3 are thus satisfied with $d_l(t) := t^{\frac{1}{2}-\beta}$ and $d_\phi(t) := t^{-\frac{1}{2}}$. Applying this theorem yields

$$\log \left(\max_{|z|=R} \|W(z)\| \right) \asymp \left[\frac{1}{d_l d_\phi} \right]^- (R) \asymp R^{\frac{1}{\beta}}.$$

□

We come to our second corollary, where we produce a variety of examples with prescribed growth of the Nevanlinna matrix. Thereby, the speed of growth is always determined by the off-diagonal, and the diagonal can be as large or as small as we please.

5.11 Corollary *Let g be regularly varying with $\text{ind } g \in (0, 1)$.*

(i) *Let $\omega_n \neq 0$, $\omega_n \rightarrow 0$, be such that $\lim_{n \rightarrow \infty} \frac{\omega_{n+1}}{\omega_n}$ exists in $(-1, 1]$. Then there exist Jacobi parameters b_n and a_n , such that*

$$b_n \sim g^-(n), \quad \frac{a_n}{b_n} \sim \omega_n, \quad \log \left(\max_{|z|=R} \|W(z)\| \right) \asymp g(R).$$

(ii) *Let $\omega \in (-2, 2)$. Then there exist Jacobi parameters b_n and a_n , such that*

$$b_n \sim g^-(n), \quad \frac{a_n}{b_n} \rightarrow \omega, \quad \log \left(\max_{|z|=R} \|W(z)\| \right) \asymp g(R).$$

(iii) *Let $\omega = -2$ or $\omega = 2$ and assume $\text{ind } g \in (0, \frac{1}{2})$. Then there exist Jacobi parameters b_n and a_n , such that*

$$b_n \sim g^-(n), \quad \frac{a_n}{b_n} \rightarrow \omega, \quad \log \left(\max_{|z|=R} \|W(z)\| \right) \asymp g(R).$$

In all cases of Corollary 5.11 we have $\lim_{n \rightarrow \infty} (a_n/b_n) \in [-2, 2]$. If this limit exists but is not in $[-2, 2]$, we cannot have limit circle case by Wouk's theorem (see e.g. [2, p.25]).

Proof We specify lengths and angles, which is done differently in different cases.

▷ Assume that $\omega_n \neq 0$, $\omega_n \rightarrow 0$, and $\gamma := \lim_{n \rightarrow \infty} \frac{\omega_{n+1}}{\omega_n}$ exists in $(-1, 1]$. Set

$$l_n := \frac{1}{g^-(n)}, \quad \phi_{n+1} := n \frac{\pi}{2} + \frac{1}{1+\gamma} \sum_{\substack{k=1 \\ |\omega_k| < \pi}}^n \omega_k, \quad n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \frac{\pi}{2} + \frac{\omega_n}{1+\gamma}$ for all sufficiently large n .

▷ Assume that $\omega \in (-2, 2)$. Let $\psi \in (0, \pi)$ be such that $\cos \psi = -\frac{\omega}{2}$, and set

$$l_n := \frac{1}{\sin \psi \cdot g^-(n)}, \quad \phi_{n+1} := n\psi, \quad n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \psi$.

▷ Assume that $\omega = -2$ and $\text{ind } g \in (0, \frac{1}{2})$. Set

$$l_n := \frac{n}{g^-(n)}, \quad \phi_{n+1} := \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \frac{1}{n}$.

▷ Assume that $\omega = 2$ and $\text{ind } g \in (0, \frac{1}{2})$. Set

$$l_n := \frac{n}{g^-(n)}, \quad \phi_{n+1} := n\pi - \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N},$$

so that $\phi_{n+1} - \phi_n = \pi - \frac{1}{n}$.

Let b_n and a_n be the Jacobi parameters given by (5.5). Then, in all cases, $b_n \sim g^-(n)$. Multiplying the two equations from (5.5), shows that

$$\frac{a_{n-1}}{b_{n-1}} = -\sqrt{\frac{l_{n+1}}{l_n}} \cdot \left(\cos(\phi_{n+1} - \phi_n) + \cos(\phi_n - \phi_{n-1}) \frac{\sin(\phi_{n+1} - \phi_n)}{\sin(\phi_n - \phi_{n-1})} \right),$$

and this implies the asserted property of $\frac{a_n}{b_n}$. Corollary 5.2 gives the lower bound:

$$\log \left(\max_{|z|=R} \|W(z)\| \right) \gtrsim g(R).$$

In cases (i) and (ii) we have $|\sin(\phi_{j+1} - \phi_j)| \asymp 1$, so we can use the third item of Corollary 4.7 to get the upper bound. In the remaining case the index of g is restricted to $(0, \frac{1}{2})$, which corresponds to $\delta_j > 2$ in Corollary 4.7. In this case the upper bound follows from the first item of Corollary 4.7. $\square$

Appendix A. Regularly Varying Functions

In complex analysis the growth of an analytic function is compared with functions of the form $\exp(\omega(r))$. The most classical comparison functions are powers $\omega(r) = r^\rho$, and this leads to the notions of order and type. A refined comparison scale was introduced already at a very early stage by E. Lindelöf [11] who considered comparison functions behaving for $r \rightarrow \infty$ like

$$r^\alpha \cdot (\log r)^{\beta_1} \cdot (\log \log r)^{\beta_2} \cdot \dots \cdot \underbrace{(\log \dots \log r)^{\beta_m}}_{m^{\text{th}} \text{ iterate}},$$

where $\alpha > 0$ and $\beta_1, \dots, \beta_m \in \mathbb{R}$. Functions which are nowadays commonly used as comparison functions are regularly varying functions in Karamata sense, cf. [7, Chapter 7] (for other levels of generality see also [10, 21]). Lindelöf's comparison functions are examples of functions of that kind.

Let us now recall Karamata's definition of regular variation.

A.1 Definition A function $\omega: [1, \infty) \rightarrow (0, \infty)$ is called *regularly varying* at ∞ with index $\alpha \in \mathbb{R}$, if it is measurable and

$$\forall \lambda \in (0, \infty): \lim_{r \rightarrow \infty} \frac{\omega(\lambda r)}{\omega(r)} = \lambda^\alpha.$$

We write $\text{ind } \omega$ for the index of regular variation of function ω . A regularly varying function with index 0 is also called *slowly varying*.

Regularly varying functions ω are used to quantify growth for $r \rightarrow \infty$, and hence the values of $\omega(r)$ for small r are irrelevant. This allows to change ω on any finite interval without changing the essence of results, and this freedom can often be used to assume ω has some additional practical properties.

We cite a number of fundamental theorems on regularly varying functions. Proofs can be found, e.g., in [7] or [23]. We start with a *representation theorem*.

A.2 Theorem (Representation theorem) *Let $\alpha \in \mathbb{R}$. A function $\omega : [1, \infty) \rightarrow (0, \infty)$ is regularly varying with index α if and only if it has a representation of the form*

$$\omega(r) = r^\alpha \cdot c(r) \exp \left(\int_1^r \epsilon(u) \frac{du}{u} \right), \quad r \in [1, \infty),$$

where c, ϵ are measurable, $\lim_{r \rightarrow \infty} c(r) = c \in (0, \infty)$, and $\lim_{r \rightarrow \infty} \epsilon(r) = 0$.

If ω is slowly varying (i.e., $\alpha = 0$) and eventually nondecreasing (nonincreasing), then ϵ may be taken eventually nonnegative (nonpositive).

It is a legitimate intuition that regularly varying functions fill in the scale of powers, and that a regularly varying function with index α behaves roughly like the power r^α . The following results, which we will use frequently, express this intuition very clearly. The first is a variant of the *Potter bounds*, and the second is the classical *Karamata Theorem* about asymptotic integration.

A.3 Theorem (Potter bounds (variant)) *Let ω be regularly varying with index $\alpha \in \mathbb{R}$. Then*

- (i) $\forall \epsilon > 0: r^{\alpha-\epsilon} \ll \omega(r) \ll r^{\alpha+\epsilon}$,
- (ii) $\lim_{r \rightarrow \infty} \frac{\log \omega(r)}{\log r} = \alpha$,
- (iii) *For all $\epsilon > 0$ the quotients $\frac{\omega(r)}{r^{\alpha-\epsilon}}$ and $\frac{r^{\alpha+\epsilon}}{\omega(r)}$ are $\sim$ to eventually increasing functions.*

A.4 Theorem (Karamata's Theorem) *Let ω be regularly varying with index $\alpha \in \mathbb{R}$.*

- (i) *Assume that $\alpha \geq -1$. Then the function $x \mapsto \int_1^x \omega(t) dt$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x \rightarrow \infty} \left(x \omega(x) / \int_1^x \omega(t) dt \right) = \alpha + 1.$$

- (ii) *Assume that $\alpha \leq -1$ and $\int_1^\infty \omega(t) dt < \infty$. Then the function $x \mapsto \int_x^\infty \omega(t) dt$ is regularly varying with index $\alpha + 1$, and*

$$\lim_{x \rightarrow \infty} \left(x \omega(x) / \int_x^\infty \omega(t) dt \right) = -(\alpha + 1).$$

A regularly varying function ω with positive index is – at least asymptotically – invertible. In fact, if

$$\omega^-(x) := \sup \{ t \in [1, \infty) \mid \omega(t) < x \},$$

we have the following result, cf. [7, Theorem 1.5.12].

A.5 Theorem *Let α be regularly varying with index $\alpha > 0$. Then α^- is regularly varying with index $\frac{1}{\alpha}$, and*

$$(\alpha \circ \alpha^-)(x) \sim (\alpha^- \circ \alpha)(x) \sim x. \quad (\text{A.1})$$

Any regularly varying function α^- with the property (A.1) is called an *asymptotic inverse* of α , and asymptotic inverses are determined uniquely up to $\sim$. We recall a useful formula for computing asymptotic inverses for functions of a particular form.

A.6 Remark Assume that $\rho > 0$ and that ℓ is regularly varying. Set $g := \ell \circ \log$ (for sufficiently large t). Then

$$\alpha(t) := t^\rho g(t), \quad \alpha^-(t) := \rho^{\frac{\text{ind } \ell}{\rho}} \cdot \left(\frac{t}{g(t)} \right)^{\frac{1}{\rho}}$$

are asymptotic inverses of each other.

Another practical observation is the following.

A.7 Remark Let ℓ be regularly varying with $\text{ind } \ell > 0$, and assume we have a function g with $g \approx \ell$. Then g is regularly varying with $\text{ind } g = \text{ind } \ell$ and $g^- \approx \ell^-$.

By the Potter bounds every regularly varying function α is bounded and bounded away from zero on every interval $[r_1, r_2]$ sufficiently far to the right. Sometimes it is needed for technical reasons to assume this property for all compact intervals in the domain of α . Of course, this is no loss in generality; remember that modification on a finite interval does not change the essence of the function α .

A.8 Lemma *Let $\alpha: [1, \infty) \rightarrow (0, \infty)$ be slowly varying and assume that $\alpha \lesssim 1$ locally. For $R \geq \alpha(1)$ set*

$$\mathfrak{b}(R) := \sup \left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{\alpha(s)}{R} \leq 1 \right\} \in [1, \infty].$$

Then $\mathfrak{b}$ grows faster than any power, i.e., $R^\rho \leq \mathfrak{b}(R)$ for every $\rho > 0$ and R sufficiently large.

Proof Let $\rho > 0$ and set $\epsilon := \frac{1}{\rho}$. By Theorem A.3, there exists $M > 0$ such that $\alpha(r) \leq r^\epsilon$ for all $r \geq M$. If $R \geq \sup_{r \in [1, M]} \alpha(r)$, this means that

$$\left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{s^\epsilon}{R} \leq 1 \right\} \subseteq \left\{ t \in [1, \infty) \mid \sup_{1 \leq s \leq t} \frac{\alpha(s)}{R} \leq 1 \right\}.$$

The assertion follows if we take suprema of both sets. □

A.9 Lemma Let a be regularly varying with index $\alpha > 0$ and assume that $a \asymp 1$ locally. Then there exists a continuously differentiable and regularly varying function δ , where $\delta'(t) > 0$ for $t \in [1, \infty)$,

$$\begin{aligned} a(t) &\asymp \delta(t), & t \in [1, \infty), \\ a(t) &\sim s(t), & t \rightarrow \infty. \end{aligned}$$

Proof We use the smooth variation theorem [7, Theorem 1.8.2]. This gives a function δ that is continuously differentiable, whose derivative is positive for all sufficiently large t , and such that $a \sim \delta$ for $t \rightarrow \infty$. W.l.o.g. we assume $\delta'(t) > 0$ for all $t \in [1, \infty)$.

Choose $t_0 \geq 1$ such that $\frac{1}{2}a(t) \leq \delta(t) \leq \frac{3}{2}a(t)$ for all $t \geq t_0$. By our assumption, $a \asymp 1$ on $[1, t_0]$. Since δ is continuous, $\delta \asymp 1 \asymp a$ on $[1, t_0]$. Summing up, we have $\delta \asymp a$ on $[1, \infty)$. $\square$

Another lemma in a similar direction is the following. The proof is immediate from the representation theorem and we do not go into details.

A.10 Lemma Let a be regularly varying, and assume that a is $\approx$ to some nonincreasing function. Then there exist

$$\epsilon[a], \kappa[a]: [1, \infty) \rightarrow (0, \infty)$$

such that

- ▷ $\epsilon[a]$ is locally integrable, $\lim_{t \rightarrow \infty} \epsilon[a](t) = 0$, and $\epsilon[a] \leq 0$ if $\text{ind } a = 0$,
- ▷ $\kappa[a]$ is eventually constant,
- ▷ The function

$$\mathcal{S}[a](t) := \kappa[a](t) \cdot t^{\text{ind } a} \cdot \exp \left(\int_1^t \epsilon[a](u) \frac{du}{u} \right)$$

is nonincreasing, continuous, and $\approx a$.

We speak of any function $\mathcal{S}[a]$ as in the lemma as a *nonincreasing smoothening* of a .

Of course, $\kappa[a]$ and $\epsilon[a]$ are far from unique. We mention two particular instances of the freedom of choice in $\mathcal{S}[a]$.

A.11 Remark (i) If $\mathcal{S}[a]$ is some nonincreasing smoothening of a and $\alpha > 0$, then also $\alpha \cdot \mathcal{S}[a]$ is a nonincreasing smoothening of a . This corresponds to multiplying $\kappa[a]$ by α .

(ii) If $\mathcal{S}[a]$ is some nonincreasing smoothening of a and $\alpha > \lim_{t \rightarrow \infty} \mathcal{S}[a](t)$, then also $\min\{\mathcal{S}[a](t), \alpha\}$ is a nonincreasing smoothening of a . This is seen by modifying $\kappa[a]$ on a finite interval.

One more property of this construction is as follows. Assume we have two functions a_1, a_2 that are both regularly varying and $\approx$ to some nonincreasing function. If

$\mathcal{S}[a_1]$ and $\mathcal{S}[a_2]$ are nonincreasing smoothenings of a_1 and a_2 , respectively, then $\mathcal{S}[a_1] \cdot \mathcal{S}[a_2]$ is a nonincreasing smoothening of $a_1 \cdot a_2$. This corresponds to taking

$$\kappa[a_1 a_2] := \kappa[a_1] \cdot \kappa[a_2], \quad \epsilon[a_1 a_2] := \epsilon[a_1] + \epsilon[a_2].$$

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