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ORIGINAL ARTICLE

Asymptotic Thorin Distributions of Multivariate Generalized Splines via Markov-Krein Transform

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Abstract

In this paper, we investigate multivariate generalized gamma convolutions, commonly referred to as the Thorin class, which forms a distinguished subclass of the class of infinitely divisible distributions. The Thorin class contains all elementary gamma distributions and is closed under convolution and weak convergence; moreover, it is the smallest class of probability measures on $\mathbb{R}^p$ enjoying these properties. By developing multivariate Markov–Krein transformations, we establish new and significant connections between multivariate generalized gamma convolutions and multivariate generalized spline distributions. In particular, using the Markov–Krein formula, we show that the asymptotic behavior of multivariate generalized splines naturally leads to Thorin measures. Conversely, we prove that Thorin measures can be characterized through the asymptotics of generalized spline distributions. Finally, within this framework, we derive a representation of the Fourier–Laplace transform of Thorin measures, obtained via the asymptotic properties of multivariate generalized splines and the Markov–Krein transform.

Keywords Markov–Krein transform · Generalized spline distributions · Multivariate generalized gamma convolutions · Thorin-class

Mathematics Subject Classification 44A60 · 60B10 · 60E10 · 65D07

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1 Introduction

We say that two measures μ and ν on $\mathbb{R}$ are related by the *Markov–Krein relation* if, for every $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{(1-zt)^{\theta}} \, \mu(dt) = \exp\biggl(-\int_{\mathbb{R}} \log(1-zu) \, \nu(du)\biggr),$$

where μ is a probability measure, ν is a finite positive measure, and $\theta = \nu(\mathbb{R})$ denotes the total mass of ν.

The Markov–Krein correspondence was introduced by M. G. Krein in the context of the classical moment problem. A comprehensive treatment of its foundations and early developments can be found in the monograph by Krein and Nudel'man [35].

Over the years, this correspondence has emerged as a fundamental tool in both analysis and probability theory. It plays a central role in the study of the Markov moment problem and appears naturally in the theory of interlacing sequences and continuous diagrams, notably in asymptotic representation theory [36]. It also provides a unifying framework for the analysis of Dirichlet processes and their generalizations [16].

From a probabilistic viewpoint, the Markov–Krein correspondence has been used to characterize the distributions of linear functionals of the Dirichlet process via its representation as a normalized gamma process. This approach was further developed by James [15], who related these distributions to a class of Beta–Gamma processes. More recently, inverse Markov–Krein transforms have been employed in random matrix theory to describe the high-temperature limits of classical β-ensembles, where the limiting spectral measures of Gaussian, Laguerre, and Jacobi type naturally arise [21].

Finally, the Markov–Krein correspondence reveals deep connections between Dirichlet distributions and projections of orbital measures associated with unitary group actions, as developed, for instance, in [13, 32].

Moreover, an explicit expression for the Markov–Krein correspondence was derived, and its validity was established through boundary values of holomorphic functions, highlighting its intrinsic links with classical problems in analysis and probability [14]. Several particular cases and variants of the Markov–Krein correspondence have also been investigated in the literature, arising in different analytical and probabilistic settings [33, 34].

Its asymptotic behavior is closely related to the broad class of *Generalized Gamma Convolutions* (GGCs) developed and studied by Bondesson [4]. The GGC class encompasses a wide range of probability distributions extensively used in statistical modeling, probability theory, and stochastic process analysis. Within this family, positive stable and tempered stable distributions form a central subclass and have been widely used to model heavy-tailed data; see, for example, [22, 26]. The positive α-stable family gives rise to several important distributions as special or limiting cases: the inverse Gaussian distribution corresponds to the $\alpha = 1/2$ stable subordinator, while the gamma distribution emerges as a limiting tempered stable case as the stability index tends to zero. Other notable GGC subclasses include the generalized inverse Gaussian (GIG) and variance–gamma distributions, widely used in variance–mean mixture models and financial econometrics [1, 18, 20], the normal inverse Gaussian (NIG) distribution for semi-heavy-tailed modeling [2, 5, 17], and Tweedie and Poisson–inverse Gaussian distributions for modeling overdispersed count data and generalized linear models [25, 29, 30].

In the multivariate context, Kerov and Tsilevich [16] investigated a specific instance of the Markov–Krein correspondence related to the Cauchy–Stieltjes transform. They studied the images of classical Dirichlet measures under linear or affine transformations and showed that both additive and multiplicative Cauchy–Stieltjes transforms behave well under affine changes of variables.

The present work extends the univariate theory to the multivariate framework by introducing a broader notion of the multivariate Markov–Krein transform. Two measures μ and ν on $\mathbb{R}^p$, with $p \geq 1$, are said to be *Markov–Krein related* if, for every $z \in i\mathbb{R}^p$,

$$\int_{\mathbb{R}^p} \frac{1}{\big(1 - \langle z, t\rangle\big)^{\theta}}\, \mu(dt) = \exp\Big(- \int_{\mathbb{R}^p} \log\big(1 - \langle z, u\rangle\big)\, \nu(du)\Big),$$

where $\theta = \nu(\mathbb{R}^p)$ and $\langle \cdot, \cdot\rangle$ denotes the standard scalar product in $\mathbb{C}^p$.

Equivalently, for $z \neq 0$, replacing z by $\frac{z}{\|z\|^2}$, the relation becomes:

$$\int_{\mathbb{R}^p} \frac{1}{\langle z, z - t\rangle^{\theta}} \mu(dt) = \exp\left(- \int_{\mathbb{R}^p} \log(\langle z, z - u\rangle)\nu(du)\right).$$

Here, $\|\cdot\|$ denotes the norm associated with the scalar product.

2 Multivariate Generalized Spline Distribution and Markov-Krein Transform

Splines play a central role in approximation theory, numerical analysis, and probability. In the univariate setting, the densities of these distributions [7] arise naturally as piecewise polynomial functions ensuring a given degree of smoothness at specified knots. The advantages of smooth locally supported basis functions can be successfully exploited for the construction of local approximation schemes [9, 37]

In higher dimensions, the notion of splines becomes more subtle due to the lack of a canonical ordering of knots and the geometric complexity of multivariate domains. A fruitful approach, initiated by [8, 19] and others, consists in defining multivariate splines through the projection of measures supported on simplices in higher-dimensional spaces.

In the univariate framework, the classical notion of spline has been extended to a broader class of probability measures, known as *generalized splines*. These compactly supported measures admit densities that can be explicitly characterized [13, 33]. Moreover, they exhibit deep connections with projections of orbital measures under the action of the unitary group on the space of Hermitian matrices [13, 31, 32, 36] as well as with Markov transforms [14] and generalized gamma convolutions.

In this section, we present work that aims to extend the probabilistic framework of generalized splines to the multivariate setting.

Definition 1 Let $\kappa = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{R}_+^*)^n$ with $n \geq 2$. for all $i = 1, \ldots, n$, and let $n \geq 2$. The *Dirichlet distribution* with parameter κ, denoted by $\mathsf{D}_n^{(\kappa)}$, is the probability measure on the simplex

$$\Delta_{n-1} = \Big\{\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \;\Big|\; \tau_i \geq 0,\; \tau_1 + \cdots + \tau_n = 1\Big\},$$

defined by

$$\int_{\Delta_{n-1}} f(\tau)\, \mathsf{D}_n^{(\kappa)}(d\tau) = \frac{1}{C_n(\kappa)} \int_{\Delta_{n-1}} f(\tau)\, \tau_1^{\kappa_1 - 1} \cdots \tau_n^{\kappa_n - 1}\, \alpha(d\tau),$$

where α denotes the restriction of the Lebesgue measure to the hyperplane

$$\Big\{\tau \in \mathbb{R}^n : \tau_1 + \cdots + \tau_n = 1\Big\},$$

and

$$C_n(\kappa) = (n-1)!\, \frac{\Gamma(\kappa_1) \cdots \Gamma(\kappa_n)}{\Gamma(|\kappa|)}, \qquad |\kappa| = \kappa_1 + \cdots + \kappa_n.$$

Now, let $A_1, \ldots, A_n \in \mathbb{R}^p$ be given by

$$A_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{pi} \end{pmatrix}, \qquad 1 \leq i \leq n,$$

and let $A = (A_1, \ldots, A_n) \in \mathcal{M}_{p\times n}(\mathbb{R})$ denote the matrix with columns $A_1, \ldots, A_n$.

We define the probability measure $M_n(\kappa; A)$ on $\mathbb{R}^p$ as the distribution of the random vector

$$T = A\tau, \qquad \tau \sim \mathsf{D}_n^{(\kappa)}.$$

Equivalently, for any continuous function $F : \mathbb{R}^p \to \mathbb{R}$,

$$\int_{\mathbb{R}^p} F(t)\, M_n(\kappa; A)(dt) = \frac{1}{C_n(\kappa)} \int_{\Delta_{n-1}} F(A\tau)\, \tau_1^{\kappa_1-1} \cdots \tau_n^{\kappa_n-1}\, d\tau_1 \cdots d\tau_n$$
$$= \frac{1}{C_n(\kappa)} \int_{\Delta_{n-1}} F\Big(\sum_{i=1}^n a_{1i}\tau_i, \ldots, \sum_{i=1}^n a_{pi}\tau_i\Big)\, \tau_1^{\kappa_1-1} \cdots \tau_n^{\kappa_n-1}\, d\tau_1 \cdots d\tau_n.$$

The support of $M_n(\kappa; A)$ is contained in the convex hull Σ of $\{A_1, \ldots, A_n\}$, and in particular within the closed ball

$$B\big(0_{\mathbb{R}^p},\, \|A\|\big), \qquad \|A\| = \max_{1\leq i\leq p,\, 1\leq j\leq n} |a_{ij}|.$$

If $\kappa_1 = \cdots = \kappa_n = 1$, then $M_n(\kappa; A)$ corresponds to a multivariate spline distribution.

1. For p=1 [13, 31], $n \geq 3$, the measure μ admits a spline density with knots $A_1 < \cdots < A_n$, supported on the interval $[A_1, A_n]$. Its restriction to each subinterval $[A_i, A_{i+1}]$ is a polynomial of degree at most $n-2$, and μ is of class C^{n-3} on $\mathbb{R}$. Moreover, these properties together with the normalization condition
$$\int_{A_1}^{A_n} \mu(x)\,dx = 1$$
uniquely determine the probability density μ
2. $p \geq 2$ [8, 12, 24]
We recall the definition introduced by Dahmen. Let $a_1, \ldots, a_n$ be n points in $\mathbb{R}^{n-1}$ forming the vertices of a simplex Σ of volume 1. For $p < n$, let proj denote the canonical projection from $\mathbb{R}^{n-1}$ onto $\mathbb{R}^p$, and set $A_i = \mathrm{proj}(a_i)$ for $i = 1, \ldots, n$. Let M a function defined on $\mathbb{R}^p$ by:
$$M(x) = \mathrm{vol}_p\big\{y \in \Sigma \mid \mathrm{proj}(y) = x\big\}.$$
The function $M(x)$ is the density, with respect to the Lebesgue measure on $\mathbb{R}^p$, of the measure $M_n(\kappa; A)$. Moreover, $M(x)$ is a *piecewise polynomial function* and it is sufficiently regular (of class C^r for some $r \geq 0$ depending on n and p)
The measure $M_n(\kappa; A)$ is the projection onto $\mathbb{R}^p$ of the restriction to Σ of the Lebesgue measure m on $\mathbb{R}^{n-1}$. More precisely, for any continuous function f on $\mathbb{R}^p$, one has
$$\int_{\mathbb{R}^p} f(x)\, M_{n,p}(a_1, \ldots, a_n; dx) = \int_{\Sigma} f\big(\mathrm{proj}(y)\big)\, m(dy).$$

If $\kappa_i > 0$, we shall refer to $M_n(\kappa; A)$ as a *multivariate generalized spline distribution*.

In the following, we will show that multivariate generalized splines satisfy a multivariate Markov–Krein relation

Lemma 1 *Let f be an integrable function on $(\mathbb{R}_+)^n$. Then*
$$\int_{(\mathbb{R}_+)^n} f(x)\, m(dx) = \frac{1}{(n-1)!} \int_0^\infty \int_{\Delta_{n-1}} f(r\tau)\, \alpha(d\tau)\, r^{n-1} dr,$$
where m is the Lebesgue measure on $\mathbb{R}^n$ and α is the restriction of the Lebesgue measure to the simplex
$$\Delta_{n-1} = \Big\{\tau \in \mathbb{R}^n \;\Big|\; \tau_i \geq 0,\ \sum_{i=1}^n \tau_i = 1\Big\}.$$

Proof We parameterize the simplex Δ_{n-1} by coordinates $(u_1, \ldots, u_{n-1})$, where $u_i > 0$ and $u_1 + \cdots + u_{n-1} < 1$. We then define

$$u_n = 1 - (u_1 + \cdots + u_{n-1}).$$

Consider the mapping

$$\Delta_{n-1} \times [0, +\infty) \longrightarrow (\mathbb{R}_+)^n,$$

given by

$$x_1 = ru_1, \quad x_2 = ru_2, \ \ldots, \ x_{n-1} = ru_{n-1}, \quad x_n = r\big(1 - (u_1 + \cdots + u_{n-1})\big).$$

The Jacobian determinant of this transformation is

$$J = \begin{vmatrix} r & 0 & \cdots & 0 & u_1 \\ 0 & r & \cdots & 0 & u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ -r & -r & \cdots & -r & 1 - (u_1 + \cdots + u_{n-1}) \end{vmatrix}.$$

By adding each of the first $n-1$ rows to the last row, the determinant simplifies to

$$J = r^{n-1}.$$

Moreover,

$$m\big\{u \in \mathbb{R}^{n-1} : u_i > 0, \ u_1 + \cdots + u_{n-1} < 1\big\} = \frac{1}{(n-1)!}.$$

□

Theorem 1 *Let $A = (A_1, \ldots, A_n) \in \mathcal{M}_{p \times n}(\mathbb{R})$ be a matrix with columns $A_i \in \mathbb{R}^p$, and let $\kappa = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{R}_+^*)^n$. Then, for all $z \in B\big(0_{\mathbb{R}^p}, 1/\|A\|\big)$,*

$$\int_{\mathbb{R}^p} \frac{1}{(1 - z^\top x)^{|\kappa|}} M_n(\kappa; A)(dx) = \prod_{i=1}^{n} \big(1 - z^\top A_i\big)^{-\kappa_i},$$

where $|\kappa| = \kappa_1 + \cdots + \kappa_n$ and $\|A\| = \max_{i,j} |a_{ji}|$. Here, $^\top$ denotes the transpose of a vector.

Proof Consider

$$I(A, z) = \int_{\mathbb{R}_+^n} \exp\big(-(1_{\mathbb{R}^n} - A^\top z)^\top x\big)\, x_1^{\kappa_1 - 1} \cdots x_n^{\kappa_n - 1}\, dx.$$

Expanding the inner product gives

$$I(A,z)=\prod_{i=1}^{n}\int_0^\infty x_i^{\kappa_i-1}\exp\Big(-x_i(1-z^\top A_i)\Big)dx_i=\prod_{i=1}^{n}\frac{\Gamma(\kappa_i)}{(1-z^\top A_i)^{\kappa_i}}.$$

On the other hand, applying Lemma 1 with $x=r\xi$, $\xi\in\Delta_{n-1}$, we have

$$I(A,z)=\frac{1}{(n-1)!}\int_{\Delta_{n-1}}\int_0^\infty r^{|\kappa|-1}\prod_{i=1}^{n}\xi_i^{\kappa_i-1}\exp\Big(-r\sum_{i=1}^{n}\xi_i(1-z^\top A_i)\Big)dr\,d\xi.$$

Integrating in r yields

$$\begin{aligned}I(A,z)&=\frac{\Gamma(|\kappa|)}{(n-1)!}\int_{\Delta_{n-1}}\frac{\prod_{i=1}^{n}\xi_i^{\kappa_i-1}}{\left(1-z^\top\sum_{i=1}^{n}\xi_i A_i\right)^{|\kappa|}}\,d\xi\\&=\Gamma(\kappa_1)\cdots\Gamma(\kappa_n)\int_{\mathbb{R}^p}\frac{1}{(1-z^\top x)^{|\kappa|}}M_n(\kappa;A)(dx).\end{aligned}$$

Comparing the two expressions of $I(A,z)$ gives the stated formula. □

Remark 1 . Theorem 1 can be seen as a particular case of the Markov–Krein relation when the discrete measure

$$\nu_A=\sum_{i=1}^{n}\kappa_i\,\delta_{A_i}$$

is considered. In this setting, $M_n(\kappa;A)$ and ν_A satisfy, for $z\in B(0_{\mathbb{R}^p},1/\|A\|)$,

$$\begin{aligned}&\int_{\mathbb{R}^p}\frac{1}{(1-z^\top x)^{|\kappa|}}\,M_n(\kappa;A)(dx)\\&\quad=\exp\Big(-\int_{\mathbb{R}^p}\log(1-z^\top u)\,\nu_A(du)\Big),\end{aligned}$$

which shows that the Markov–Krein relation reduces to the formula of Theorem 1 for this choice of ν_A.

Now we will present the multivariate Markov-Krein transform and their use to generalized spline distribution.

For $l=(l_1,\ldots,l_p)\in\mathbb{N}^p$ and $z=(z_1,\ldots,z_p)\in\mathbb{C}^p$, let

$$|l|=\sum_{i=1}^{p}l_i,\qquad l!=\prod_{i=1}^{p}l_i!,\qquad z^l=\prod_{i=1}^{p}z_i^{l_i},$$

and define the differential operator

$$D_l(z) = \frac{\partial^{|l|}}{\partial z_1^{l_1} \cdots \partial z_p^{l_p}}, \qquad |l| > 0.$$

We introduce the relation $\prec$ on $\mathbb{N}^p$ as follows: for $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$, we write $\alpha \prec \lambda$ if one of the following conditions holds:

1. $|\alpha| < |\lambda|$;
2. $|\alpha| = |\lambda|$ and $\alpha_1 < \lambda_1$;
3. $|\alpha| = |\lambda|, \alpha_1 = \lambda_1, \ldots, \alpha_k = \lambda_k$ and $\alpha_{k+1} < \lambda_{k+1}$ for some $1 \le k < p$.

Lemma 2 ([6]) *Let*

$$F(z_1, \ldots, z_p) = f\big(G(z_1, \ldots, z_p)\big),$$

where G is holomorphic in a neighborhood of $z^0 = (z_1^0, \ldots, z_p^0)$ with values in $\mathbb{C}$, and f is holomorphic in a neighborhood of $G(z^0)$. Then for $\lambda \in \mathbb{N}^p$ with $m = |\lambda|$, one has

$$D_\lambda F(z^0) = \sum_{r=1}^{m} D_r f\big(G(z^0)\big) \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\big[D_{l_j} G(z^0)\big]^{k_j}}{k_j!\,(l_j!)^{k_j}},$$

where

$$\begin{aligned} p(\alpha, r) = \Big\{&(k, l) \in \mathbb{N}^m \times (\mathbb{N}^p)^m \;\Big|\; \exists\, 1 \le s \le m, \text{ with } I_0 \\ &= \{1, \ldots, m-s\},\ I_+ = \{m-s+1, \ldots, m\} : \\ &k_i = l_i = 0\ \forall i \in I_0, \quad k_i > 0,\ 0 \prec l_i\ \forall i \in I_+, \\ &\sum_{i=1}^{m} k_i = r, \quad \sum_{i=1}^{m} k_i l_i = \alpha \Big\}. \end{aligned}$$

In particular, if $F(z) = \exp(G(z))$, then

$$D_\lambda F(z^0) = \exp\big(G(z^0)\big) \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\big[D_{l_j} G(z^0)\big]^{k_j}}{k_j!\,(l_j!)^{k_j}}.$$

Suppose now that μ and ν are probability measures with compact support contained in $B(0_{\mathbb{R}^p}, M)$.

For $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{N}^p$ and $l = (l_1, \ldots, l_p) \in \mathbb{N}^p$, we denote by h_λ^μ the λ-moment of μ and h_l^ν the l-moment of ν:

$$h_\lambda^\mu = \int_{\mathbb{R}^p} t_1^{\lambda_1} \cdots t_p^{\lambda_p}\, \mu(dt), \qquad h_l^\nu = \int_{\mathbb{R}^p} u_1^{l_1} \cdots u_p^{l_p}\, \nu(du).$$

Proposition 1 *For* $\lambda \in \mathbb{N}^p$ *with* $m = |\lambda|$, *one has*

$$h^{\mu}_{\lambda} = \frac{1}{\theta_{|\lambda|}} \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\left((|l_j|-1)!\, h^{\nu}_{l_j}\right)^{k_j}}{k_j!\, (l_j!)^{k_j}},$$

where

$$\theta_{|\lambda|} = \theta(\theta+1)\cdots(\theta+|\lambda|-1).$$

Proof The Markov–Krein transform can be expressed in the form

$$F(z) = \exp(G(z)),$$

where

$$F(z) = \int_{\mathbb{R}^p} \frac{1}{(1-\langle z,t\rangle)^{\theta}}\, \mu(dt), \qquad G(z) = -\int_{\mathbb{R}^p} \log\bigl(1-\langle z,u\rangle\bigr)\, \nu(du).$$

Since both μ and ν have compact support included in $B(0_{\mathbb{R}^p}, M)$, for all $t \in B(0_{\mathbb{R}^p}, M)$ we have

$$|\langle z,t\rangle| \le \|z\|\,\|t\| < M\|z\|,$$

and consequently $|\langle z,t\rangle| < 1$ whenever $z \in B(0_{\mathbb{R}^p}, 1/M)$. Thus F and G are holomorphic on $B(0_{\mathbb{R}^p}, 1/M)$.

Differentiating, we obtain

$$D_{\lambda} F(z) = \theta(\theta+1)\cdots(\theta+|\lambda|-1) \int_{\mathbb{R}^p} \frac{t^{\lambda}}{(1-\langle z,t\rangle)^{\theta+|\lambda|}}\, \mu(dt),$$

and

$$D_{\lambda} G(z) = (|\lambda|-1)! \int_{\mathbb{R}^p} \frac{u^{\lambda}}{(1-\langle z,u\rangle)^{|\lambda|}}\, \nu(du),$$

where $t^{\lambda} = \prod_{i=1}^{p} t_i^{\lambda_i}$ and $u^{\lambda} = \prod_{i=1}^{p} u_i^{\lambda_i}$. Applying Lemma 2 at $z=0$ gives

$$D_{\lambda} F(0) = \exp(G(0)) \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\left[D_{l_j} G(0)\right]^{k_j}}{k_j!\, (l_j!)^{k_j}}.$$

Moreover,

$$D_{\lambda} F(0) = \theta_{|\lambda|}\, h^{\mu}_{\lambda}, \qquad D_{\lambda} G(0) = (|\lambda|-1)!\, h^{\nu}_{\lambda}, \qquad G(0) = 0.$$

Substituting into the previous relation yields

$$\theta_{|\lambda|}\, h^{\mu}_{\lambda} = \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\left((|l_j|-1)!\, h^{\nu}_{l_j}\right)^{k_j}}{k_j!\, (l_j!)^{k_j}},$$

which proves the result. □

Theorem 2 *Let ν be a nonzero positive measure on $\mathbb{R}^p$ with compact support. Then there exists a unique probability measure μ on $\mathbb{R}^p$ with compact support such that ν and μ are linked through the multivariate Markov–Krein relation.*

Proof Let ν be a measure with compact support on $\mathbb{R}^p$. If such a measure μ exists, its moments are uniquely determined by the preceding formula given in Proposition 1.

Let the support of ν be contained in $B(0_{\mathbb{R}^p}, M)$. There exists a sequence (ν_n) of discrete measures supported in $B(0_{\mathbb{R}^p}, M)$ of the form

$$\nu_n = \sum_{i=1}^{n} \kappa_i^{(n)}\, \delta_{A_i^{(n)}}, \qquad A_i^{(n)} \in \mathbb{R}^p,$$

such that $\nu_n(\mathbb{R}^p) = \nu(\mathbb{R}^p) = \theta$, and ν_n converges vaguely to ν. For every continuous function f on $B(0_{\mathbb{R}^p}, M)$ we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^p} f(u)\, \nu_n(du) \;=\; \int_{\mathbb{R}^p} f(u)\, \nu(du).$$

The measure ν_n is linked to a measure $\mu_n = M_n(\kappa^{(n)}, A^{(n)})$ via the Markov–Krein transform, where $A^{(n)} = (A_1^{(n)}, \ldots, A_n^{(n)})$ is the matrix of support points and $\kappa^{(n)} = (\kappa_1^{(n)}, \ldots, \kappa_n^{(n)})$ are the corresponding weights. The relation reads

$$\int_{\mathbb{R}^p} \frac{1}{\big(1 - \langle z, x\rangle\big)^{|\kappa^{(n)}|}}\, M_n(\kappa^{(n)}; A^{(n)}; dx) = \exp\left(- \int_{\mathbb{R}^p} \log\big(1 - \langle z, u\rangle\big)\, \nu_{A^{(n)}}(du) \right).$$

Denote by $h_\lambda^{\mu_n}$ the λ-moment of μ_n and by $h_l^{\nu_n}$ the l-moment of ν_n. Then $h_\lambda^{\mu_n}$ can be expressed as a polynomial in the moments $h_l^{\nu_n}$:

$$h_\lambda^{\mu_n} = \frac{1}{\theta_{|\lambda|}} \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\big((|l_j| - 1)!\, h_{l_j}^{\nu_n}\big)^{k_j}}{k_j!\,(l_j!)^{k_j}}.$$

Since $\nu_n \to \nu$, for each $l \in \mathbb{N}^p$ the sequence $h_l^{\nu_n}$ converges to h_l^{ν}, the l-moment of ν. Consequently, $h_\lambda^{\mu_n}$ converges to h_λ^{μ}, where

$$h_\lambda^{\mu} = \frac{1}{\theta_{|\lambda|}} \sum_{r=1}^{m} \sum_{p(\alpha,r)} (\alpha!) \prod_{j=1}^{m} \frac{\big((|l_j| - 1)!\, h_{l_j}^{\nu}\big)^{k_j}}{k_j!\,(l_j!)^{k_j}}.$$

It follows that (μ_n) converges to a probability measure μ on $\mathbb{R}^p$ with compact support. Moreover, the moments of μ are expressed in terms of the moments of ν according to the above relation. Thus, ν and μ are related through the Markov–Krein transform, and uniqueness of μ follows from uniqueness of the moment problem on compact sets. □

3 Multivariate Generalized Gamma Convolutions and Thorin-Class

For probability distributions on the real line, [27] introduced the smallest class containing all Gamma distributions, closed under convolution and weak convergence and represents a subclass of the Bondesson class. He called this class the *generalized Gamma convolutions*, denoted by $\mathcal{T}(\mathbb{R})$.

The multivariate analogues of $\mathcal{T}(\mathbb{R})$, denoted by $\mathcal{T}(\mathbb{R}^p)$, are defined as subclasses of $ID(\mathbb{R}^p)$, the class of infinitely divisible distributions on $\mathbb{R}^p$. A distribution $\mu \in \mathcal{T}(\mathbb{R}^p)$ is characterized by the fact that its Lévy measure has radial components satisfying the same structural properties as those arising in the one-dimensional Thorin class. Thus, $\mathcal{T}(\mathbb{R}^p)$ is the natural multivariate extension of the Thorin class.

Let (A, ν, γ) denote the Lévy–Khintchine triplet of a measure $\mu \in ID(\mathbb{R}^p)$. Its characteristic function is given by

$$\widehat{\mu}(z) = \exp\left(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^p} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \frac{1}{1 + \|x\|^2} \right) \nu(dx) \right).$$

Here A is a symmetric non-negative definite $p \times p$ matrix, $\gamma \in \mathbb{R}^p$, and ν is the Lévy measure of μ. A measure ν is a Lévy measure if and only if $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^p} (1 \wedge \|x\|^2)\, \nu(dx) < \infty.$$

When the stronger condition

$$\int_{\mathbb{R}^p} (1 \wedge \|x\|)\, \nu(dx) < \infty$$

holds, the characteristic function reduces to

$$\widehat{\mu}(z) = \exp\left(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma^{'}, z\rangle + \int_{\mathbb{R}^p} \left(e^{i\langle z,x\rangle} - 1 \right) \nu(dx) \right),$$

where $\gamma^{'}$ is called the *drift* of μ.

Let $S_{p-1} = \{\xi \in \mathbb{R}^p : \|\xi\| = 1\}$ denote the unit sphere of $\mathbb{R}^p$ with respect to the norm $\|\cdot\|$. It is well known (see [3]) that any Lévy measure ν on $\mathbb{R}^p$ with $0 < \nu(\mathbb{R}^p) \le \infty$ admits a polar decomposition of the form

$$\int_{\mathbb{R}^p} f(x)\, \nu(dx) = \int_{S_{p-1}} \int_0^\infty f(r\xi)\, \nu_\xi(dr)\, \lambda(d\xi),$$

for all continuous functions f on $\mathbb{R}^p$. Here, λ is a σ-finite measure on S_{p-1}, called the spherical component, and $\{\nu_\xi\}_{\xi \in S_{p-1}}$ is a family of measures on $(0, \infty)$, referred to as the radial components.

The polar decomposition is unique in the following sense. If (λ, ν_ξ) and (λ', ν'_ξ) are two polar decompositions of ν satisfying the above properties, then there exists a measurable function $c : S_{p-1} \to (0, \infty)$ such that

$$\lambda'(d\xi) = c(\xi)\,\lambda(d\xi), \qquad \nu'_\xi(dr) = \frac{1}{c(\xi)}\,\nu_\xi(dr),$$

for λ-almost every $\xi \in S_{p-1}$. Thus, the polar decomposition is unique up to a measurable redistribution between the spherical and radial components.

A distinguished subclass of infinitely divisible distributions is the class $\mathcal{T}(\mathbb{R}^p)$ introduced by [3]. A distribution $\mu \in ID(\mathbb{R}^p)$ belongs to $\mathcal{T}(\mathbb{R}^p)$ if its Lévy measure admits a polar decomposition whose radial components are of the form

$$\nu_\xi(dr) = \frac{k_\xi(r)}{r}\,dr,$$

where k_ξ is a completely monotone function on $(0, \infty)$ for λ-almost every $\xi \in S_{p-1}$. This condition provides a multivariate extension of the classical Thorin measures.

An explicit expression for the characteristic function of distributions in $\mathcal{T}(\mathbb{R}^p)$ can be obtained when the Lévy measure admits a polar decomposition with exponentially radial components. More precisely, consider a probability measure $\mu \in \mathcal{T}(\mathbb{R}^p)$ whose Lévy measure ν admits a polar decomposition (λ, ν_ξ) of the form

$$\nu_\xi(dr) = \frac{e^{-\beta(\xi)r}}{r}\,dr, \qquad \xi \in S_{p-1},$$

where λ is a finite measure on the unit sphere S_{p-1}, and $\beta : S_{p-1} \to \mathbb{R}_+$ is a measurable function. In this case, the characteristic function of μ is given by

$$\widehat{\mu}(z) = \exp\left(\int_{S_{p-1}} \ln\left(\frac{\beta(\xi)}{\beta(\xi) - \langle z, \xi\rangle}\right)\lambda(d\xi)\right), \text{ for } z \in \mathbb{R}^p, \tag{1}$$

whenever the integral is well defined; see [23].

Elementary Gamma distributions constitute the fundamental building blocks of the class $\mathcal{T}(\mathbb{R}^p)$. Given a nonzero vector $B \in \mathbb{R}^p$, parameters $\kappa > 0$ and $a > 0$, and a Gamma random variable $Y \sim \gamma(\kappa, a)$, the random vector $\Gamma_B = YB$ defines an elementary Gamma distribution on $\mathbb{R}^p$ [3, Definition 2.4]. Its characteristic function is explicitly given by

$$\widehat{\gamma_B}(\kappa, a)(z) = \left(1 - ia^{-1}\langle z, B\rangle\right)^{-\kappa}, \qquad z \in \mathbb{R}^p.$$

Using the Lévy–Khintchine representation of the Gamma distribution, this expression may equivalently be written as

$$\widehat{\gamma_B}(\kappa, a)(z) = \exp\left(\kappa \int_0^\infty \frac{e^{ir\langle z, B\rangle} - 1}{r}\,e^{-ar}\,dr\right),$$

which highlights the infinitely divisible structure of Γ_B.

Moreover, the characteristic function admits a polar representation of the form

$$\widehat{\gamma_B}(\kappa, a)(z) = \exp\left(\int_{S_{p-1}} \int_0^{\infty} \frac{e^{ir\langle z,\xi\rangle} - 1}{r}\, e^{-\beta(\xi)r}\, dr\, \lambda(d\xi)\right),$$

where the spherical measure is given by $\lambda = \kappa\, \delta_{B/\|B\|}$ and $\beta : S_{p-1} \to \mathbb{R}_+$ such that $\beta\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|}$ for $x \in \mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}$. Consequently, elementary Gamma distributions belong to the class $\mathcal{T}(\mathbb{R}^p)$, and their Lévy measures exhibit the class of multivariate Thorin measures.

A particularly transparent representation arises when the spherical measure λ is discrete. Assume that

$$\lambda(d\xi) = \sum_{i=1}^{n} \kappa_i\, \delta_{A_i/\|A_i\|}(d\xi), \qquad A_i \in \mathbb{R}^p \setminus \{0\}, \quad \kappa_i > 0.$$

Then the characteristic function simplifies to the product form

$$\widehat{\mu}(z) = \prod_{i=1}^{n} \big(1 - \langle z, A_i\rangle\big)^{-\kappa_i}, \qquad z \in \mathbb{R}^p.$$

This representation shows that μ coincides with the convolution

$$\mu = \gamma_{A_1}(\kappa_1, 1) * \cdots * \gamma_{A_n}(\kappa_n, 1),$$

that is, μ is the distribution of a finite sum of independent elementary Gamma variables. Consequently, the class $\mathcal{T}(\mathbb{R}^p)$ is closed under finite convolution of elementary Gamma distributions.

As a direct result of the explicit product representation of characteristic functions in the Thorin class, a natural connection emerges via Theorem 1 between multivariate Gamma convolutions and generalized multivariate splines through the Markov–Krein identity.

This leads by formula (1) to the following representation, relating multivariate generalized splines by means of a Markov–Krein type transform, expressed on the unit sphere.

Proposition 2 *Let $A = (A_1, \ldots, A_n) \in \mathcal{M}_{p\times n}(\mathbb{R})$ be a matrix whose columns satisfy $A_i \in \mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}$, and let $\kappa = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{R}_+^*)^n$. Denote*

$$|\kappa| = \sum_{i=1}^{n} \kappa_i, \qquad \|A\| = \max_{i,j} |a_{ji}|, \qquad \lambda(d\xi) = \sum_{i=1}^{n} \kappa_i\, \delta_{A_i/\|A_i\|}(d\xi).$$

Then, for every $z \in B\big(0_{\mathbb{R}^p},\ 1/\|A\|\big)$, the generalized multivariate spline $M_n(\kappa; A)$ satisfies the Markov–Krein type identity

$$\int_{\mathbb{R}^p} \frac{1}{\big(1-\langle z,x\rangle\big)^{|\kappa|}}\, M_n(\kappa; A)(dx) = \exp\left(-\int_{S_{p-1}} \ln\left(1-\left\langle z, \frac{\xi}{\beta(\xi)}\right\rangle\right) \lambda(d\xi)\right),$$

Proof The proof follows immediately from Theorem 1 and equation (1) for

$$\lambda(d\xi) = \sum_{i=1}^{n} \kappa_i\, \delta_{A_i/\|A_i\|}(d\xi), \qquad A_i \in \mathbb{R}^p \setminus \{0\}, \quad \kappa_i > 0$$

and $\beta\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|}$ for $x \in \mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}$. □

Elementary Gamma distributions thus play a central structural role in the class $\mathcal{T}(\mathbb{R}^p)$. Beyond the fact that finite convolutions of elementary Gamma laws belong to $\mathcal{T}(\mathbb{R}^p)$, a fundamental result due to Barndorff–Nielsen shows that this class is in fact generated by such distributions. More precisely, $\mathcal{T}(\mathbb{R}^p)$ is the smallest class of probability measures on $\mathbb{R}^p$ that is closed under convolution and weak convergence and that contains all elementary Gamma distributions. This characterization identifies multivariate Thorin distributions as limits of generalized Gamma convolutions and provides their canonical building blocks.

4 Asymptotic of Spline Distribution

In this section, we consider a sequence of measures (ν_n) on $\mathbb{R}^p$ with compact support, and the corresponding sequence of their multivariate Markov–Krein transforms (μ_n). We will study the convergence of the sequence (μ_n) under the assumption that $\theta_n = \nu_n(\mathbb{R}^p) \to \infty$ as $n \to \infty$.

Lemma 3 *Let (μ_n) be a sequence of measures such that $\mu_n \to \mu$ weakly, and let (θ_n) be a sequence with $\theta_n \to +\infty$ as $n \to \infty$. Define, for $z \in \mathbb{R}^p$,*

$$F_n(z) = \int_{\mathbb{R}^p} \left(1 - \frac{\langle z,t\rangle}{\theta_n}\right)^{-\theta_n} \mu_n(dt).$$

Then (F_n) converges uniformly on every compact subset of $\mathbb{R}^p$ to the function

$$F(z) = \int_{\mathbb{R}^p} e^{\langle z,t\rangle}\mu(dt).$$

Proof Fix a compact set $K \subset \mathbb{R}^p$. Choose $R > 0$ such that $|z| \le R$ for all $z \in K$. Since $\theta_n \to \infty$, there exists N_0 such that for all $n \ge N_0$ we have $\theta_n > 2RM$. Then for any $z \in K$, $t \in B(0, M)$, and $n \geq N_0$:

$$\left|\frac{\langle z, t\rangle}{\theta_n}\right| \leq \frac{RM}{\theta_n} < \frac{1}{2},$$

so $1 - \frac{\langle z,t\rangle}{\theta_n} > \frac{1}{2}$ and the integrand is well-defined.

The function $g_n(s) = \left(1 - \frac{s}{\theta_n}\right)^{-\theta_n}$ converges uniformly to e^s on the interval $[-RM, RM]$ as $n \to \infty$. Therefore, for any $\epsilon > 0$, there exists $N_1 \geq N_0$ such that for all $n \geq N_1$ and all $s \in [-RM, RM]$:

$$\left|\left(1 - \frac{s}{\theta_n}\right)^{-\theta_n} - e^s\right| < \epsilon.$$

Since $|\langle z, t\rangle| \leq RM$ for all $z \in K$ and $t \in B(0, M)$, we have:

$$\sup_{\substack{z\in K\\ t\in B(0,M)}} \left|\left(1 - \frac{\langle z, t\rangle}{\theta_n}\right)^{-\theta_n} - e^{\langle z,t\rangle}\right| < \epsilon.$$

Define $f_n(z) = \int_{\mathbb{R}^p} e^{\langle z,t\rangle} \mu_n(dt)$. For $n \geq N_1$ and $z \in K$:

$$|F_n(z) - f_n(z)| \leq \int_{B(0,M)} \left|\left(1 - \frac{\langle z, t\rangle}{\theta_n}\right)^{-\theta_n} - e^{\langle z,t\rangle}\right| \mu_n(dt) < \epsilon\mu_n(\mathbb{R}^p).$$

Since $\mu_n \to \mu$ weakly and all measures are supported in $B(0, M)$, their total masses converge: $\mu_n(\mathbb{R}^p) \to \mu(\mathbb{R}^p)$. Thus $\sup_n \mu_n(\mathbb{R}^p) = C < \infty$. Hence for $n \geq N_1$:

$$\sup_{z\in K} |F_n(z) - f_n(z)| < C\epsilon,$$

so $F_n - f_n \to 0$ uniformly on K.

Finally, since $\mu_n \to \mu$ weakly, the Laplace transforms f_n converge to

$$F(z) = \int_{\mathbb{R}^p} e^{\langle z,t\rangle} \mu(dt),$$

uniformly on compact subsets of $\mathbb{R}^p$. Therefore, for sufficiently large n, we have

$$|f_n(z) - F(z)| < \frac{\epsilon}{2}, \qquad \forall z \in K.$$

Combining this with the above estimate gives $|F_n(z) - F(z)| < \epsilon$ uniformly on K, which establishes the claim. □

Let

$$A^{(n)} = \left(A_1^{(n)}, \ldots, A_n^{(n)}\right), \qquad A_i^{(n)} \in \mathbb{R}^p,\ 1 \le i \le n,$$

and let

$$\kappa^{(n)} = \left(\kappa_1^{(n)}, \ldots, \kappa_n^{(n)}\right) \in (\mathbb{R}_+^*)^n.$$

Define the discrete measures

$$\nu_n = \sum_{i=1}^{n} \kappa_i^{(n)} \delta_{A_i^{(n)}}, \qquad \tilde{A}_i^{(n)} = \frac{1}{\theta_n} A_i^{(n)}, 1 \le i \le n, \qquad \theta_n = |\kappa^{(n)}| = \sum_{i=1}^{n} \kappa_i^{(n)}$$

and the probability measure

$$\mu_n = M_n(\kappa^{(n)}, A^{(n)}).$$

Proposition 3 *Assume that $\theta_n \to +\infty$ as $n \to \infty$, and that the sequence (μ_n) converges weakly to a probability measure μ. Then μ is a Thorin distribution.*

Proof Define:

$$F_n(z) = \int_{\mathbb{R}^p} \left(1 - \frac{\langle z, t\rangle}{\theta_n}\right)^{-\theta_n} \mu_n(dt)$$

by Lemma 3

$$\lim_{n\to+\infty} F_n(iy) = F(iy) := \int_{\mathbb{R}^p} e^{i\langle t,y\rangle} \mu(dt).$$

On the other hand, by Markov-Krein formula we have,

$$F_n(z) = \exp\left(-\int_{\mathbb{R}^p} \log\big(1 - \langle z, u\rangle\big) \tilde{\nu}_n(du)\right) = \widehat{\tilde{\mu}}_n(z).$$

where $\tilde{\nu}_n$ is the image of ν_n by the dilation of ratio $1/\theta_n$, and

$$\tilde{\mu}_n = \gamma_{\tilde{A}_1^{(n)}}(\kappa_1^n, 1) * \cdots * \gamma_{\tilde{A}_n^{(n)}}(\kappa_n^n, 1).$$

For $z \in i\mathbb{R}^p$

$$F_n(z) = \int_{\mathbb{R}^p} e^{\langle z,t\rangle} \tilde{\mu}_n(dt).$$

By the Lévy–Cramér Theorem,

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu$$

for weak convergence. Since the set $\mathcal{T}(\mathbb{R}^p)$ of Thorin distributions is closed by weak convergence, it follows that μ is a Thorin–Bondesson distribution.

□

The following theorem describes a representation for the Fourier-Laplace transform of the Thorin-Bondesson distribution μ, limit of the sequence of measures (μ_n) generalized spline distribution.

Define

$$\beta_n = \sum_{i=1}^{n} \kappa_i^{(n)} \tilde{A}_i^{(n)},$$

and, for every $i, j \in \{1, \ldots, p\}$, define the measure

$$\sigma_n^{i,j}(du) := u_i u_j \tilde{\nu}_n(du),$$

where u_i denotes the i-th component of the vector $u \in \mathbb{R}^p$.

$$\sigma_n(du) := \langle z, u\rangle^2 \tilde{\nu}_n(du) = \sum_{1\le i,j\le p} z_i z_j \sigma_n^{i,j}(du), \qquad z \in \mathbb{C}^p$$

Theorem 3 *Assume that $\theta_n \to +\infty$ as $n \to \infty$, the sequence (β_n) converge, and that the sequence of measure $(\sigma_n^{i,j})$ converges weakly for every $i, j \in \{1, \ldots, p\}$.*

Then the sequences of measure (μ_n) converge weakly to a Thorin measure μ whose the Fourier-Laplace transform is given by:

$$\int_{\mathbb{R}^p} e^{\langle z,t\rangle} \mu(dt) = \exp\left(\langle z, \beta\rangle + \int_{\mathbb{R}^p} \frac{\ln(1-\langle z, u\rangle)+\langle z, u\rangle}{\langle z, u\rangle^2} \sum_{1\le i,j\le p} z_i z_j \sigma_{i,j}(du) \right),$$

where $\beta = \lim_{n\to+\infty} \sum_{i=1}^{n} \kappa_i^{(n)} \tilde{A}_i^{(n)}$, and $\sigma_{i,j}$ is a limit of the sequence $(\sigma_n^{i,j})$(weack convergence).

The above formula can be written also,

$$\int_{\mathbb{R}^p} e^{\langle z,t\rangle} \mu(dt) = \exp\left(\langle z, \beta\rangle + \frac{1}{2}\langle z, Az\rangle - \int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} \frac{\ln(1-\langle z, u\rangle)+\langle z, u\rangle}{\langle z, u\rangle^2} \right.$$
$$\left. \sum_{1\le i,j\le p} z_i z_j \sigma_{i,j}(du) \right),$$

where A is a symetric matrix, $A = (\gamma_{i,j})_{1\le i,j\le p}$, $\gamma_{i,j} = \sigma_{i,j}(\{0_{\mathbb{R}^p}\})$.

Proof By Markov-Krein formula, for $z \in i\mathbb{R}^p$, $z \neq 0$

$$F_n(z) = \int_{\mathbb{R}^p} \left(1 - \frac{\langle z, t\rangle}{\theta_n}\right)^{-\theta_n} \mu_n(dt) = \exp\left(- \int_{\mathbb{R}^p} \ln(1 - \langle z, u\rangle)\ \tilde{\nu}_n(du) \right).$$

Moreover,

$$\begin{aligned} F_n(z) &= \exp\left(- \int_{\mathbb{R}^p} \frac{\ln\big(1-\langle z, u\rangle\big)+\langle z, u\rangle}{\langle z, u\rangle^2} \langle z, u\rangle^2 \tilde{\nu}_n(du) + \int_{\mathbb{R}^p} \langle z, u\rangle\ \tilde{\nu}_n(du) \right) \\ &= \exp\left(- \int_{\mathbb{R}^p} \frac{\ln(1-\langle z, u\rangle)+\langle z, u\rangle}{\langle z, u\rangle^2} \langle z, u\rangle^2 \tilde{\nu}_n(du) + \langle z, \beta_n\rangle \right). \end{aligned}$$

Observe that,

$$\lim_{u\to 0}\frac{\ln(1-\langle z,u\rangle)+\langle z,u\rangle}{\langle z,u\rangle^2}=-\frac{1}{2}.$$

Hence, the function

$$g(u)=\frac{\ln(1-\langle z,u\rangle)+\langle z,u\rangle}{\langle z,u\rangle^2}$$

admits a continuous extension to the whole space $\mathbb{R}^p$. In particular, the extended function g is continuous and bounded on $\mathbb{R}^p$.

Since

$$\lim_{n\to\infty}\beta_n=\beta,\qquad\text{and}\qquad\lim_{n\to\infty}\sigma_n^{i,j}=\sigma^{i,j}\quad\text{(weakly)},$$

we obtain that

$$\lim_{n\to+\infty}F_n(z)=\exp\left(\langle z,\beta\rangle-\int_{\mathbb{R}^p}\frac{\ln(1-\langle z,u\rangle)+\langle z,u\rangle}{\langle z,u\rangle^2}\,(\sum_{1\le i,j\le p}z_iz_j\sigma^{i,j})(du)\right)$$

We consider the first and second order moments of the measures μ_n, for $i, j\in\{1,\dots,p\}$,

$$h_i^{(n)}=\int_{\mathbb{R}^p}t_i\,\mu_n(dt),\qquad h_{ij}^{(n)}=\int_{\mathbb{R}^p}t_it_j\,\mu_n(dt),$$

and the first and second order moments of the measure $\tilde{\nu}_n$, denoted as follows,

$$p_i^{(n)}=\int_{\mathbb{R}^p}t_i\,\tilde{\nu}_n(dt),\qquad p_{ij}^{(n)}=\int_{\mathbb{R}^p}t_it_j\,\tilde{\nu}_n(dt),$$

More explicitly, if we write $\tilde{A}_k^{(n)}=\big(\tilde{A}_{k,1}^{(n)},\dots,\tilde{A}_{k,p}^{(n)}\big)$, we obtain

$$p_i^{(n)}=\sum_{k=1}^{n}\kappa_k^{(n)}\,\tilde{A}_{k,i}^{(n)},\qquad p_{ij}^{(n)}=\sum_{k=1}^{n}\kappa_k^{(n)}\,\tilde{A}_{k,i}^{(n)}\,\tilde{A}_{k,j}^{(n)}.$$

By differentiating both sides of the previous equality (4) with respect to z_i and evaluating at $z=0$, we obtain the following relation between the first- and second-order moments of μ_n and $\tilde{\nu}_n$:

$$h_i^{(n)}=p_i^{(n)},\qquad h_{ij}^{(n)}=\frac{\theta_n+1}{\theta_n}\left(p_{ij}^{(n)}+p_i^{(n)}p_j^{(n)}\right).$$

Since the sequences $\big(\sum_{i=1}^n\kappa_i^{(n)}\,\tilde{A}_i^{(n)}\big)$ and $(\sigma_n^{i,j})$ converge weakly, it follows that the sequences $(p_i^{(n)})$ and $(p_{ij}^{(n)})$ also converge. Consequently, the second order moments $h_{ij}^{(n)}$ of the sequence (μ_n) are bounded, then the sequence (μ_n) is relatively compact. Let $\tilde{\mu}$ denote the limit of a convergent subsequence of (μ_n). Let $\tilde{\mu}$ denote the limit of a convergent subsequence of (μ_n). Since $\theta_n \to +\infty$ as $n \to \infty$, it follows by passing to the limit and using Lemma 3,

$$\int_{\mathbb{R}^p} e^{\langle z,t\rangle}\,\tilde{\mu}(dt)=\exp\left(\langle z,\beta\rangle+\int_{\mathbb{R}^p}\frac{\ln(1-\langle z,u\rangle)+\langle z,u\rangle}{\langle z,u\rangle^2}\,(\sum_{1\le i,j\le p} z_i z_j\,\sigma^{i,j})(du)\right).$$

This shows that there exists only one possible limit for any converging subsequence. Therefore, the sequence (μ_n) converges.

□

Lemma 4 *Let (ν_n) be a sequence of discrete measures of the form*

$$\nu_n = \sum_{i=1}^{n} \kappa_i^{(n)} \delta_{A_i^{(n)}},$$

where $A_i^{(n)} \in \mathbb{R}^p$ and $\kappa_i^{(n)} > \kappa > 0$ for all n and i. Assume that ν_n converges vaguely to a measure ν. Then ν is also a discrete measure of the form

$$\nu = \sum_{j=1}^{\infty} \kappa_j \delta_{A_j},$$

where $A_j \in \mathbb{R}^p$ and $\kappa_j \geq \kappa$ for all j.

Proof Let $\mathbb{A}$ denote the set of atoms of ν. Since ν is σ-finite, $\mathbb{A}$ is countable. For any atom $A \in \mathbb{A}$, there exists $R > 0$ such that $\partial B(A, R) \cap \mathbb{A} = \emptyset$, where $\partial B(A, R) = \{x \in \mathbb{R}^p : \|x - A\| = R\}$ is the boundary of the ball. This condition ensures that no atom lies exactly on the boundary, which is important because vague convergence guarantees convergence only for sets whose boundaries carry no measure. Then, by vague convergence,

$$\nu(B(A, R)) = \lim_{n\to\infty} \nu_n(B(A, R)) \geq \kappa.$$

Constructing a decreasing sequence (R_n) tending to zero with $\partial B(A, R_n) \cap \mathbb{A} = \emptyset$, we obtain

$$\nu(\{A\}) = \lim_{n\to\infty} \nu(B(A, R_n)) \geq \kappa,$$

showing that every atom has measure at least κ and that each ball contains only finitely many atoms, so $\mathbb{A}$ is discrete. Next, consider any compact set $\mathcal{C}$ not containing atoms of ν. For each $x \in \mathcal{C}$, there exists $R_x > 0$ with $\nu(B(x, R_x)) < \kappa$; otherwise x would be an atom. The balls $B(x, R_x)$ cover $\mathcal{C}$, and by compactness, a finite subcover exists. Summing over this subcover yields $\nu(\mathcal{C}) = 0$, showing that ν has no non-atomic part. Finally, taking an increasing sequence of balls exhausting $\mathbb{R}^p$ and enumerating all atoms in these balls, we conclude that ν can be written as $\nu = \sum_{j=1}^{\infty} \kappa_j \delta_{A_j}$ with $A_j \in \mathbb{R}^p$ and $\kappa_j \geq \kappa > 0$. □

Theorem 4 *Assume that the numbers $\kappa_i^{(n)}$ satisfy $\kappa_i^{(n)} > \kappa$. Assume furthermore that the sequence $\sigma_n^{i,j}$ of measures converge weackly to a measure $\sigma^{i,j}$ for every $i, j \in \{1, p\}$*

(i) Then $\sigma^{i,j}$ has the form

$$\sigma^{i,j} = \sum_{k=1}^{\infty} \kappa_k A_{k,i} A_{k,j} \, \delta_{A_k} \; + \; \gamma_{i,j} \delta_{0_{\mathbb{R}^p}},$$

where (A_k) is a sequence in $\mathbb{R}^p$, $A_{k,i}$ denotes the i-th component of A_k, $\gamma_{i,j} = \sigma^{i,j}(\{0_{\mathbb{R}^p}\})$, $\kappa_k \geq \kappa > 0$, and $\sum_{k \geq 1} \kappa \|A_k\|^2$ converges.

(ii) Assume moreover that, (β_n) converge, Then the measure

$$\mu_n = M_n\big(\kappa^{(n)}, A^{(n)}\big)$$

converges (weackly) to a Thorin distribution such that

$$\int_{\mathbb{R}^p} e^{\langle z,t \rangle} \, \mu(dt) = e^{\langle z,\beta \rangle} e^{\frac{1}{2}\langle z, \mathcal{A} z \rangle} \prod_{i=1}^{+\infty} \left(\frac{e^{-\langle z, A_i \rangle}}{1 - \langle z, A_i \rangle} \right)^{\kappa_i}.$$

Where $\mathcal{A} = (\gamma_{i,j})_{1 \leq i,j \leq p}$,

Proof The sequence $\sigma_n^{i,j}$ of measures converge weakly to a measure $\sigma^{i,j}$ for every $i, j \in \{1, \ldots, p\}$, then a sequence $\eta_n = \|u\|^2 \tilde{\nu}_n(du)$ converge weackly and there exists $M \in \mathbb{R}$ such that:

$$\sum_{k=1}^{n} \kappa_k^{(n)} \|\tilde{A}_i^{(n)}\|^2 \leq M.$$

Since $\kappa_i^{(n)} > \kappa$, for all i and all $n \in \mathbb{N}$, we deduce that

$$\big(\|\tilde{A}_i^{(n)}\|\big)^2 \leq \frac{M}{\kappa}.$$

Thus, the sequence $\tilde{\nu}_n$ is with compact support.

Let f be a continuous function on $\mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}$ with compact support, for every $i, j \in \{1, \ldots, p\}$, define $g(u) = \dfrac{f(u)}{u_i u_j}$

Then

$$\int_{\mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}} f(u) \tilde{\nu}_n(du) = \int_{\mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}} g(u) \sigma_n^{i,j}(du)$$

Hence,

$$\lim_{n\to\infty}\int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} f(u)\tilde{\nu}_n(du) = \lim_{n\to\infty}\int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} g(u)\sigma_n^{i,j}(du) = \int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} g(u)\sigma^{i,j}(du)$$
$$= \int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} f(u)\tilde{\nu}(du).$$

Thus, the sequence $(\tilde{\nu}_n)$ converges vaguely to the measure $\tilde{\nu}(du) = \frac{1}{u_i u_j}\sigma^{i,j}(du)$.

By Lemma 4, there exist two sequences $(\kappa_i) \subset \mathbb{R}_+$ and $(A_i) \subset \mathbb{R}^p$ such that

$$\nu = \sum_{k=1}^{\infty} \kappa_k\, \delta_{A_k}.$$

Then the restriction of $\sigma^{i,j}$ to $\mathbb{R}^p \setminus \{0_{\mathbb{R}^p}\}$ satisfies

$$\sigma^{i,j}_{|\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} = \sum_k \kappa_k\, A_{k,i} A_{k,j} \delta_{A_k}.$$

Let $\gamma_{i,j} = \sigma^{i,j}(0_{\mathbb{R}^p})$. Then

$$\sigma^{i,j} = \sum_k \kappa_k\, A_{k,i} A_{k,j} \delta_{A_k} + \gamma_{i,j}\delta_{0_{\mathbb{R}^p}}.$$

Therefore

$$\sigma = \sum_{1\le i,j\le p} z_i z_j\, \sigma^{i,j} = \sum_k \kappa_k\, \langle z, A_k\rangle^2\, \delta_{A_k} \;+\; \langle z, \mathcal{A}z\rangle\, \delta_{0_{\mathbb{R}^p}}.$$

The sequences $(\kappa_j^{(n)})$ are bounded below by a strictly positive real number, so

$$\lim_{n\to\infty} \kappa_j^{(n)} = +\infty.$$

By Theorem 3, the measure $\mu_n = M_n((A^{(n)}); \kappa^{(n)})$ converges weakly to a Thorin measure μ such that

$$\int_{\mathbb{R}^p} e^{\langle z,t\rangle}\, \mu(dt) = \exp\left(\langle z, \beta\rangle - \int_{\mathbb{R}^p} \frac{\ln(1-\langle z,u\rangle) + \langle z,u\rangle}{\langle z,u\rangle^2}\, \sigma(du)\right).$$

$$\int_{\mathbb{R}^p} \frac{\ln(1-\langle z,u\rangle) + \langle z,u\rangle}{\langle z,u\rangle^2}\, \sigma(du) = g(0)\sigma(0_{\mathbb{R}^p}) + \int_{\mathbb{R}^p\setminus\{0_{\mathbb{R}^p}\}} g(u)\sigma(du) +$$
$$= \sum_k \kappa_k \ln(1-\langle z, A_k\rangle) + \sum_k \kappa_k \langle z, A_k\rangle - \frac{1}{2}\langle z, \mathcal{A}z\rangle$$
$$= \ln\left(\prod_k (1-\langle z, A_k\rangle)^{\kappa_k}\right) + \sum_k \kappa_k\, \langle z, A_k\rangle - \frac{1}{2}\langle z, \mathcal{A}z\rangle.$$

Consequently, the Fourier transform of the measure μ is obtained by applying the exponential. □

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References

1. Barndorff-Nielsen OE (1977) Exponentially decreasing distributions for the logarithm of particle size. Proceedings of the Royal Society of London A 353:401–419
2. Barndorff-Nielsen OE (1997) Normal inverse Gaussian distributions and stochastic volatility modelling. Scand J Stat 24:1–13
3. Barndorff-Nielsen OE, Maejima M, Sato KI (2006) Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. Bernoulli 12:1–33
4. Bondesson L (1992) Generalized Gamma Convolutions and Related Classes of Distributions and Densities. Lecture Notes in Statistics, 76. Springer-Verlag, New York
5. Charfi S, Mselmi F (2022) Modeling exchange rate volatility: application of GARCH models with a Normal Tempered Stable distribution. Quantitative Finance and Economics 6:206–222
6. Constantine GM, Savits TH (1996) A multivariate Faá di Bruno formula with applications. Trans Am Math Soc 348:503–520
7. Curry HB, Schoenberg IJ (1966) On Pólya frequency functions. IV. The fundamental spline functions and their limits. Journal d'Analyse Mathématique 17:71–107
8. Dahmen W (1980) On multivariate B-splines. SIAM J Numer Anal 17(2):179–191
9. de Boor C, Fix G (1973) Spline approximation by quasi-interpolants. J Approx Theory 7:19–45
10. de Boor C (1976) Splines as linear combinations of B-splines. In: Lorentz GG, Chui CK, Schumaker LL (eds) Approximation Theory II. Academic Press, New York, pp 1–47
11. de Boor C, Höllig K, Riemenschneider S (1993) Box Splines. Springer-Verlag
12. de Boor C, DeVore RA, Ron A (1993) Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$. Trans Am Math Soc 341:787–806
13. Fourati F (2011) *Distributions de Dirichlet, mesures orbitales et transformation de Markov–Krein*. Thèse de doctorat, Université Paris VI & Université El Manar, Tunis
14. Fourati F, Faraut J (2016) Markov-Krein transform. Colloq Math 144(1):137–156
15. James LF (2005) Functionals of Dirichlet processes, the Cifarelli-Regazzini identity and beta-gamma processes. Ann Stat 33:647–660
16. Kerov SV, Tsilevich NV (2004) The Markov-Krein correspondence in several dimensions. J Math Sci 121:2345–2359
17. Louati M, Masmoudi A, Mselmi F (2020) The normal tempered stable regression model. Communications in Statistics - Theory and Methods 49(2):500–512
18. Madan DB, Carr P, Chang EC (1998) The Variance Gamma Process and Option Pricing. Eur Finance Rev 2:79–105
19. Micchelli CA (1979) A constructive approach to Kergin interpolation in $\mathbb{R}^k$. Rocky Mountain Journal of Mathematics 9:485–497
20. Mselmi F (2021) Generalized linear model for subordinated Lévy processes. Scand J Stat 49(2):772–801
21. Nakano F, Trinh HD, Trinh KD (2016) Classical beta-ensembles and related eigenvalues processes at high temperature and the Markov-Krein transform. J Math Phys 66:053304
22. Nolan JP (2020) Stable Distributions – Models for Heavy Tailed Data. Birkhäuser

23. Pérez-Abreu V, Stelzer R (2012) Class of infinitely divisible multivariate and matrix Gamma distributions and cone-valued generalised Gamma convolutions. *arXiv preprint*, arXiv:1201.1461
24. Risler J-J (1992) Mathematical methods for CAD. In *Handbook of Numerical Analysis, Vol. IV*, P. G. Ciarlet and J.-L. Lions (eds.), Elsevier, pp. 523–572
25. Saha D, Alluri P, Dumbaugh E, Gan A (2020) Application of the Poisson-Tweedie distribution in analyzing crash frequency data. Accident Analysis & Prevention 137:105–115
26. Samorodnitsky G, Taqqu MS (1994) Stable Non-Gaussian Random Processes. Chapman & Hall
27. Thorin O (1978) An extension of the notion of a generalized Γ-convolution. Scand Actuar J 1978:141–149
28. Vershik A, Yor M, Tsilevich N (2004) On the Markov-Krein identity and quasi-invariance of the gamma process. J Math Sci 121:2303–2310
29. Willmot GE (1987) The Poisson-inverse Gaussian distribution. ASTIN. Bulletin 17:185–201
30. Zha L, Lord D, Zou Y (2016) The Poisson Inverse Gaussian generalized linear regression model. J Transportation Safety & Security 8:18–35
31. Faraut J (2005) Noyau de Peano et intégrales orbitales, *Proceedings of the Symposium of the Tunisian Mathematical Society*, *Global Journal of Pure and Applied Mathematics*, **1** 306–320
32. Okounkov A, Olshanski G (1997) Shifted Jack polynomials, binomial formula, and applications. Math Res Lett 4:69–78
33. Diaconis P, Kemperman J (1996) Some new tools for Dirichlet priors, in Bayesian Statistics 5. Oxford University Press, New York, pp 97–106
34. Cifarelli DM, Regazzini E (1990) Distribution functions of means of a Dirichlet process. Ann Stat 18:429–442
35. Krein MG, Nudel'man AA (1997) *The Markov Moment Problem and Extremal Problems*, American Mathematical Society,
36. Kerov S (2003) Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis, Translations of Mathematical Monographs, Vol. 219, American Mathematical Society,
37. Lyche T, Schumaker LL (1975) Local spline approximation methods. J Approx Theory 15:294–325