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ORIGINAL ARTICLE

Controllability of a fluid–structure interaction system governed by the heat and damped beam equations

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Abstract
In this article, we study the null-controllability and observability properties of a fluid–structure interaction system, governed by the heat equation coupled with the damped beam equation. To do so, we prove a global Carleman estimate for the coupled system, making explicit the dependence on the damping parameter. In the course of the study, we demonstrate several inequalities for the beam equation alone, valid for different damping parameter regimes.

Keywords Carleman estimates · Observability inequality · Null-controllability · Fluid–structure interaction system

Mathematics Subject Classification 76D05 · 35Q30 · 74F10 · 76D55 · 76D27 · 93B05 · 93B07 · 93C10

Contents

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1 Introduction and main results

Fluid–structure interaction (FSI) models are central to understanding the interplay between fluid dynamics and deformable boundaries in physical, biological, engineering systems, etc. In particular, the cardiovascular system illustrates such interactions, where pulsatile blood flow interacts with the elastic properties of large vessels walls. The comprehensive modeling of these phenomena has been the focus of foundational works detailing the complex coupling between fluid dynamics and vascular wall mechanics in precise geometries. The model consists in a coupling between the 2D-Navier–Stokes equations in a channel and a damped beam equation that describes the deformation of the boundary of the domain. We refer to [1] and [2] for the model derivation, and we refer to [3] (and references therein) for the mathematical well posedness of the model. Controllability issues have been tackled very recently in [4, 5] by coupling microlocal analysis and global Carleman inequalities.

In contrast to these realistic models, the present work deals with the null-controllability of a simplified fluid–structure interaction system where the Navier–Stokes equations are replaced by the heat equation. The proposed model retains the core framework of a fluid equation coupled at the boundary with a structural equation such as the damped beam equation, thereby capturing the basic dynamic interaction between a diffusive field and a deformable interface. The aim is not to fully capture the biomechanical behavior of blood vessels, but rather to explore the essential mathematical features of a coupled system within a tractable setting. Specifically, our approach relies entirely on global Carleman inequalities, and allows us to consider arbitrarily large structural damping for the beam—features absent from the previously cited works (see Sect. 1.4).

In the present study, we focus on a two-dimensional interaction system set in a rectangular domain, with periodic boundary conditions on the lateral boundary of the rectangle. Mathematically, we set $\mathcal{I} = \mathbb{R}/2\pi\mathbb{Z}$ the one-dimensional torus, and, for some $T > 0$, define

$$\Omega := \mathcal{I} \times (0, 1), \quad \Gamma_1 = \mathcal{I} \times \{1\}, \quad \Gamma_0 = \mathcal{I} \times \{0\}, \quad Q = (0, T) \times \Omega.$$

In the following, we will also identify $\mathcal{I}$ and Γ_1 when it does not lead to confusion. The precise meaning of this identification is given in Sect. 2.

In this geometrical setting, we consider, for some initial conditions $(w_0, \zeta_0, \zeta_1) \in L^2(\Omega) \times H^2(\mathcal{I}) \times L^2(\mathcal{I})$ and some source terms $G \in L^2(Q)$ and $H \in L^2((0,T)\times\mathcal{I})$, the forward system of equations

$$\begin{cases} \partial_t w - \Delta w = G & \text{in } Q, \\ w = 0 & \text{on } (0,T)\times\Gamma_0, \\ w = \partial_t \zeta & \text{on } (0,T)\times\Gamma_1, \\ \partial_t^2 \zeta - \alpha \partial_{x_1}^2 \partial_t \zeta + \alpha \partial_t \zeta + \partial_{x_1}^4 \zeta + \zeta = -\partial_n w + H & \text{in } (0,T)\times\mathcal{I}, \\ w(0,\cdot) = w_0, \text{ in } \Omega, \quad \zeta(0,\cdot) = \zeta_0, \ \partial_t \zeta(0,\cdot) = \zeta_1 & \text{in } \mathcal{I}. \end{cases} \tag{1.1}$$

In (1.1), the first heat equation in the domain Ω plays the role of a fluid equation submitted to some external forces G. The condition $w = \partial_t \zeta$ on the boundary indicates that the velocity of the fluid coincides with the velocity of the beam's displacement. The fourth line in (1.1) represents the structural response of the beam under the external force H and under the force exerted by the fluid that is represented by $-\partial_n w$. The coefficient $\alpha > 0$ is a damping parameter.

We emphasize that the terms ζ and $\alpha\partial_t\zeta$ in the fourth equation of system (1.1) are included for technical reasons only. This simplifies the presentation of the paper by straightforwardly ensuring the positivity of the underlying elliptic operators (see (2.3) and (2.4) below). Note that in the original fluid–structure system, the incompressibility condition ensures that the beam deformation ζ has a zero mean value (see the introduction of [6]), and the elliptic operators are inherently positive without the additional terms.

System (1.1) is well posed (see Sect. 2): It admits a unique solution (w, ζ) satisfying

$$(w, \zeta, \partial_t \zeta) \in C([0,T]; L^2(\Omega) \times H^2(\mathcal{I}) \times L^2(\mathcal{I})) \tag{1.2}$$

and the following regularity for all $\varepsilon > 0$:

$$\begin{gathered} w \in L^2(\varepsilon, T; H^2(\Omega)) \cap H^1(\varepsilon, T; L^2(\Omega)), \\ \zeta \in L^2(\varepsilon, T; H^4(\mathcal{I})) \cap H^1(\varepsilon, T; H^2(\mathcal{I})) \cap H^2(\varepsilon, T; L^2(\mathcal{I})). \end{gathered} \tag{1.3}$$

1.1 Control problem

We now focus on our main problem of interest, the null-controllability of the interaction system. More precisely, we consider ω, a nonempty open subset of Ω, and J, a nonempty open subset of $\mathcal{I}$. In the context of controllability, the version of system (1.1) we are interested in reads

$$\begin{cases} \partial_t w - \Delta w = \mathbb{1}_\omega g & \text{in } Q, \\ w = 0 & \text{on } (0,T)\times\Gamma_0, \\ w = \partial_t \zeta & \text{on } (0,T)\times\Gamma_1, \\ \partial_t^2 \zeta - \alpha \partial_{x_1}^2 \partial_t \zeta + \alpha \partial_t \zeta + \partial_{x_1}^4 \zeta + \zeta = -\partial_n w + \mathbb{1}_J h & \text{in } (0,T)\times\mathcal{I}, \\ w(0,\cdot) = w_0, \text{ in } \Omega, \quad \zeta(0,\cdot) = \zeta_0, \ \partial_t \zeta(0,\cdot) = \zeta_1 & \text{in } \mathcal{I}, \end{cases} \tag{1.4}$$

where $g \in L^2((0,T)\times\omega)$ and $h \in L^2((0,T)\times J)$ are the distributed controls acting, respectively, on the heat equation and on the damped beam equation. The functions $\mathbb{1}_\omega$ and $\mathbb{1}_J$ denote the characteristic functions on ω and J, respectively.

The geometrical setting of our study is illustrated in Fig. 1 below:

Definition 1.1 We say that system (1.4) is null-controllable at time $T > 0$ if there exists a constant $C_T > 0$ such that, for all initial conditions $w_0 \in L^2(\Omega)$, $\zeta_0 \in H^2(\mathcal{I})$ and $\zeta_1 \in L^2(\mathcal{I})$, there exist $g \in L^2((0,T)\times\omega)$ and $h \in L^2((0,T)\times J)$ such that (w,ζ) solution of (1.4) satisfies $(w(T),\zeta(T),\partial_t\zeta(T)) = (0,0,0)$, and

$$\|g\|_{L^2((0,T)\times\omega)} + \|h\|_{L^2((0,T)\times J)} \leq C_T \left(\|w_0\|_{L^2(\Omega)} + \|\zeta_0\|_{H^2(\mathcal{I})} + \|\zeta_1\|_{L^2(\mathcal{I})}\right).$$

In this study, we prove that system (1.4) is null-controllable in any arbitrary small time $T > 0$, for any choice of the damping parameter $\alpha > 0$.

Theorem 1.2 *Let $T > 0$ and $\alpha > 0$. System (1.4) is null-controllable at time T, and there exist constants $C > 0$ and $\tau > 0$—independent of T and α—such that the cost C_T can be chosen as*

$$C_T = C \exp\left(e^{C\lambda_\alpha}\left(1+\frac{1}{T}\right)\right),$$

where

$$\lambda_\alpha = \begin{cases} \tau\alpha^{-2} & \text{if } \alpha \leq 1, \\ \tau\alpha^{\frac{2}{3}} & \text{if } \alpha \geq 1. \end{cases}$$

Remark 1.3 In Theorem 1.2, the dependence of the constant C_T on the time horizon T is improved compared to [4], where the constant takes the form $C_T = Ce^{\frac{C}{T^2}}$. This is because [4] utilizes the Carleman inequality for the damped beam equation presented in [7], where the dependence of the Carleman parameter s on T is suboptimal. We improve this dependence in Theorem 1.12 and in Theorem 1.13, see Remark 1.14.

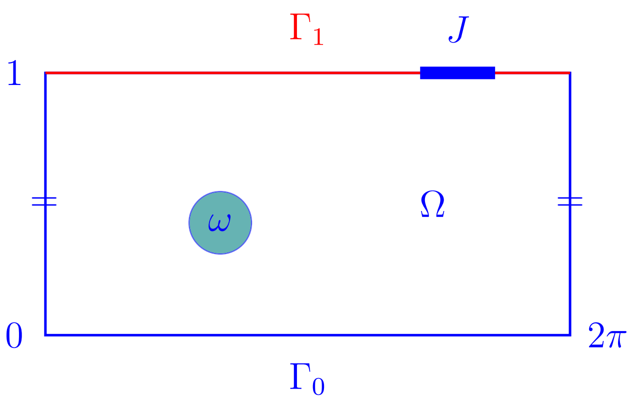

Fig. 1 Geometrical setting, with control domains ω and J

To prove this theorem, we use the classical duality strategy, and prove an observability inequality for the adjoint system, which in our context reads (see Sect. 2):

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } Q, \\ u = 0, & \text{on } (0,T) \times \Gamma_0, \\ u = \partial_t \eta, & \text{on } (0,T) \times \Gamma_1, \\ \partial_t^2 \eta - \alpha \partial_{x_1}^2 \partial_t \eta + \alpha \partial_t \eta + \partial_{x_1}^4 \eta + \eta = -\partial_n u & \text{in } (0,T) \times \mathcal{I}, \\ u(0,\cdot) = u_0, \text{ in } \Omega, \quad \eta(0,\cdot) = \eta_0, \ \partial_t \eta(0,\cdot) = \eta_1 & \text{in } \mathcal{I}. \end{cases} \tag{1.5}$$

Here, the initial conditions satisfy $u_0 \in L^2(\Omega)$, $\eta_0 \in H^2(\mathcal{I})$, $\eta_1 \in L^2(\mathcal{I})$, and as a consequence system (1.5) is well posed.

Definition 1.4 Let $T > 0$. We say that system (1.5) is observable at time T from the observation sets ω and J if there exists a constant $K_T > 0$ such that any solution of system (1.5) satisfies

$$\|u(T)\|_{L^2(\Omega)} + \|\eta(T)\|_{H^2(\mathcal{I})} + \|\partial_t \eta(T)\|_{L^2(\mathcal{I})} \le K_T \left(\|u\|_{L^2((0,T)\times\omega)} + \|\partial_t \eta\|_{L^2((0,T)\times J)} \right). \tag{1.6}$$

Very classically, system (1.4) is null-controllable at time T if and only if system (1.5) is observable at time T, and in that case we can choose $C_T = K_T$ (see Sect. 2). We prove the following result:

Theorem 1.5 *For all $T > 0$, and all $\alpha > 0$, the system (1.5) is observable at time T from the observation sets ω and J. Furthermore, there exist constants $C > 0$ and $\tau > 0$—independent of T and α—such that K_T can be chosen as*

$$K_T = C \exp\left(e^{C\lambda_\alpha} \left(1 + \frac{1}{T} \right) \right),$$

where

$$\lambda_\alpha = \begin{cases} \tau \alpha^{-2} & \text{if } \alpha \le 1, \\ \tau \alpha^{\frac{2}{3}} & \text{if } \alpha \ge 1. \end{cases}$$

The main ingredient in our proof of Theorem 1.5 is a new global Carleman estimate for the whole heat-damped beam system (1.5). This Carleman estimate, which constitutes the main novelty of our study, is of interest in its own right. We present this new result in the next section.

Another novelty of Theorem 1.5 is its validity for any positive damping parameter α. Maybe unexpectedly, new difficulties appear for large parameters α, for which we need the coupling between the heat equation and the damped beam equation to obtain the global Carleman estimate.

1.2 Carleman estimate for the coupled system

We now present our new global Carleman estimate for the system (1.5). First, we need to introduce the following weight functions. We recall that the existence of such functions is proved in [4, Section 2.2]. We consider the function ℓ defined by

$$\ell(t) := t(T-t), \quad \text{for } t \in (0,T).$$

Let ω_0 and J_0 be two nonempty open sets satisfying $\omega_0 \Subset \omega$ and $J_0 \Subset J$. We consider two smooth functions ψ_Ω and $\psi_\mathcal{I}$ satisfying

$$\psi_\Omega > 0 \text{ in } \Omega, \quad \psi_\Omega = 0 \text{ and } \partial_n \psi_\Omega = -1 \text{ on } \partial\Omega, \ \nabla\psi_\Omega(x) = 0 \Rightarrow x \in \omega_0. \tag{1.7}$$

$$\psi_\mathcal{I} > 0 \text{ on } \mathcal{I}, \quad \psi'_\mathcal{I}(x_1) = 0 \Rightarrow x_1 \in J_0. \tag{1.8}$$

Let us consider

$$\Psi := \|\psi_\Omega\|_{L^\infty(\Omega)} + \|\psi_\mathcal{I}\|_{L^\infty(\mathcal{I})} \tag{1.9}$$

and for $k \geq 2$ and $\lambda \geq \mu > 0$, let us define the following functions:

$$\begin{aligned}
\varphi(t,x_1,x_2) &= \frac{e^{\mu\psi_\mathcal{I}(x_1)+\lambda\psi_\Omega(x_1,x_2)+k\lambda\Psi} - e^{(k+2)\lambda\Psi}}{\ell(t)^{k/2}}, \\
\xi(t,x_1,x_2) &= \frac{e^{\mu\psi_\mathcal{I}(x_1)+\lambda\psi_\Omega(x_1,x_2)+k\lambda\Psi}}{\ell(t)^{k/2}},
\end{aligned} \tag{1.10}$$

$$\varphi_0(t,x_1) = \frac{e^{\mu\psi_\mathcal{I}(x_1)+k\lambda\Psi} - e^{(k+2)\lambda\Psi}}{\ell(t)^{k/2}}, \quad \xi_0(t,x_1) = \frac{e^{\mu\psi_\mathcal{I}(x_1)+k\lambda\Psi}}{\ell(t)^{k/2}}. \tag{1.11}$$

Since ψ_Ω vanishes on $\partial\Omega$ we observe that $\varphi_0 = \varphi|_{\partial\Omega}$ and $\xi_0 = \xi|_{\partial\Omega}$.
We also introduce the following weight functions depending only on the time variable t:

$$\varphi_1(t) = \frac{e^{k\lambda\Psi} - e^{(k+2)\lambda\Psi}}{\ell^{k/2}(t)}, \quad \xi_1(t) = \frac{e^{k\lambda\Psi}}{\ell^{k/2}(t)}, \tag{1.12}$$

$$\varphi_2(t) = \frac{e^{(k+1)\lambda\Psi} - e^{(k+2)\lambda\Psi}}{\ell^{k/2}(t)}, \quad \xi_2(t) = \frac{e^{(k+1)\lambda\Psi}}{\ell^{k/2}(t)}. \tag{1.13}$$

Remark 1.6 To derive the results presented in this paper, the additional parameter k is not strictly required; we could fix $k = 2$ throughout the analysis. However, there are situations—in particular for the Stokes system—where choosing the parameter k large enough proves to be highly useful. This is evident, for example, when using energy estimates with a weight that is frozen in space as in [8, Proof of Theorem 1, Step 3]) or in [9, page 364]. For this reason, we opt to keep k as a free parameter, as it may be of interest for future research.

The Carleman estimate for the adjoint system (1.5) is given in the following theorem. Set

$$H(s,\lambda,u) := s^{-1}\iint_{(0,T)\times\Omega} \xi^{-1}\, e^{2s\varphi}\left(|\partial_t u|^2+|\Delta u|^2\right)+s\lambda^2\iint_{(0,T)\times\Omega} \xi\, e^{2s\varphi}|\nabla u|^2$$
$$+s^3\lambda^4\iint_{(0,T)\times\Omega} \xi^3\, e^{2s\varphi}|u|^2+s^3\lambda^3\iint_{\Sigma}\xi_0^3\, e^{2s\varphi_0}|u|^2+s\lambda\iint_{\Sigma}\xi_0 e^{2s\varphi_0}|\partial_n u|^2,$$

and

$$B_\alpha(s,\mu,\eta) := \frac{\alpha^2}{1+\alpha^2}\frac{1}{s}\iint_{(0,T)\times\mathcal{I}}\frac{1}{\xi_0}\, e^{2s\varphi_0}(|\partial_{x_1}^4\eta|^2+|\partial_t^2\eta|^2)+\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}}$$
$$\frac{1}{\xi_0}\, e^{2s\varphi_0}(|\partial_t\partial_{x_1}^2\eta|^2)$$
$$+\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}\Big(s^7\mu^8\xi_0^7|\eta|^2+s^5\mu^6\xi_0^5|\partial_{x_1}\eta|^2+s^3\mu^4\xi_0^3|\partial_{x_1}^2\eta|^2$$
$$+(1+\alpha^2)s^3\mu^4\xi_0^3|\partial_t\eta|^2+s\mu^2\xi_0|\partial_{x_1}^3\eta|^2+(1+\alpha^2)s\mu^2\xi_0|\partial_t\partial_{x_1}\eta|^2\Big).$$

Theorem 1.7 *Let ω and J be nonempty open subsets of Ω and $\mathcal{I}$, respectively, and let $k\geq 2$. There exist constants $c_0>0$, $c_1\geq 1$, $\mu_0\geq 1$ and $\widehat{s_0}\geq 1$ such that, for all $\alpha>0$, for all $\mu\geq\mu_0$ and all λ satisfying $\lambda\geq c_1\mu$, $\lambda\geq c_1(1+1/\alpha^2)$ and $\lambda\in[c_1\alpha^{2/3}\mu^{\frac{4}{3}}, \frac{1+\alpha^2}{c_1}\mu^2]$, for all $T>0$ and for all $s\geq\widehat{s_0}(1+\alpha)(T^k+T^{k-1})$, the following inequality holds for all $u\in C^1([0,T];C(\overline{\Omega}))$ and $\eta\in C^2([0,T];C^4(\mathcal{I}))$ such that $u=\partial_t\eta$ on $(0,T)\times\Gamma_1$ and $u=0$ on $(0,T)\times\Gamma_0$:*

$$H(s,\lambda,u)+B_\alpha(s,\mu,\eta)\leq c_0\Big(s^3\lambda^4\iint_{(0,T)\times\omega}\xi^3\ \ e^{2s\varphi}|u|^2$$
$$+\left(\alpha^{-8}+\alpha^4\right)s^7\mu^8\iint_{(0,T)\times J}\xi_0^7\, e^{2s\varphi_0}|\eta|^2$$
$$+\iint_{(0,T)\times\Omega} e^{2s\varphi}|\partial_t u-\Delta u|^2$$
$$+\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\alpha\partial_t\eta+\partial_{x_1}^4\eta+\eta+\partial_n u|^2\Big). \quad (1.14)$$

Remark 1.8 In Theorem 1.7 one can choose λ proportional to $\alpha^{2/3}$ if $\alpha\geq 1$ and λ proportional to $1/\alpha^2$ if $\alpha<1$. More precisely, we can choose $\theta>0$ large enough and $\tau\in[c_1\theta^{4/3},\theta^2/c_1]$ such that the assumptions of Theorem 1.7 are satisfied with $\lambda=\tau\alpha^{2/3}$ and $\mu=\theta$ for all $\alpha\geq 1$, or with $\lambda=\frac{\tau}{\alpha^2}$ and $\mu=\frac{\theta}{\alpha}$ for all $0<\alpha<1$.

The proof of Theorem 1.7 is provided in Sect. 3.3. Our proof strategy involves two steps: First, we establish separate Carleman estimates for the heat equation and the damped beam equation. These two estimates are then combined to derive the global estimate presented above. The Carleman estimates for the damped beam equation are proved in Sect. 5. The combination argument is provided in Sect. 3.3.

Remark 1.9 The approach consisting in merging several Carleman estimates to deal with coupled equations has been widely used to deal with systems of coupled parabolic equations defined in the same geometric domain and with distributed coupling terms that are small-order perturbations of the principal operators (see e.g [10–12]). Here, the situation is more complex: The equations are not defined in the same domain and the coupling is achieved through trace conditions.

The proof of (1.14) relies on the merging of the Carleman inequalities (3.16) for the heat operator and (1.17) for the beam operator. The main difficulties rely in the treatment of three terms: the boundary integrals $\iint_\Sigma \frac{\partial v}{\partial n}\partial_t v$ and $\iint_\Sigma \xi_0|\partial_{x_1} v|^2$ in the Carleman inequality for the heat operator (see (3.15)) and, for large α, the integral $\alpha^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t\eta|^2$ at the right side of the Carleman inequality for the beam operator. Estimating these terms cannot be achieved by only playing with the Carleman parameter s but requires a sharp choice of the second parameters λ and μ.

The most problematic term arising from the Carleman inequality for the heat operator is $\iint_\Sigma \frac{\partial v}{\partial n}\partial_t v$. Since we have the coupling condition $u = \partial_t\eta$ on Γ_1, it requires an estimation of $\partial_t^2\eta$ via the Carleman inequality for the beam operator. This step crucially utilizes the parabolic regularity of the beam equation and therefore the fact that $\alpha > 0$. The absorption of this term is achieved only through a gain of a power of λ, specifically requiring $\lambda \gtrsim 1 + \frac{1}{\alpha^2}$.

For the term $\iint_\Sigma \xi_0|\partial_{x_1} v|^2$, the parabolicity condition $\alpha > 0$ is not required for absorption. Instead, absorption is achieved through a specific balance between λ and μ, given by the condition $\lambda \lesssim (1+\alpha^2)\mu^2$.

The term $\alpha^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t\eta|^2$ only arises for large values of α. Specifically, it originates from the term $\alpha\partial_{x_1}^2\partial_t\eta$ in the beam equation, which generates the terms αI_7 and αI_8 when computing the Carleman inequality for the beam operator (see (5.32)). For $\alpha < \alpha_* \approx 1.45$, these can be directly absorbed into the left-hand side of the Carleman estimates and thus do not present any particular difficulty (see Theorem 1.12). However, for large values of α, αI_7 and αI_8 can no longer be absorbed, which leads to the remaining global term $\alpha^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t\eta|^2$ (see Sect. 5.2). Fortunately, thanks to the boundary condition $u = \partial_t\eta$, we can absorb this term in the left-hand side of the Carleman estimate for the heat operator in the regime $\lambda^3 \gtrsim \alpha^2\mu^4$—which explains the condition $\lambda \gtrsim \alpha^{2/3}\mu^{4/3}$. Note that the works in [4, 7] focus exclusively on the case where $\alpha = 1$, which is why this issue does not arise in their analysis.

Note that in Theorem 1.7 we obtain naturally an observation term in η for the damped beam equation. To prove the observability inequality (1.6), we need to get an observation term in $\partial_t\eta$. To do so, we rely on the strategy used in [4, Section 5]. We prove the following:

Proposition 1.10 *Let ω and J be nonempty open subsets of Ω and $\mathcal{I}$, respectively, and let λ and s satisfy the conditions stated in Theorem 1.7. There exist $c_0 > 0$ such that any solution of (1.5) satisfies:*

$$\int_0^T s^3\lambda^2\,\xi_1^3\,e^{2s\varphi_1}\left(\|u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^2(\mathcal{I})} + \|\partial_t\eta\|^2_{L^2(\mathcal{I})}\right)$$

$$\leq c_0\Big((\alpha^{-4}+\alpha^{16})\iint_{(0,T)\times\omega} s^{11}\lambda^4\xi_2^{11}\, e^{4s\varphi_2-2s\varphi_1}|u|^2$$
$$+(\alpha^{-8}+\alpha^{10})\iint_{(0,T)\times J} s^7\lambda^8\xi_2^7\, e^{2s\varphi_2}|\partial_t\eta|^2\Big), \tag{1.15}$$

where φ_1, φ_2, ξ_1 and ξ_2 are defined in (1.12) *and* (1.13).

The proof of Proposition 1.10 is given in Sect. 4. The Theorem 1.5 follows then by a standard dissipation argument, see Theorem 4.1 below.

1.3 Carleman estimates for the one-dimensional beam operator

This subsection is dedicated to deriving Carleman estimates for the one-dimensional beam operator. In Theorem 1.12, we state a Carleman inequality that is essential for proving Theorem 1.7. This inequality includes a global observation term in $\partial_t\eta$, which will be absorbed through the coupling with the Carleman inequality for the heat operator (see Sect. 3.3).

Additionally, we present two other results that may be of independent interest:

- Theorem 1.11 provides a Carleman inequality valid for $\alpha = 0$, though it involves higher-order observation terms. This result arises as an intermediate step in the proof of Theorem 1.12.
- Theorem 1.13 presents a Carleman inequality for $\alpha \in (0,\alpha^*)$, where $\alpha^* \approx 1.45$, without requiring the additional global observation term in $\partial_t\eta$. Although‘ Theorem 1.13 is not used in this paper, it is worthy of interest as it improves upon the results in [7].

The proofs of the following results are given in Sect. 5.

1.3.1 The beam operator

We first obtain the following Carleman estimates.

Theorem 1.11 *Let J be a nonempty open subset of $\mathcal{I}$ and let $k \geq 2$. There exist $c_0 > 0$, $\mu_0 \geq 1$ and $\widehat{s}_0 \geq 1$ such that for all $\mu \geq \mu_0$, for all $\alpha \geq 0$, for all $T > 0$ and for all $s \geq \widehat{s}_0(1+\alpha)(T^k+T^{k-1})$, the following inequality holds for all function $\eta \in C^2([0,T]; C^4(\mathcal{I}))$:*

$$\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}\Big(s^7\mu^8\xi_0^7|\eta|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}\eta|^2 + s^3\mu^4\xi_0^3|\partial_{x_1}^2\eta|^2$$
$$+(1+\alpha^2)s^3\mu^4\xi_0^3|\partial_t\eta|^2 + s\mu^2\xi_0|\partial_{x_1}^3\eta|^2 + (1+\alpha^2)s\mu^2\xi_0|\partial_t\partial_{x_1}\eta|^2\Big)$$
$$\leq c_0\Big(\iint_{(0,T)\times J} e^{2s\varphi_0}\Big(s^7\mu^8\xi_0^7|\eta|^2 + (1+\alpha^2)s^3\mu^4\xi_0^3|\partial_t\eta|^2 + s\mu^2\xi_0|\partial_{x_1}^3\eta|^2$$
$$+(1+\alpha^2)s\mu^2\xi_0|\partial_t\partial_{x_1}\eta|^2\Big) + \alpha^2 s^3\mu^4\iint_{(0,T)\times\mathcal{I}} \xi_0^3\, e^{2s\varphi_0}|\partial_t\eta|^2$$

$$+ \iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0} |\partial_t^2 \eta - \alpha \partial_{x_1}^2 \partial_t \eta + \partial_{x_1}^4 \eta|^2 \Big). \tag{1.16}$$

This Carleman estimate is valid for $\alpha = 0$, but with observation terms of order strictly greater than zero (i.e, $\partial_{x_1}^3 \eta$, $\partial_{x_1} \partial_t \eta$) on J. As a consequence, we cannot obtain the expected controllability result for the beam equation from this inequality.

However, note that in the case $\alpha = 0$, Theorem 1.16 implies immediately the following unique continuation property: if η satisfies $\partial_t^2 \eta + \partial_{x_1}^4 \eta = 0$ on $(0, T) \times \mathcal{I}$, and $\eta \equiv 0$ on $(0, T) \times J$, then $\eta \equiv 0$ on $(0, T) \times \mathcal{I}$.

The proof of Theorem 1.11 is given in Sect. 5.2.

1.3.2 The damped beam operator

We now focus on the case with active damping, *i.e.,* $\alpha > 0$. In that context, we obtain the following Carleman estimate.

Theorem 1.12 *Let $\alpha > 0$. Let J be a nonempty open subset of $\mathcal{I}$ and let $k \geq 2$. There exist $c_0 > 0$, $\mu_0 \geq 1$ and $\widehat{s_0} \geq 1$ such that for all $\mu \geq \mu_0$, for all $\alpha > 0$, for all $T > 0$ and for all $s \geq \widehat{s_0}(1+\alpha)(T^k + T^{k-1})$, the following inequality holds for all function $\eta \in C^2([0, T]; C^4(\mathcal{I}))$:*

$$\begin{aligned}
&\frac{\alpha^2}{1+\alpha^2} \frac{1}{s} \iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0} e^{2s\varphi_0} (|\partial_{x_1}^4 \eta|^2 + |\partial_t^2 \eta|^2) + \frac{\alpha^2}{s} \iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0} e^{2s\varphi_0} |\partial_t \partial_{x_1}^2 \eta|^2 \\
&+ \iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0} \Big(s^7 \mu^8 \xi_0^7 |\eta|^2 + s^5 \mu^6 \xi_0^5 |\partial_{x_1} \eta|^2 + s^3 \mu^4 \xi_0^3 |\partial_{x_1}^2 \eta|^2 \\
&+ (1+\alpha^2) s^3 \mu^4 \xi_0^3 |\partial_t \eta|^2 + s\mu^2 \xi_0 |\partial_{x_1}^3 \eta|^2 + (1+\alpha^2) s\mu^2 \xi_0 |\partial_t \partial_{x_1} \eta|^2 \Big) \\
&\leq c_0 \Big(\Big(\alpha^{-8} + \alpha^4\Big) \iint_{(0,T)\times J} s^7 \mu^8 \xi_0^7 \, e^{2s\varphi_0} |\eta|^2 + \iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0} |\partial_t^2 \eta \\
&- \alpha \partial_{x_1}^2 \partial_t \eta + \partial_{x_1}^4 \eta|^2 + \alpha^2 s^3 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 \, e^{2s\varphi_0} |\partial_t \eta|^2 \Big).
\end{aligned} \tag{1.17}$$

We now make several comments on the result: First of all, estimate (1.17) is better than estimate (1.16) because, in (1.17), we retrieve an observation of uniquely η on J. However, it is also clear that the estimate (1.17) is uniquely valid for α positive, as the inverse of α^8 appears on the right-hand side.

Nonetheless, estimate (1.17) is not sufficient to obtain the observability of the damped beam equation. This is due to the presence of the last term in the right-hand side, involving an observation of $\partial_t \eta$ on the whole domain $\mathcal{I}$. For large values of the parameter α, this term cannot be absorbed into the left-hand side, regardless of the choice of the parameters s and μ. In contrast, the same term can be absorbed for α small enough. It is therefore natural to try to determine the range of parameters α that can be directly absorbed into the left-hand side. An answer is provided in Theorem 1.13 below. In our study, we get rid of this term by coupling the Carleman estimate (1.17) with a Carleman estimate for the heat equation. Indeed, the term $\partial_t \eta$ is directly linked to the heat solution via the boundary conditions (see (1.5)). In other words, we deeply use the coupling between the heat equation and the beam equation.

In [7], the author obtains a Carleman estimate for the one-dimensional damped beam equation similar to (1.17) but without the last term on the right-hand side involving $\partial_t \eta$, with the particular choice of damping parameter $\alpha = 1$. It is natural to ask whether we can generalize the result of [7] for any positive value of the parameter α, which would give the observability of the beam equation alone. It turns out that this is not the case, as the techniques developed in [7] fail for large values of the parameter α. More precisely, setting

$$\alpha^* = \sqrt{\frac{6(2-\beta^*)}{4+\beta^*}} \quad \text{where } \beta^* \text{ is the unique real root of } \; 36\beta^3 + 111\beta^2 + 77\beta - 58, \tag{1.18}$$

which gives $\beta^* = 0,438 \pm 10^{-3}$ and $\alpha^* = 1,453 \pm 10^{-3}$, we obtain using the strategy of [7] the following Carleman estimate.

Theorem 1.13 *Let $\alpha \in (0, \alpha^*)$. Let J be a nonempty open subset of $\mathcal{I}$ and let $k \geq 2$. There exist $c_0 > 0$, $\mu_0 \geq 1$ and $s_0 \geq 1$ such that for all $\mu \geq \mu_0$, for all $T > 0$ and for all $s \geq s_0(T^k + T^{k-1})$, the following inequality holds for all function $\eta \in C^2([0,T]; C^4(\mathcal{I}))$:*

$$\begin{aligned} &\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}\Big(s^7\mu^8\xi_0^7|\eta|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}\eta|^2 + s^3\mu^4\xi_0^3|\partial_{x_1}^2\eta|^2 + s^3\mu^4\xi_0^3|\partial_t\eta|^2 + s\mu^2\xi_0|\partial_{x_1}^3\eta|^2 \\ &\quad + s\mu^2\xi_0|\partial_t\partial_{x_1}\eta|^2\Big) + \frac{1}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0} e^{2s\varphi_0}(|\partial_{x_1}^4\eta|^2 + |\partial_t^2\eta|^2 + |\partial_t\partial_{x_1}^2\eta|^2) \\ &\leq c_0\Big(\iint_{(0,T)\times J} s^7\mu^8\xi_0^7\, e^{2s\varphi_0}|\eta|^2 + \iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|\partial_t^2\eta - \alpha\partial_{x_1}^2\partial_t\eta + \partial_{x_1}^4\eta|^2\Big). \end{aligned} \tag{1.19}$$

This result classically leads to the observability of the corresponding damped beam equation, for any α in the range $(0, \alpha^*)$. To the best of our knowledge, obtaining a global Carleman estimate of type (1.19), valid for large values of the damping parameter α, remains an open problem. We underline that a Carleman estimate can be obtained by factorizing the structurally damped beam operator as a product of two heat-like operators. But this leads to an inequality with strictly lower exponents of s than as in (1.19) (see, e.g., [13]) that does not permit to handle coupled system such as (1.5).

Remark 1.14 Although the dependence of s is not explicitly given in [7], a check of the proof shows that s can be assumed to satisfy $s \geq s_0(T^4 + T^2)$ there, for some s_0 independent of T. This would lead to a cost of the control proportional to $\exp(C/T^2)$. Here we improve this dependence in T: For $k \geq 2$ the parameter s can be chosen $s \geq s_0(T^k + T^{k-1})$ which leads to a cost of the control proportional to $\exp(C/T)$ (see Theorem 4.1).

The improvement arises from how certain terms in the conjugated operators are treated in [7] compared to the present study. Specifically, we retain the terms $\mathcal{M}_{12}z$, $\mathcal{M}_{22}z$ (see (5.4) and (5.6) in Sect. 5) within the main operator, whereas in [7], they are first considered as source terms, and must be absorbed *a posteriori*. This absorption process incurs a loss of one power of T in the control cost. The fact that we keep $\mathcal{M}_{12}z$, $\mathcal{M}_{22}z$ in the main operator yields the additional cross terms $\mathcal{R}_1$ defined in (5.11) that are estimated in Subsect. 5.5. It is this estimation process that improves the dependence of s in T and yields the controllability cost $e^{C/T}$.

1.4 Scientific context

The main contributions that are closely connected to the present work are [4, 5] where the authors investigate the local null-controllability of an incompressible fluid in a periodic channel, described by the Navier–Stokes equations, with a structure located on the boundary of the fluid domain, described by a damped beam equation. In [4], two distributed controls are considered: one in the fluid domain and one on the beam. In [5], the authors manage to suppress the control on the structure by using a Fourier decomposition, but this approach is very specific to the geometry. We underline that in [4, 5] only the particular case of a structural damping coefficient $\alpha = 1$ is considered. We may also mention [14] where the damped beam equation is replaced by a heat equation.

The controllability properties of fluid–structure interaction systems have been tackled mainly in the case where the structure is described by a finite dimensional equation. For the case of a rigid body immersed in a fluid, the first results can be found in [15, 16] for a fluid described by the 2D-Navier–Stokes equations and for some specific geometric assumptions on the solid. These are removed in [17] for a simplified fluid model, the 3D case without specific geometric assumptions is treated in [18], the case of a rigid body immersed in a Boussinesq flow can be found in [19] and the case of Navier slip matching conditions on the solid is treated in [20]. For a fluid coupled with a structure described by a system of ordinary differential equations corresponding to a finite dimensional approximation of equations modeling deformations of an elastic body, we refer to [21, 22].

Let us also mention some works concerning the controllability of the beam equation. In [7], the null-controllability of the damped beam equation is proved, when the damping parameter is equal to one. The proof relies on a Carleman inequality of type (1.19) for $\alpha = 1$. In [13], a Carleman inequality for the beam equation is obtained by factorizing the structurally damped beam equation as a product of two heat-like equations. However, this results in a Carleman inequality with strictly lower exponents for s than in (1.19). We also mention the reference [23] where the authors proved the null-controllability for the beam equation with structural damping using the moments method, and the reference [24] where the author proves the same result using a spectral approach.

The outline of the article is as follows: In Sect. 2, we give the functional setting associated with our controllability problem. In Sect. 3, we obtain the Carleman estimate for the coupled system, by coupling a Carleman estimate for the heat equation with a Carleman estimate for the damped beam equation. We deduce the observability inequality of the adjoint system in Sect. 4. Finally, Sect. 5 is devoted to the proofs of several Carleman estimates for the damped or undamped beam equation.

2 Functional setting

This section is dedicated to the study of the well posedness of system (1.4) and of the regularity of its solutions, and to the justification of the equivalence between the null-controllability of (1.4) and the observability of (1.5) (in the sense of Definition 1.1 and Definition 1.4). For that we introduce an abstract framework in order to rewrite system (1.4) as an evolution equation.

We define the following operators associated with the damped beam equation in (1.4). Let $\Lambda : L^2(\mathcal{I}) \to L^2(\partial\Omega)$ defined by

$$(\Lambda\zeta)(x_1, x_2) = \begin{cases} \zeta(x_1) & \text{if } (x_1, x_2) \in \Gamma_1, \\ 0 & \text{if } (x_1, x_2) \in \Gamma_0. \end{cases} \tag{2.1}$$

The adjoint $\Lambda^* : L^2(\partial\Omega) \to L^2(\mathcal{I})$ is given by

$$(\Lambda^* w)(x_1) = w(x_1, 1) \quad (x_1 \in \mathcal{I}). \tag{2.2}$$

We introduce the following linear operators A_i with domains $\mathcal{D}(A_i)$ (for $i = 1, 2$) defined on $L^2(\mathcal{I})$,

$$A_1\zeta = \partial_{x_1}^4 \zeta + \zeta, \qquad \mathcal{D}(A_1) = H^4(\mathcal{I}) \tag{2.3}$$

$$A_2\zeta = \alpha\left(-\partial_{x_1}^2 \zeta + \zeta\right), \qquad \mathcal{D}(A_2) = H^2(\mathcal{I}). \tag{2.4}$$

One can check that

$$\mathcal{D}(A_1^{1/2}) = H^2(\mathcal{I}).$$

Then the damped beam equation in (1.4) writes as:

$$\partial_t^2 \zeta + A_1\zeta + A_2\partial_t\zeta = -\Lambda^*\partial_n w + \mathbb{1}_J h.$$

We define the space

$$\mathscr{H} = \left\{ w \in L^2(\Omega), \ \ \zeta_1 \in \mathcal{D}(A_1^{1/2}), \ \ \zeta_2 \in L^2(\mathcal{I}) \right\}$$

equipped with the scalar product:

$$\left((w, \zeta_1, \zeta_2), (\widetilde{w}, \widetilde{\zeta}_1, \widetilde{\zeta}_2)\right)_{\mathscr{H}} = \int_\Omega w \, \widetilde{w} + \int_{\mathcal{I}} A_1^{1/2}\zeta_1 \, A_1^{1/2}\widetilde{\zeta}_1 + \int_{\mathcal{I}} \zeta_2 \, \widetilde{\zeta}_2.$$

We define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ (w, \zeta_1, \zeta_2) \in H^2(\Omega) \times H^4(\mathcal{I}) \times H^2(\mathcal{I}) \ : \ w = \Lambda\zeta_2 \ \text{ on } \partial\Omega \right\} \tag{2.5}$$

and

$$\mathcal{A} \begin{bmatrix} w \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \Delta w \\ \zeta_2 \\ -A_1\zeta_1 - A_2\zeta_2 - \Lambda^*\partial_n w \end{bmatrix}. \tag{2.6}$$

From an easy adaptation of the proof of [25, Propositions 3.4, 3.5 and 3.11], we deduce the following result.

Proposition 2.1 *The operator $\mathcal{A}$ defined by (2.5), (2.6) is the infinitesimal generator of an analytic and uniformly stable semigroup of contractions on $\mathscr{H}$.*

A consequence of Proposition 2.1 is that for $F \in L^2(0,T;\mathscr{H})$ and $Y_0 \in \mathscr{H}$, the linear system

$$\frac{dY}{dt} = \mathcal{A}Y + F \ \text{ in } \ (0,T), \quad Y(0) = Y_0, \tag{2.7}$$

admits a unique "classical solution" that is a solution Y that belongs to $H^1(\varepsilon,T;\mathscr{H}) \cap L^2(\varepsilon,T;\mathcal{D}(\mathcal{A})) \cap C([0,T];\mathscr{H})$ for all $\varepsilon > 0$ (see [26, Definition 3.1 p.129 and Proposition 3.8 p.145]).

In particular, for $(w_0,\zeta_0,\zeta_1) \in L^2(\Omega) \times H^2(\mathcal{I}) \times L^2(\mathcal{I})$ and $(G,H) \in L^2(Q) \times L^2((0,T)\times\mathcal{I})$, if we set

$$Y = \begin{bmatrix} w \\ \zeta \\ \partial_t\zeta \end{bmatrix}, \qquad Y_0 = \begin{bmatrix} w_0 \\ \zeta_0 \\ \zeta_1 \end{bmatrix} \tag{2.8}$$

and

$$F = \begin{bmatrix} G \\ 0 \\ H \end{bmatrix}$$

then system (1.1) can be rewritten in the form (2.7), and it implies the existence and uniqueness of a solution to (1.1) satisfying (1.2) and (1.3).

Using that 0 is in the resolvent set of $\mathcal{A}$ (by Proposition 2.1), we deduce the following result:

Proposition 2.2 *There exists a constant $c > 0$, such that for all nonnegative values of the parameter α, any solution $Y = (w,\zeta,\partial_t\zeta)$ of (2.7) with $F = 0$ satisfies*

$$\|\partial_t[w,\zeta,\partial_t\zeta](t)\|_{\mathscr{H}} = \|\mathcal{A}[w,\zeta,\partial_t\zeta](t)\|_{\mathscr{H}} \geq \frac{c}{1+\alpha}\|[w,\zeta,\partial_t\zeta](t)\|_{\mathscr{H}} \qquad \forall t \in (0,T). \tag{2.9}$$

Proposition 2.2 can be proved similarly as [25, Proposition 3.5]. In particular, by following the same steps as in the proof of [25, Proposition 3.5] we observe that the constant in (2.9) can be chosen of the form $\frac{c}{1+\alpha}$ with $c > 0$ independent of α.

If $Y_0 = 0$ we also have the following useful maximal regularity property: For $F \in L^2(0,T;\mathscr{H})$, the solution of

$$\frac{dY}{dt} = \mathcal{A}Y + F \ \text{ in } \ (0,T), \quad Y(0) = 0, \tag{2.10}$$

belongs to $H^1(0,T;\mathscr{H}) \cap L^2(0,T;\mathcal{D}(\mathcal{A}))$ and satisfies

$$\|Y\|_{L^2(0,T;\mathcal{D}(\mathcal{A}))} + \|Y\|_{H^1(0,T;\mathscr{H})} \leq M_\alpha \|F\|_{L^2(0,T;\mathscr{H})}. \tag{2.11}$$

Note that it is not straightforward to explicit the dependence in α of the constant M_α in (2.11) from the resolvent estimate in [25, Proposition 3.11]. However, by adapting the approach of [27, Section 3] one can obtain that it is of the form $M_\alpha = c(\alpha^{-1} + \alpha)$ with $c > 0$ independent of $\alpha > 0$.

Next, we introduce the control operator $B : L^2(\omega) \times L^2(J) \to \mathscr{H}$ defined by

$$B(g, h) = \begin{bmatrix} \mathbb{1}_\omega g \\ 0 \\ \mathbb{1}_J h \end{bmatrix}. \tag{2.12}$$

Hence, with (2.8) system (1.4) can be rewritten in the following form:

$$\frac{dY}{dt} = \mathcal{A}Y + B(g, h) \text{ in } (0, T), \quad Y(0) = Y_0. \tag{2.13}$$

Then Definition 1.1 can be equivalently stated as follows:

Definition 2.3 We say that system (2.13) is null-controllable at time $T > 0$ if there exists a constant $C_T > 0$ such that, for all $Y_0 \in \mathscr{H}$, there exists a control $(g, h) \in L^2(0, T; L^2(\omega) \times L^2(J))$ such that the solution Y of system (2.13) satisfies $Y(T) = 0$ and

$$\|g\|_{L^2((0,T)\times\omega)} + \|h\|_{L^2((0,T)\times J)} \leq C_T \|Y_0\|_{\mathscr{H}}.$$

Proposition 2.4 *The adjoint of the operator $\mathcal{A}$ is given by $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$ and*

$$\mathcal{A}^* \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \Delta u \\ -\eta_2 \\ A_1\eta_1 - A_2\eta_2 - \Lambda^*\partial_n u \end{bmatrix}. \tag{2.14}$$

Proof First, let us prove $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A})$. For that, suppose that $[u, \eta_1, \eta_2] \in \mathcal{D}(\mathcal{A}^*)$ and set $[F, G, H] = \mathcal{A}^*[u, \eta_1, \eta_2] \in \mathscr{H}$. We have

$$(\mathcal{A}[w, \zeta_1, \zeta_2], [u, \eta_1, \eta_2])_{\mathscr{H}} = ([w, \zeta_1, \zeta_2], [F, G, H])_{\mathscr{H}} \quad \forall\, [w, \zeta_1, \zeta_2] \in \mathcal{D}(\mathcal{A}).$$

The above equality rewrites

$$\begin{aligned} &\int_\Omega \Delta w\, u + \int_{\mathcal{I}} A_1^{1/2}\zeta_2\, A_1^{1/2}\eta_1 - \int_{\mathcal{I}} (\Lambda^*\partial_n w + A_1\zeta_1 + A_2\zeta_2)\eta_2 \\ &\quad = \int_\Omega w\, F + \int_{\mathcal{I}} A_1^{1/2}\zeta_1\, A_1^{1/2} G + \int_{\mathcal{I}} H\, \zeta_2. \end{aligned} \tag{2.15}$$

First, with $w = 0$ and $\zeta_2 = 0$ we find

$$\int_{\mathcal{I}} A_1\zeta_1(G + \eta_2) = 0 \quad \forall \zeta_1 \in H^4(\mathcal{I}).$$

from which we deduce that $\eta_2 = -G \in \mathcal{D}(A_1^{1/2})$.

Next, with $\zeta_1 = \zeta_2 = 0$ and $w \in C_c^\infty(\Omega)$, we obtain

$$\int_\Omega u \Delta w = \int_\Omega F w \quad \forall w \in C_c^\infty(\Omega).$$

from which we deduce $\Delta u = F \in L^2(\Omega)$. It implies that the trace and the normal derivative of u are well defined in $H^{-1/2}(\partial\Omega)$ and in $H^{-3/2}(\partial\Omega)$, respectively, and, that the following formula is true for all $w \in H^2(\Omega)$ (see [28, Lemma 1.5.3.3], and the subsequent discussion),

$$\int_\Omega \Delta w\, u - \int_\Omega \Delta u\, w = \langle \partial_n w, u\rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} - \langle w, \partial_n u\rangle_{H^{3/2}(\partial\Omega), H^{-3/2}(\partial\Omega)}. \tag{2.16}$$

Thus, by setting $F = \Delta u$ in (2.15), and with $\zeta_1 = 0$, $\zeta_2 \in H^2(\mathcal{I})$ and $w \in H^2(\Omega)$ such that $w = \Lambda\zeta_2$ on $\partial\Omega$, we deduce

$$\int_\Omega \Delta w\, u - \int_\Omega w\, \Delta u + \int_{\mathcal{I}} A_1^{1/2}\zeta_2\, A_1^{1/2}\eta_1 - \int_{\partial\Omega} \partial_n w\, \Lambda\eta_2 - \int_{\mathcal{I}} A_2\zeta_2\, \eta_2 = \int_{\mathcal{I}} \zeta_2\, H. \tag{2.17}$$

Then with (2.16), the equality (2.17) rewrites:

$$\begin{aligned} &\langle \partial_n w, u\rangle_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)} - \langle w, \partial_n u\rangle_{H^{3/2}(\partial\Omega), H^{-3/2}(\partial\Omega)} \\ &\quad + \int_{\mathcal{I}} A_1^{1/2}\zeta_2\, A_1^{1/2}\eta_1 - \int_{\partial\Omega} \partial_n w\, \Lambda\eta_2 - \int_{\mathcal{I}} A_2\zeta_2\, \eta_2 = \int_{\mathcal{I}} \zeta_2\, H. \end{aligned} \tag{2.18}$$

Let $d \in H^{1/2}(\partial\Omega)$. By using the surjectivity of the map $w \in H^2(\Omega) \mapsto (w, \partial_n w) \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, we can choose $w \in H^2(\Omega)$ such that $\partial_n w = d$ on $\partial\Omega$. Hence by applying (2.18) for such a w and with $\zeta_2 = 0$ on $\mathcal{I}$, we finally deduce that

$$\langle d, u\rangle_{H^{1/2}(\Gamma_1), H^{-1/2}(\Gamma_1)} - \int_{\partial\Omega} d\, \Lambda\eta_2 = 0 \quad \forall d \in H^{1/2}(\partial\Omega). \tag{2.19}$$

Then we deduce that $u = \Lambda\eta_2$ on $\partial\Omega$. Moreover, since $\eta_2 \in H^2(\mathcal{I})$ it is clear that $u = \Lambda\eta_2 \in H^2(\partial\Omega)$. Since we also have $\Delta u \in L^2(\Omega)$, elliptic regularity results yield $u \in H^2(\Omega)$. In particular, we have $\partial_n u \in H^{1/2}(\partial\Omega)$ and $\Lambda^*\partial_n u \in H^{1/2}(\mathcal{I})$.

Finally, by coming back to (2.18), that holds for all $w \in H^2(\Omega)$ and $\zeta_2 \in H^2(\mathcal{I})$ such that $w = \Lambda\zeta_2$ on $\partial\Omega$, and by taking into account that $u = \Lambda\eta_2$ on $\partial\Omega$, we find that

$$-\int_{\partial\Omega} \Lambda\zeta_2\, \partial_n u + \int_{\mathcal{I}} A_1^{1/2}\zeta_2\, A_1^{1/2}\eta_1 - \int_{\mathcal{I}} A_2\zeta_2\, \eta_2 = \int_{\mathcal{I}} \zeta_2\, H.$$

Then $A_1\eta_1 = H + A_2\eta_2 + \Lambda^*\partial_n u \in L^2(\mathcal{I})$ from which we deduce $\eta_1 \in \mathcal{D}(A_1)$.

If we summarize the above results, we have proved $(u, \eta_1, \eta_2) \in H^2(\Omega) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})$ and $u = \Lambda\eta_2$ on $\partial\Omega$, and that $F = \Delta u$, $G = -\eta_2$ and $H = A_1\eta_1 - A_2\eta_2 - \Lambda^*\partial_n u$. Then with (2.5) it means that $(u, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A})$, and we have proved $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A})$. Moreover, since $(F, G, H) = \mathcal{A}^*(u, \eta_1, \eta_2)$ we have that (2.14) is true.

To conclude, the converse inclusion $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}^*)$ follows from an application of the Green formula. $\square$

Next, the adjoint $B^* : \mathscr{H} \to L^2(\omega) \times L^2(J)$, of the operator B defined in (2.12), is given by

$$B^* \begin{bmatrix} u \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} u|_\omega \\ \eta_2|_J \end{bmatrix}. \tag{2.20}$$

Let us consider the following adjoint system:

$$\frac{dV}{dt} = \mathcal{A}^* V \ \text{ in } \ (0, T). \tag{2.21}$$

Definition 2.5 We say that the pair $(\mathcal{A}^*, B^*)$ is final-state observable in time $T > 0$ if, and only if, there exist $K_T > 0$ such that any solution of (2.21) satisfies

$$\|V(T)\|^2_{\mathscr{H}} \leq K_T \int_0^T \|B^* V\|^2_{L^2(\omega)\times L^2(J)}. \tag{2.22}$$

It is well-known (see, e.g., [29, Theorem 11.2.1, p.357]) that the null-controllability of (2.13) in time $T > 0$ is equivalent to the final-state observability of the pair $(\mathcal{A}^*, B^*)$ in time $T > 0$, and in that case the constants C_T and K_T are actually equal.

In view of (2.14), the equality (2.21) is equivalent to

$$\begin{cases} \partial_t u = \Delta u & \text{in } Q, \\ u = \Lambda \eta_2 & \text{on } (0, T) \times \partial\Omega, \\ \partial_t \eta_1 = -\eta_2 & \text{in } (0, T) \times \mathcal{I}, \\ \partial_t \eta_2 = A_1 \eta_1 - A_2 \eta_2 - \Lambda^* \partial_n u & \text{in } (0, T) \times \mathcal{I}. \end{cases}$$

Hence, we deduce that $(u, \eta) = (u, -\eta_1)$ satisfies

$$\frac{d}{dt} \begin{bmatrix} u \\ \eta \\ \partial_t \eta \end{bmatrix} = \mathcal{A} \begin{bmatrix} u \\ \eta \\ \partial_t \eta \end{bmatrix} \ \text{ in } \ (0, T) \tag{2.23}$$

or equivalently,

$$\begin{cases} \partial_t u = \Delta u & \text{in } Q, \\ u = \Lambda \partial_t \eta & \text{on } (0, T) \times \partial\Omega, \\ \partial_t^2 \eta + A_1 \eta + A_2 \partial_t \eta = -\Lambda^* \partial_n u & \text{in } (0, T) \times \mathcal{I}. \end{cases} \tag{2.24}$$

As a consequence, the final-state observability of the pair $(\mathcal{A}^*, B^*)$ in time $T > 0$ is equivalent to the existence of $K_T > 0$ such that any solution of (2.24) satisfies

$$\|[u, \eta, \partial_t \eta](T)\|_{\mathscr{H}}^2 \leq K_T \left(\int_0^T \int_\omega |u|^2 + \int_0^T \int_J |\partial_t \eta|^2 \right). \tag{2.25}$$

Finally, since the first fourth equalities of (1.5) can be rewritten as (2.24), we deduce that the pair $(\mathcal{A}^*, B^*)$ is final-state observable in time $T > 0$ if, and only if, system (1.5) is observable at time T from the observation sets ω and J in the sense of Definition 1.4. As a consequence, system (1.4) is null-controllable at time T if and only if system (1.5) is observable at time T from the observation sets ω and J, and in that case we can choose $C_T = K_T$. The remaining part of the article is therefore devoted to the proof of the observability inequality (1.6).

3 Carleman estimate for the coupled heat and damped beam system

In this section, we state the Carleman estimate for the heat equation that we combine with Theorem 1.12 to deduce Theorem 1.7, and we prove the observability inequality for the adjoint system (1.5).

Notation. In what follows, we use the notation $X \lesssim Y$ and $X \gtrsim Y$ if there exists a constant $C > 0$ which is independent of the parameters $(s_0, s, \lambda, \mu, T, \alpha)$ such that we have the inequalities $X \leq CY$ and $X \geq CY$, respectively.

3.1 About some properties of the weight functions

We now state some properties that are satisfied by the functions φ, ξ, φ_0 and ξ_0 defined by equations (1.10) and (1.11) in the introduction, and that we need in the following calculations.

Let $b > a$, $i = 1, 2$ and $j \in \mathbb{N}$. The following properties are true.

$$\begin{aligned} |\partial_{x_i}^j \varphi| + |\partial_{x_i}^j \xi| &\lesssim \lambda^j \xi \quad \text{if } j \neq 0, \\ |\partial_t \partial_{x_i}^j \varphi| + |\partial_t \partial_{x_i}^j \xi| &\lesssim \lambda^j T \xi^{1+\frac{2}{k}}, \\ |\partial_t^2 \partial_{x_i}^j \varphi| + |\partial_t^2 \partial_{x_i}^j \xi| &\lesssim \lambda^j T^2 \xi^{1+\frac{4}{k}}, \\ \xi^a &\leq T^{(b-a)k} \xi^b, \end{aligned} \tag{3.1}$$

and

$$|\partial_{x_1}^j \varphi_0| + |\partial_{x_1}^j \xi_0| \lesssim \mu^j \xi_0 \quad \text{if } j \neq 0, \tag{3.2}$$

$$|\partial_t \partial_{x_1}^j \xi_0| + |\partial_t \partial_{x_1}^j \varphi_0| \lesssim \mu^j T \xi_0^{1+\frac{2}{k}}, \tag{3.3}$$

$$|\partial_t^2 \partial_{x_1}^j \xi_0| + |\partial_t^2 \partial_{x_1}^j \varphi_0| \lesssim \mu^j T^2 \xi_0^{1+\frac{4}{k}}, \tag{3.4}$$

$$\xi_0^a \leq T^{(b-a)k} \xi_0^b. \tag{3.5}$$

For the following, we suppose that for some $s_0 \geq 1$ we have $s \geq s_0(T^k + T^{k-1})$. Then, the above estimates yield

$$|\partial_t \partial_{x_i}^j \xi| + |\partial_t \partial_{x_i}^j \varphi| \lesssim \lambda^j \, (s/s_0)\, \xi^2, \tag{3.6}$$
$$|\partial_t^2 \partial_{x_i}^j \xi| + |\partial_t^2 \partial_{x_i}^j \varphi| \lesssim \mu^j \, (s/s_0)^2 \, \xi^3, \tag{3.7}$$
$$\xi^a \lesssim (s/s_0)^{b-a} \, \xi^b. \tag{3.8}$$

and

$$|\partial_t \partial_{x_1}^j \xi_0| + |\partial_t \partial_{x_1}^j \varphi_0| \lesssim \mu^j \, (s/s_0)\, \xi_0^2, \tag{3.9}$$
$$|\partial_t^2 \partial_{x_1}^j \xi_0| + |\partial_t^2 \partial_{x_1}^j \varphi_0| \lesssim \mu^j \, (s/s_0)^2 \, \xi_0^3, \tag{3.10}$$
$$\xi_0^a \lesssim (s/s_0)^{b-a} \, \xi_0^b. \tag{3.11}$$

From (1.11) we observe that:

$$\partial_{x_1}\varphi_0 = \mu \psi_{\mathcal{I}}' \xi_0 \quad \text{and} \quad \partial_{x_1}^2 \varphi_0 = (\mu \psi_{\mathcal{I}}'' + \mu^2 |\psi_{\mathcal{I}}'|^2)\xi_0.$$

Hence, from (1.8) we deduce that there exists $\mu_0 \geq 1$ large enough such that

$$\begin{matrix} \mu\xi_0(t,x_1) \lesssim |\partial_{x_1}\varphi_0(t,x_1)| \\ \mu^2\xi_0(t,x_1) \lesssim \partial_{x_1}^2\varphi_0(t,x_1) \end{matrix} \quad \text{for all } \mu \geq \mu_0 \text{ and } (t,x_1) \in [0,T] \times \mathcal{I} \setminus J_0. \tag{3.12}$$

3.2 Carleman estimate for the heat equation

Let $\psi_{\mathcal{I}}$ and ψ_Ω be the weight functions introduced in Subsection 1.2 and set $\psi(x_1,x_2) = \frac{\mu}{\lambda}\psi_{\mathcal{I}}(x_1) + \psi_\Omega(x_1,x_2)$. The definitions (1.10) rewrites

$$\varphi(t,x_1,x_2) = \frac{e^{\lambda\psi(x_1,x_2)+k\lambda\Psi} - e^{(k+2)\lambda\Psi}}{\ell(t)^{k/2}}, \quad \xi(t,x_1,x_2) = \frac{e^{\lambda\psi(x_1,x_2)+k\lambda\Psi}}{\ell(t)^{k/2}}. \tag{3.13}$$

From $\lambda \geq \mu$, we have that ψ and its derivatives are bounded uniformly with respect to λ and μ and from (1.7) we have:

$$\psi > 0 \text{ in } \Omega, \quad \text{and} \quad \partial_n \psi = -1. \tag{3.14}$$

Moreover, from the last statement in (1.7) we deduce that for $\frac{\mu}{\lambda}$ small enough the infimum of $|\nabla\psi(x)|$ over $\Omega \backslash \omega_0$ is positive and independent of μ, λ.

By following standard calculations, these properties allow to obtain a Carleman estimate for the heat operator with an observation term localized in ω_0 and with nonhomogeneous boundary terms. To give a precise statement, let us introduce the following boundary integral, where we have denoted $\Sigma = (0,T) \times \partial\Omega$:

$$
\begin{aligned}
I_\Sigma(s,\lambda,\mu,v) = &-\iint_\Sigma \frac{\partial v}{\partial n}\,\partial_t v + s^3\iint_\Sigma \xi_0^3(\lambda^3+\lambda\mu^2(\psi_{\mathcal{I}}')^2)|v|^2 \\
&- 2s\iint_\Sigma \xi_0(\lambda^2+\mu^2(\psi_{\mathcal{I}}')^2)\partial_n v v + \lambda s^2\iint_\Sigma \xi_0\partial_t\varphi|v|^2 \\
&+ \lambda s\iint_\Sigma \xi_0|\partial_n v|^2 - \lambda s\iint_\Sigma \xi_0|\partial_{x_1} v|^2 - 2s\mu\iint_\Sigma \xi_0(\psi_{\mathcal{I}}')\partial_n v\partial_{x_1} v.
\end{aligned}
\tag{3.15}
$$

Theorem 3.1 *Let ω be a nonempty open subset of Ω and let $k \geq 2$. There exist $c > 0$, $c_0 > 0$, $c_1 \geq 1$ and $s_0 \geq 1$ such that for all $\lambda \geq c_1\mu \geq 1$, for all $T > 0$ and for all $s \geq s_0(T^k + T^{k-1})$, the following inequality holds for all $u \in C^1([0,T]; C^2(\overline{\Omega})$ such that with $u = 0$ on $(0,T)\times\Gamma_0$:*

$$
\begin{aligned}
&c\left(s^{-1}\iint_{(0,T)\times\Omega}\xi^{-1}e^{2s\varphi}\left(|\partial_t u|^2+|\Delta u|^2\right) + s\lambda^2\iint_{(0,T)\times\Omega}\xi e^{2s\varphi}|\nabla u|^2\right. \\
&\left.\quad + s^3\lambda^4\iint_{(0,T)\times\Omega}\xi^3 e^{2s\varphi}|u|^2\right) + 2I_\Sigma(s,\lambda,\mu,v=e^{s\varphi}u) \\
&\leq c_0\left(\iint_{(0,T)\times\Omega}e^{2s\varphi}|\partial_t u-\Delta u|^2 + s^3\lambda^4\iint_{(0,T)\times\omega}\xi^3 e^{2s\varphi}|u|^2\right),
\end{aligned}
\tag{3.16}
$$

where $I_\Sigma(s,\lambda,\mu,v=e^{s\varphi}u)$ is given by (3.15).

Such Carleman estimate for the heat equation can easily be obtained by following the classical techniques developed in [30, 31]. The only particularity here is that we keep all the boundary terms coming from the integration by parts, which gives the term I_Σ (see for instance [4, A.2]), and will be used in the combination with the Carleman estimate for the damped beam equation.

3.3 Carleman estimate for the coupled system—Proof of Theorem 1.7

Proof of Theorem 1.7 We denote by I_Σ^ℓ, $\ell = 1,\ldots,7$, the ℓth term on the right-hand side of (3.15). Let $\varepsilon > 0$. First, using Young's inequality, we deduce:

$$
|I_\Sigma^1| = \left|\iint_\Sigma \partial_n v\;\partial_t v\right| \lesssim \frac{1}{\varepsilon s\lambda}\iint_\Sigma \frac{1}{\xi_0}|\partial_t v|^2 + \varepsilon s\lambda\iint_\Sigma \xi_0\,|\partial_n v|^2.
$$

Above and in the two following estimates, the constant behind the symbol $\lesssim$ is also independent of ε. Next, using $\lambda \geq \mu$, Young's inequality and (3.11), we obtain

$$
|I_\Sigma^3| = \left|2s\iint_\Sigma \xi_0(\lambda^2+\mu^2(\psi_{\mathcal{I}}')^2)\partial_n v v\right| \lesssim \varepsilon s\lambda\iint_\Sigma \xi_0|\partial_n v|^2 + \frac{1}{\varepsilon}\frac{s^3}{s_0^2}\lambda^3\iint_\Sigma \xi_0^3|v|^2
$$

and

$$|I_\Sigma^6 + I_\Sigma^7| = \left| \lambda s \iint_\Sigma \xi_0 |\partial_{x_1} v|^2 + 2\mu s \iint_\Sigma \xi_0 (\psi'_{\mathcal{I}}) \partial_n v \partial_{x_1} v \right| \lesssim \varepsilon s \lambda \iint_\Sigma \xi_0 |\partial_n v|^2$$
$$+ \frac{s\lambda}{\varepsilon} \iint_\Sigma \xi_0 \, |\partial_{x_1} v|^2.$$

Next, using (3.9), we deduce that,

$$|I_\Sigma^4| = \left| s^2 \lambda \iint_\Sigma \xi_0 \; \partial_t \varphi_0 \; |v|^2 \right| \lesssim \frac{s^3}{s_0} \lambda \iint_\Sigma \xi_0^3 \; |v|^2.$$

Then by combining the above inequalities, by fixing $\varepsilon > 0$ small enough and for $s_0 \geq 1$ large enough, from (3.16) we deduce that

$$\begin{aligned} & s^{-1} \iint_{(0,T)\times\Omega} \xi^{-1} e^{2s\varphi} \left(|\partial_t u|^2 + |\Delta u|^2 \right) + s\lambda^2 \iint_{(0,T)\times\Omega} \xi e^{2s\varphi} |\nabla u|^2 \\ & \quad + s^3 \lambda^4 \iint_{(0,T)\times\Omega} \xi^3 e^{2s\varphi} |u|^2 + s^3 \lambda^3 \iint_\Sigma \xi_0^3 \; |v|^2 + \lambda s \iint_\Sigma \xi_0 |\partial_n v|^2 \\ & \lesssim \left(\iint_{(0,T)\times\Omega} e^{2s\varphi} |\partial_t u - \Delta u|^2 + s^3 \lambda^4 \iint_{(0,T)\times\omega} \xi^3 e^{2s\varphi} |u|^2 \right. \\ & \quad \left. + \frac{1}{s\lambda} \iint_\Sigma \frac{1}{\xi_0} |\partial_t v|^2 + \lambda s \iint_\Sigma \xi_0 |\partial_{x_1} v|^2 \right). \end{aligned} \tag{3.17}$$

From $v = e^{s\varphi} u$, we deduce $\partial_n v = s \partial_n \varphi v + e^{s\varphi} \partial_n u$. Then with (3.11), we get

$$\lambda s \iint_\Sigma \xi_0 \; e^{2s\varphi_0} |\partial_n u|^2 \lesssim \lambda s \iint_\Sigma \xi_0 |\partial_n v|^2 + s^3 \lambda^3 \iint_\Sigma \xi_0^3 \; |v|^2.$$

and (3.17) yields

$$\begin{aligned} H(s,\lambda,u) \leq c_0 \Big(& \iint_{(0,T)\times\Omega} e^{2s\varphi} |\partial_t u - \Delta u|^2 + s^3 \lambda^4 \iint_{(0,T)\times\omega} \xi^3 e^{2s\varphi} |u|^2 \\ & + \frac{1}{s\lambda} \iint_\Sigma \frac{1}{\xi_0} |\partial_t v|^2 + \lambda s \iint_\Sigma \xi_0 |\partial_{x_1} v|^2 \Big). \end{aligned} \tag{3.18}$$

Next, from $v = e^{s\varphi} u$ and $u = \Lambda(\partial_t \eta)$ on Σ, where Λ is defined in (2.1), we deduce

$$\partial_t v = \partial_t (e^{s\varphi_0} u) = s(\partial_t \varphi_0) e^{s\varphi_0} u + e^{s\varphi_0} \partial_t u = \Lambda(s(\partial_t \varphi_0) e^{s\varphi_0} \partial_t \eta + e^{s\varphi_0} \partial_t^2 \eta) \quad \text{on } \Sigma,$$
$$\partial_{x_1} v = s\mu(\psi'_{\mathcal{I}}) \xi_0 e^{s\varphi_0} u + e^{s\varphi_0} \partial_{x_1} u = \Lambda(s\mu(\psi'_{\mathcal{I}}) \xi_0 e^{s\varphi_0} \partial_t \eta + e^{s\varphi_0} \partial_t \partial_{x_1} \eta) \quad \text{on } \Sigma,$$

and then with (3.11) and $\lambda \geq \mu$,

$$\frac{1}{s\lambda}\iint_{\Sigma}\frac{1}{\xi_0}|\partial_t v|^2+\lambda s\iint_{\Sigma}\xi_0\,|\partial_{x_1}v|^2\lesssim\frac{1}{s\lambda}\iint_{(0,T)\times\mathcal{I}}\frac{1}{\xi_0}\,e^{2s\varphi_0}|\partial_t^2\eta|^2$$
$$+s^3\lambda\mu^2\iint_{(0,T)\times\mathcal{I}}\xi_0^3\,e^{2s\varphi_0}|\partial_t\eta|^2+s\lambda\iint_{(0,T)\times\mathcal{I}}\xi_0\,e^{2s\varphi_0}|\partial_t\partial_{x_1}\eta|^2$$
$$\lesssim\left(\frac{1+\alpha^2}{\lambda\alpha^2}+\frac{\lambda}{(1+\alpha^2)\mu^2}\right)B_\alpha(s,\mu,\eta). \tag{3.19}$$

Moreover, from $u=\Lambda(\partial_t\eta)$ on Σ we deduce

$$\alpha^2 s^3\mu^4\iint_{(0,T)\times\mathcal{I}}\xi_0^3\,e^{2s\varphi_0}|\partial_t\eta|^2=\alpha^2 s^3\mu^4\iint_{\Sigma}\xi_0^3\,e^{2s\varphi_0}|u|^2\le\frac{\alpha^2\mu^4}{\lambda^3}H(s,\lambda,u). \tag{3.20}$$

Then by combining (3.18), (3.19), (3.20) with the Carleman estimate (1.17) for the damped beam operator in Theorem 1.12, we obtain

$$H(s,\lambda,u)+B_\alpha(s,\mu,\eta)\lesssim\iint_{(0,T)\times\Omega}e^{2s\varphi}|\partial_t u+\Delta u|^2$$
$$+\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}|\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\partial_{x_1}^4\eta|^2$$
$$+s^3\lambda^4\iint_{(0,T)\times\omega}\xi^3\,e^{2s\varphi}|u|^2+\left(\alpha^{-8}+\alpha^4\right)\iint_{(0,T)\times J_0}s^7\mu^8\xi_0^7\,e^{2s\varphi_0}|\eta|^2+$$
$$+\left(\frac{1+\alpha^2}{\lambda\alpha^2}+\frac{\lambda}{(1+\alpha^2)\mu^2}\right)B_\alpha(s,\mu,\eta)+\frac{\alpha^2\mu^4}{\lambda^3}H(s,\lambda,u). \tag{3.21}$$

Moreover, note that from (3.11) we deduce

$$\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}|\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\partial_{x_1}^4\eta|^2$$
$$\lesssim\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}|\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\alpha\partial_t\eta+\partial_{x_1}^4\eta+\eta+\partial_n u|^2$$
$$+\alpha^2(s/s_0)^3\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}\xi_0^3|\partial_t\eta|^2+(s/s_0)^7\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}\xi_0^7|\eta|^2.$$

Then by choosing s_0 large enough, we can replace $\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\partial_{x_1}^4\eta$ by $\partial_t^2\eta-\alpha\partial_{x_1}^2\partial_t\eta+\alpha\partial_t\eta+\partial_{x_1}^4\eta+\eta+\partial_n u$ in (3.21).

Finally, for the choice of parameters λ, μ given in the statement of the theorem the last two terms at the right side of the inequality (3.21) can be compensated by its left side. □

4 Observability inequality for the coupled system

In this section, we fix the parameters λ, s as in Theorem 1.7. We now prove Proposition 1.10. First, we recall that φ_1, φ_2 and ξ_1, ξ_2 appearing in inequality (1.15) are defined in (1.12) and (1.13). Observe that, by definition of Ψ in (1.9), we have

$$\varphi_1(t) \le \varphi(t,\cdot) \le \varphi_2(t), \quad \xi_1(t) \le \xi(t,\cdot) \le \xi_2(t) \ \text{ for } \ t \in (0,T), \tag{4.1}$$

and that, analogously to (3.6)-(3.11), we have for $a, b \in \mathbb{R}$, and $j = 1, 2$:

$$\begin{aligned} &\xi_j^a \lesssim (s/s_0)^{b-a}\,\xi_j^b,\ |\partial_t \xi_j| + |\partial_t \varphi_j| \lesssim (s/s_0)\,\xi_j^2,\\ &|\partial_t^2 \xi_j| + |\partial_t^2 \varphi_j| \lesssim (s/s_0)^2\,\xi_j^3. \end{aligned} \tag{4.2}$$

Proof of Proposition 1.10 First, for the following we set

$$\begin{aligned} &\varrho_0(t)=s^{3/2}\lambda\,\xi_1^{3/2}\,e^{s\varphi_1}, \quad \varrho_1(t)=s^{7/2}\lambda^4\xi_2^{7/2}\,e^{s\varphi_2}, \quad \varrho_2(t)=s^{11/2}\lambda^3\xi_2^{11/2}\,e^{2s\varphi_2-s\varphi_1},\\ &\varrho_3(t)=s^{-1/2}\lambda\,\xi_1^{-1/2}\,e^{s\varphi_1}, \quad \varrho_4(t)=s^{-5/2}\lambda\,\xi_1^{-5/2}e^{s\varphi_1} \ \text{ and } \ \varrho_5(t)=s^{3/2}\lambda^2\xi_2^{3/2}\,e^{s\varphi_2}. \end{aligned} \tag{4.3}$$

Observe in particular that $\varrho_i \in \mathscr{C}^1([0,T])$, $\varrho_i(0) = \varrho_i(T) = 0$, $i = 0,\dots,5$, and thanks to (4.2):

$$|\varrho_3'| \lesssim \varrho_0 \ \text{ and } \ |\varrho_4'| \lesssim \varrho_3. \tag{4.4}$$

On the other hand, thanks to Theorem 1.7, and with $\mu \le \lambda \le (1+\alpha^2)\mu^2/c_1$, (4.1) and (4.2) we get

$$\begin{aligned} &\frac{1}{(1+\alpha^2)^2}\int_0^T s^3\lambda^2\xi_1^3\ e^{2s\varphi_1}\left(\|u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^2(\mathcal{I})} + \|\partial_t\eta\|^2_{L^2(\mathcal{I})}\right) \lesssim H(s,\lambda,u)\\ &+B_\alpha(s,\mu,\eta) \lesssim \iint_{(0,T)\times\omega} s^3\lambda^4\ \xi_2^3\ e^{2s\varphi_2}|u|^2 + \left(\alpha^{-8}+\alpha^4\right)\iint_{(0,T)\times J} s^7\lambda^8\ \xi_2^7\ e^{2s\varphi_2}|\eta|^2. \end{aligned} \tag{4.5}$$

From (4.5), we have:

$$\begin{aligned} &\int_0^T \varrho_0^2(t)\,\left(\|u\|^2_{L^2(\Omega)} + \|\eta\|^2_{H^2(\mathcal{I})} + \|\partial_t\eta\|^2_{L^2(\mathcal{I})}\right)\\ &\quad\lesssim (1+\alpha^4)\iint_{(0,T)\times\omega}\varrho_5^2(t)|u|^2 + \left(\alpha^{-8}+\alpha^8\right)\iint_{(0,T)\times J}\varrho_1^2(t)|\eta|^2. \end{aligned} \tag{4.6}$$

We observe that

$$\frac{d}{dt}\begin{bmatrix}\partial_t u\\ \partial_t\eta\\ \partial_t^2\eta\end{bmatrix} = \mathcal{A}\begin{bmatrix}\partial_t u\\ \partial_t\eta\\ \partial_t^2\eta\end{bmatrix} \quad \text{in } \ (0,T).$$

Then, we can apply (4.6) to $[\partial_t u, \partial_t\eta, \partial_t^2\eta]$ and we get

$$\begin{aligned} &\int_0^T \varrho_0^2(t)\,\left(\|\partial_t u\|^2_{L^2(\Omega)} + \|\partial_t\eta\|^2_{H^2(\mathcal{I})} + \|\partial_t^2\eta\|^2_{L^2(\mathcal{I})}\right)\\ &\quad\lesssim (1+\alpha^4)\iint_{(0,T)\times\omega}\varrho_5^2(t)|\partial_t u|^2 + \left(\alpha^{-8}+\alpha^8\right)\iint_{(0,T)\times J}\varrho_1^2(t)|\partial_t\eta|^2. \end{aligned} \tag{4.7}$$

We integrate by parts the first term on the right-hand side of (4.7). We then have

$$\begin{aligned}\iint_{(0,T)\times\omega} \varrho_5^2(t)|\partial_t u|^2 &= -\iint_{(0,T)\times\omega} \partial_t(\varrho_5^2(t)\partial_t u)u \\ &= -\iint_{(0,T)\times\omega} \partial_t(\varrho_5^2(t))\ \partial_t u\ u - \iint_{(0,T)\times\omega} \varrho_5^2(t)\ \partial_t^2 u\ u \\ &= \frac{1}{2}\iint_{(0,T)\times\omega} \partial_t^2\big(s^3\lambda^4\ \xi_2^3\ \ e^{2s\varphi_2}\big)\ |u|^2 - \iint_{(0,T)\times\omega} \varrho_5^2(t)\ \partial_t^2 u\ u.\end{aligned}$$

From (4.2), we deduce

$$\left|\partial_t^2\big(s^3\lambda^4\ \xi_2^3\ e^{2s\varphi_2}\big)\right| \lesssim s^{11}\lambda^4\ \xi_2^{11}\ e^{2s\varphi_2}$$

and since $\lambda \geq 1$ and $\varphi_2 - \varphi_1 \geq 0$ we get

$$\iint_{(0,T)\times\omega} \varrho_5^2(t)|\partial_t u|^2 \lesssim \iint_{(0,T)\times\omega} \varrho_2^2(t)|u|^2 - \iint_{(0,T)\times\omega} \varrho_5^2(t)\ \partial_t^2 u\ u.$$

On the other hand,

$$\begin{aligned}&-\iint_{(0,T)\times\omega} \varrho_5^2(t)\ \partial_t^2 u\ u = -\iint_{(0,T)\times\omega} s^3\lambda^4\ \xi_2^3\ \ e^{2s\varphi_2}\ \partial_t^2 u\ u \\ &= -\iint_{(0,T)\times\omega} (\underbrace{s^{11/2}\lambda^3\ \xi_2^{11/2} e^{2s\varphi_2 - s\varphi_1}}_{\varrho_2}\ u) \times (\underbrace{s^{-5/2}\lambda\ \xi_2^{-5/2} e^{s\varphi_1}}_{\varrho_4})\partial_t^2 u.\end{aligned}$$

Therefore, for $\varepsilon > 0$ we get

$$\begin{aligned}(1+\alpha^4)\iint_{(0,T)\times\omega} \varrho_5^2(t)|\partial_t u|^2 &\lesssim \frac{(1+\alpha^4)^2(1+\alpha)^2 M_\alpha^4}{\varepsilon}\iint_{(0,T)\times\omega} \varrho_2^2|u|^2 \\ &+ \frac{\varepsilon}{2(1+\alpha)^2 M_\alpha^4}\iint_{(0,T)\times\Omega} \varrho_4^2\ |\partial_t^2 u|^2,\end{aligned}\tag{4.8}$$

where M_α is the constant in (2.11). Above and in what follows, the constant behind the symbol $\lesssim$ is also independent of ε.

The next step is dedicated to the estimate of the last term in (4.8). For that, we are going to use the parabolic regularizing property of (2.23). We first deduce from (2.23) that

$$\frac{d}{dt}\left(\varrho_3\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix}\right) = \mathcal{A}\left(\varrho_3\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix}\right) + \varrho_3'\begin{bmatrix} u \\ \eta \\ \partial_t\eta\end{bmatrix} \quad \text{in } (0,T), \begin{bmatrix} \varrho_3 u \\ \varrho_3\eta \\ \varrho_3\partial_t\eta\end{bmatrix}(0) = 0. \tag{4.9}$$

Thanks to (2.11), we have

$$\begin{aligned}
&\|\varrho_3[u, \eta, \partial_t\eta]\|_{H^1(0,T;\ L^2(\Omega)\times\mathcal{D}(A_1^{1/2})\times L^2(\mathcal{I}))} \\
&\quad \leq M_\alpha \|\varrho_3'[u, \eta, \partial_t\eta]\|_{L^2(0,T;\ L^2(\Omega)\times\mathcal{D}(A_1^{1/2})\times L^2(\mathcal{I}))}.
\end{aligned} \tag{4.10}$$

From (4.10) and (4.4), we find

$$\begin{aligned}
&\|\varrho_3\partial_t u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_3\partial_t\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_3\partial_t^2\eta\|_{L^2(0,T;L^2(\mathcal{I}))} \\
&\quad \lesssim M_\alpha \left(\|\varrho_0 u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_0\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_0\partial_t\eta\|_{L^2(0,T;L^2(\mathcal{I}))}\right).
\end{aligned} \tag{4.11}$$

In addition, we deduce from (2.23), that

$$\begin{aligned}
&\frac{d}{dt}\left(\varrho_4\frac{d}{dt}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix}\right) = \mathcal{A}\left(\varrho_4\frac{d}{dt}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix}\right) + \varrho_4'\frac{d}{dt}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix} \quad \text{in } (0,T), \\
&\left(\varrho_4\frac{d}{dt}\begin{bmatrix} u \\ \eta \\ \partial_t\eta \end{bmatrix}\right)(0) = 0.
\end{aligned} \tag{4.12}$$

From (2.11), (4.4) and (4.11), we obtain

$$\begin{aligned}
&\|\varrho_4\partial_t^2 u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_4\partial_t^2\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_4\partial_t^3\eta\|_{L^2(0,T;L^2(\mathcal{I}))} \\
&\quad \lesssim M_\alpha^2 \left(\|\varrho_0 u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_0\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_0\partial_t\eta\|_{L^2(0,T;L^2(\mathcal{I}))}\right).
\end{aligned} \tag{4.13}$$

Since $(u, \eta, \partial_t\eta)$ satisfies (2.23) by Proposition 2.2 we have that inequality (2.9) is true for $[u, \eta, \partial_t\eta]$. Hence, by multiplying it by $\varrho_0(t)$, and integrating over time, we obtain

$$\begin{aligned}
&\|\varrho_0 u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_0\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_0\partial_t\eta\|_{L^2(0,T;L^2(\mathcal{I}))} \\
&\quad \lesssim (1+\alpha)(\|\varrho_0\partial_t u\|_{L^2(0,T;L^2(\Omega))} + \|\varrho_0\partial_t\eta\|_{L^2(0,T;H^2(\mathcal{I}))} + \|\varrho_0\partial_t^2\eta\|_{L^2(0,T;L^2(\mathcal{I}))}).
\end{aligned} \tag{4.14}$$

Thus, thanks to (4.13) and (4.14), the term in ϱ_4 of (4.8) is absorbed by the left-hand side of (4.7) when choosing ε small enough, and we deduce

$$\begin{aligned}
&\int_0^T \varrho_0^2(t)\left(\|\partial_t u\|_{L^2(\Omega)}^2 + \|\partial_t\eta\|_{H^2(\mathcal{I})}^2 + \|\partial_t^2\eta\|_{L^2(\mathcal{I})}^2\right) \\
&\quad \lesssim (1+\alpha^{10})M_\alpha^4 \iint_{(0,T)\times\omega} \varrho_2^2(t)|u|^2 + \left(\alpha^{-8}+\alpha^8\right)\iint_{(0,T)\times J} \varrho_1^2(t)|\partial_t\eta|^2.
\end{aligned} \tag{4.15}$$

Finally, (4.14) with (4.15) and $M_\alpha \lesssim \alpha^{-1} + \alpha$ proves Proposition 1.10. □

We are now in position to prove an observability inequality for system (2.24).

Theorem 4.1 *Let ω and J be nonempty open subsets of Ω and $\mathcal{I}$, respectively. There exists $C > 0$ and $\tau > 0$ such that for all $T > 0$ any solution of (2.23) satisfies:*

$$\begin{aligned} &\|u(T,\cdot)\|^2_{L^2(\Omega)} + \|\eta(T,\cdot)\|^2_{H^2(\mathcal{I})} + \|\partial_t \eta(T,\cdot)\|^2_{L^2(\mathcal{I})} \\ &\quad \leq C e^{e^{C\lambda_\alpha}\left(1+\frac{1}{T}\right)} \Big(\iint_{(0,T)\times\omega} |u|^2 + \iint_{(0,T)\times J} |\partial_t \eta|^2 \Big), \end{aligned} \tag{4.16}$$

where $\lambda_\alpha = \tau\alpha^{-2}$ if $\alpha < 1$ and $\lambda_\alpha = \tau\alpha^{2/3}$ if $\alpha \geq 1$.

Proof First, from (1.15) and (4.3) we deduce that

$$\begin{aligned} \int_{T/4}^{3T/4} \varrho_0^2(t) \|[u,\eta,\partial_t\eta]\|^2_{\mathscr{H}} \lesssim (\alpha^{-4}+\alpha^{16}) \iint_{(0,T)\times\omega} \varrho_2^2(t)|u|^2 + (\alpha^{-8}+\alpha^{10}) \\ \iint_{(0,T)\times J} \varrho_1^2(t) |\partial_t\eta|^2. \end{aligned} \tag{4.17}$$

Set $s = \widehat{s}_0(1+\alpha)(T^k + T^{k-1})$ and $\lambda = \lambda_\alpha$ with λ_α given in the statement of the Theorem for $\tau > 0$ large enough so that it satisfies the assumptions of Theorem 1.7 (see Remark 1.8). Since $\lambda_\alpha \geq \tau$ we can choose τ large enough so that

$$\forall \alpha > 0 \quad 2 - e^{\lambda_\alpha \Psi} - e^{-\lambda_\alpha \Psi} < -\varepsilon,$$

for some $\varepsilon > 0$ independent on α. The above inequality guarantees that $2\varphi_2 - \varphi_1 \leq -\varepsilon\xi_2$, and that

$$\varrho_2^2 \leq \lambda_\alpha^6 (s\xi_2)^{11} e^{-2\varepsilon(s\xi_2)} \lesssim \lambda_\alpha^6.$$

Similarly, we can choose τ large enough so that $1 - e^{\lambda_\alpha\Psi} < -\varepsilon$ and $\varphi_2 \leq -\varepsilon\xi_2$. Thus, we deduce $\varrho_1^2 \lesssim \lambda_\alpha^8$.

Next, with $s = \widehat{s}_0(1+\alpha)(T^k + T^{k-1})$ and using the fact that $\frac{3T^2}{16} \leq \ell(t) \leq \frac{T^2}{4}$ on $(\frac{T}{4}, \frac{3T}{4})$ we deduce

$$\begin{aligned} s\varphi_1(t) &\gtrsim -(1+\alpha) e^{(k+2)\lambda_\alpha\Psi} \frac{T^k + T^{k-1}}{\ell^{k/2}(t)} \gtrsim -e^{C\lambda_\alpha} \frac{T^k + T^{k-1}}{T^k} \\ &= -e^{C\lambda_\alpha} \left(1 + \frac{1}{T}\right) \quad \text{for all } t \in (\tfrac{T}{4}, \tfrac{3T}{4}), \end{aligned}$$

and

$$s\xi_1(t) \gtrsim (1+\alpha) \frac{T^k + T^{k-1}}{T^k} = (1+\alpha)\left(1 + \frac{1}{T}\right) \quad \text{for all } t \in (\tfrac{T}{4}, \tfrac{3T}{4}),$$

and from (4.2), $\lambda_\alpha \geq \tau$ and $(1+\alpha)(1+1/T) > 1$, we deduce

$$\varrho_0^2(t) = s^3 \lambda_\alpha^2\, \xi_1^3(t)\, e^{2s\varphi_1(t)} \gtrsim (s\xi_1(t))^3 e^{2s\varphi_1(t)} \gtrsim e^{-e^{C\lambda_\alpha}\left(1+\frac{1}{T}\right)}.$$

By combining (4.17) with the above estimates of ϱ_0^2, ϱ_1^2 and ϱ_2^2, we deduce

$$\int_{T/4}^{3T/4} \|[u,\eta,\partial_t\eta]\|_{\mathscr{H}}^2 \lesssim e^{e^{C\lambda\alpha}\left(1+\frac{1}{T}\right)} \left(\iint_{(0,T)\times\omega} |u|^2 + \iint_{(0,T)\times J} |\partial_t\eta|^2\right). \tag{4.18}$$

Finally, since $(e^{\mathcal{A}t})_{t\geq 0}$ is a strongly continuous semigroup of contractions on $\mathscr{H}$ we have $\|[u,\eta,\partial_t\eta](T)\|_{\mathscr{H}} \leq \|[u,\eta,\partial_t\eta](t)\|_{\mathscr{H}}$ for all $t\in(\frac{T}{4},\frac{3T}{4})$. Then

$$\frac{T}{2}\|[u,\eta,\partial_t\eta](T)\|_{\mathscr{H}} \leq \int_{T/4}^{3T/4} \|[u,\eta,\partial_t\eta](t)\|_{\mathscr{H}},$$

and with (4.18) this concludes the proof. □

5 Carleman estimates for the damped beam equation

In this section, we obtain a Carleman estimate for the damped beam equation.

We set

$$f_\eta := \partial_t^2\eta + \partial_{x_1}^4\eta - \alpha\partial_t\partial_{x_1}^2\eta \quad\text{and}\quad z = e^{s\varphi_0}\eta.$$

Direct computations yield to

$$\mathcal{M}_1 z + \mathcal{M}_2 z = e^{s\varphi} f_\eta - \mathcal{N} z, \tag{5.1}$$

with

$$\mathcal{M}_1 z = \mathcal{M}_{11} z + \mathcal{M}_{12} z \quad\text{and}\quad \mathcal{M}_2 z = \mathcal{M}_{21} z + \mathcal{M}_{22} z, \tag{5.2}$$

where

$$\mathcal{M}_{11} z = s^4(\partial_{x_1}\varphi_0)^4 z + 6s^2(\partial_{x_1}\varphi_0)^2\partial_{x_1}^2 z + \partial_{x_1}^4 z + 2s\alpha(\partial_{x_1}\varphi_0)\partial_t\partial_{x_1} z + \partial_t^2 z, \tag{5.3}$$

$$\mathcal{M}_{12} z = s^2(\partial_t\varphi_0)^2 z + s\alpha(\partial_t\varphi_0)\partial_{x_1}^2 z + s^3\alpha(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0) z, \tag{5.4}$$

and

$$\begin{aligned}\mathcal{M}_{21} z = &-4s^3(\partial_{x_1}\varphi_0)^3\partial_{x_1} z - 4s(\partial_{x_1}\varphi_0)\partial_{x_1}^3 z - \alpha\partial_t\partial_{x_1}^2 z\\ &- s^2\alpha(\partial_{x_1}\varphi_0)^2\partial_t z - 6(1+\beta)s^3(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0) z,\end{aligned} \tag{5.5}$$

$$\mathcal{M}_{22} z = -2s(\partial_t\varphi_0)\partial_t z - 2s^2\alpha(\partial_t\varphi_0)(\partial_{x_1}\varphi)\partial_{x_1} z, \tag{5.6}$$

and

$$\begin{aligned}\mathcal{N} z = &\alpha(s(\partial_{x_1}^2\varphi_0)\partial_t z + 2s(\partial_t\partial_{x_1}\varphi_0)\partial_{x_1} z + s(\partial_t\partial_{x_1}^2\varphi_0) z - s^2(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0) z\\ &- 2s^2(\partial_t\partial_{x_1}\varphi_0)(\partial_{x_1}\varphi_0) z) - s(\partial_t^2\varphi_0) z - 6s(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z - 4s(\partial_{x_1}^3\varphi_0)\partial_{x_1} z\\ &+ 12s^2(\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)\partial_{x_1} z - s(\partial_{x_1}^4\varphi_0) z + 4s^2(\partial_{x_1}^3\varphi_0)(\partial_{x_1}\varphi_0) z + 3s^2(\partial_{x_1}^2\varphi_0)^2 z\\ &+ 6\beta s^3(\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)^2 z.\end{aligned}$$

In the above setting, β is a free real-valued parameter that will play a role only in the Proof of Theorem 1.13.

For the following, we set

$$\mathcal{R}_0 = 2 \iint_{(0,T)\times\mathcal{I}} |\mathcal{N}z|^2 \tag{5.7}$$

and from (5.1) we deduce

$$\begin{aligned} &\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})} + \|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})} + 2\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_1 z \cdot \mathcal{M}_2 z \\ &= \iint_{(0,T)\times\mathcal{I}} |e^{s\varphi} f_\eta - \mathcal{N}z|^2 \leq 2\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0} |f_\eta|^2 + \mathcal{R}_0. \end{aligned} \tag{5.8}$$

From by using (3.2),(3.9),(3.10),(3.11), and with $s_0 \geq 1$, we deduce that $\mathcal{R}_0$ defined in (5.7) satisfies

$$\begin{aligned} |\mathcal{R}_0| \lesssim \left(\frac{1}{s_0} + \frac{\alpha^2}{s_0^2}\right) \iint_{(0,T)\times\mathcal{I}} \Big(& s^7\mu^8\xi_0^7|z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3|\partial^2_{x_1} z|^2 \\ &+ (1+\alpha^2)s^3\mu^4\xi_0^3|\partial_t z|^2\Big). \end{aligned} \tag{5.9}$$

Let us denote by I_{ij} the cross product of the i'th term of $\mathcal{M}_1$ with the j'th term of $\mathcal{M}_2$. We have:

$$\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_1 z \cdot \mathcal{M}_2 z = \iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11} z \cdot \mathcal{M}_{21} z + \mathcal{R}_1,$$

with

$$\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11} z \cdot \mathcal{M}_{21} z = \sum_{i=1,j=1}^{i=5,j=5} I_{ij}, \tag{5.10}$$

and

$$\mathcal{R}_1 = \sum_{(i,j)\in\mathbb{I}} I_{ij}, \quad \mathbb{I} = \{1,\dots 8\}\times\{1,\dots 7\}\backslash\{1,\dots,5\}\times\{1,\dots,5\}. \tag{5.11}$$

We prove in Sect. 5.5 that

$$\begin{aligned} |\mathcal{R}_1| \lesssim \left(\frac{1+\alpha}{s_0} + \frac{\alpha^2}{s_0^2}\right) \iint_{(0,T)\times\mathcal{I}} \Big(& s^7\mu^8\xi_0^7|z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3|\partial^2_{x_1} z|^2 \\ &+ (1+\alpha^2)s^3\mu^4\xi_0^3|\partial_t z|^2 + s\mu^2\xi_0|\partial_t\partial_{x_1} z|^2\Big). \end{aligned} \tag{5.12}$$

From (5.8), we deduce

$$\begin{aligned}&\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})} + \|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})}\\ &\quad + 2\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z\cdot\mathcal{M}_{21}z \le 2\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2 + \mathcal{R}_0 + \mathcal{R}_1.\end{aligned} \tag{5.13}$$

5.1 Computation of the cross product

The present section is dedicated to the computation of $\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z\cdot\mathcal{M}_{21}z$ that is the computations of each I_{ij} defined in (5.10).

This leads to tedious but straightforward computations.

– Computation of I_{11} –

$$I_{11} = -4s^7\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^7 z\partial_{x_1}z = 14s^7\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^6(\partial^2_{x_1}\varphi_0)|z|^2.$$

– Computation of I_{12} –

$$\begin{aligned}I_{12} &= -4s^5\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5 z\partial^3_{x_1}z\\ &= 20s^5\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial^2_{x_1}\varphi_0)z\partial^2_{x_1}z + 2s^5\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5\partial_{x_1}|\partial_{x_1}z|^2\\ &=10s^5\iint_{(0,T)\times\mathcal{I}} \partial^2_{x_1}[(\partial_{x_1}\varphi_0)^4(\partial^2_{x_1}\varphi_0)]|z|^2 - 30s^5\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial^2_{x_1}\varphi_0)|\partial_{x_1}z|^2\\ &= -30s^5\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial^2_{x_1}\varphi_0)|\partial_{x_1}z|^2 + \mathcal{R}_{12},\end{aligned}$$

where

$$\mathcal{R}_{12} = 10s^5\iint_{(0,T)\times\mathcal{I}} \partial^2_{x_1}[(\partial_{x_1}\varphi_0)^4(\partial^2_{x_1}\varphi_0)]|z|^2.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{12}| \lesssim \frac{1}{s_0^2}s^7\mu^8\iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2. \tag{5.14}$$

– Computation of I_{13} –

$$\begin{aligned}I_{13} &= -s^4\alpha\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4\partial_t\partial^2_{x_1}zz\\ &= 4s^4\alpha\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial^2_{x_1}\varphi_0)\partial_t\partial_{x_1}zz - 2s^4\alpha\iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)(\partial_{x_1}\varphi_0)^3|\partial_{x_1}z|^2\\ &=-4s^4\alpha\iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial^2_{x_1}\varphi_0)\partial_{x_1}z\partial_t z + 2s^4\alpha\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t[(\partial_{x_1}\varphi_0)^3(\partial^2_{x_1}\varphi_0)]|z|^2\end{aligned}$$

$$
\begin{aligned}
&- 2s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)(\partial_{x_1}\varphi_0)^3|\partial_{x_1}z|^2 \\
&= -4s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_{x_1}z\partial_t z + \mathcal{R}_{13},
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{R}_{13} = 2s^4\alpha &\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t[(\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)]|z|^2 \\
&- 2s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)(\partial_{x_1}\varphi_0)^3|\partial_{x_1}z|^2.
\end{aligned}
$$

Moreover, from (3.2), (3.9) and (3.11) we deduce

$$
|\mathcal{R}_{13}| \lesssim \frac{\alpha}{s_0^3}s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2 + \frac{\alpha}{s_0}s^5\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2. \tag{5.15}
$$

– Computation of I_{14} –

$$
I_{14} = -s^6\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^6\partial_t z z = 3s^6\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5(\partial_t\partial_{x_1}\varphi_0)|z|^2 = \mathcal{R}_{14}.
$$

Moreover, from (3.2) and (3.9) we deduce

$$
|\mathcal{R}_{14}| \lesssim \frac{\alpha}{s_0}s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2. \tag{5.16}
$$

– Computation of I_{15} –

$$
I_{15} = -6(1+\beta)s^7 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^6(\partial_{x_1}^2\varphi_0)|z|^2.
$$

– Computation of I_{21} –

$$
I_{21} = -24s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5\partial_{x_1}z\partial_{x_1}^2 z = 60s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial_{x_1}^2\varphi_0)|\partial_{x_1}z|^2.
$$

– Computation of I_{22} –

$$
I_{22} = -24s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3\partial_{x_1}^2 z\partial_{x_1}^3 z = 36s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2.
$$

– Computation of I_{23} –

$$
I_{23} = -6s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2\partial_t\partial_{x_1}^2 z\partial_{x_1}^2 z
$$

$$= 6s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t \partial_{x_1}\varphi_0)(\partial_{x_1}\varphi_0)|\partial_{x_1}^2 z|^2 = \mathcal{R}_{23}.$$

Moreover, from (3.2) and (3.9) we deduce

$$|\mathcal{R}_{23}| \lesssim \frac{\alpha}{s_0} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2. \tag{5.17}$$

– Computation of I_{24} –

$$\begin{aligned} I_{24} &= -6s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4 \partial_t z \partial_{x_1}^2 z \\ &= 24s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 (\partial_{x_1}^2\varphi_0)\partial_t z \partial_{x_1} z + \mathcal{R}_{24}, \end{aligned}$$

with

$$\mathcal{R}_{24} = -12s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 (\partial_t \partial_{x_1}\varphi_0)|\partial_{x_1} z|^2.$$

Moreover, from (3.2) and (3.9) we deduce

$$|\mathcal{R}_{24}| \lesssim \frac{\alpha}{s_0} s^5\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2. \tag{5.18}$$

– Computation of I_{25} –

$$\begin{aligned} I_{25} &= -36(1+\beta)s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4 (\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z z \\ &= 36(1+\beta)s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4 (\partial_{x_1}^2\varphi_0)|\partial_{x_1} z|^2 + \mathcal{R}_{25}, \end{aligned}$$

with

$$\mathcal{R}_{25} = -18(1+\beta)s^5 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^4(\partial_{x_1}^2\varphi_0)]|z|^2.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{25}| \lesssim \frac{1}{s_0^2} s^5\mu^8 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2. \tag{5.19}$$

– Computation of I_{31} –

$$\begin{aligned} I_{31} &= -4s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 \partial_{x_1} z \partial_{x_1}^4 z \\ &= 12s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 (\partial_{x_1}^2\varphi_0)\partial_{x_1} z \partial_{x_1}^3 z + 4s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 \partial_{x_1}^2 z \partial_{x_1}^3 z \end{aligned}$$

$$= 6s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|\partial_{x_1}z|^2 - 18s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2$$
$$= -18s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2 + \mathcal{R}_{31},$$

with

$$\mathcal{R}_{31} = 6s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|\partial_{x_1}z|^2.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{31}| \lesssim \frac{1}{s_0^2} s^5\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2. \tag{5.20}$$

– Computation of I_{32} –

$$I_{32} = -4s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)\partial_{x_1}^3 z\partial_{x_1}^4 z = 2s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)|\partial_{x_1}^3 z|^2.$$

– Computation of I_{33} –

$$I_{33} = -\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^4 z\partial_t\partial_{x_1}^2 z = 0.$$

– Computation of I_{34} –

$$I_{34} = -s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2\partial_{x_1}^4 z\partial_t z$$
$$= 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^3 z\partial_t z + s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2\partial_{x_1}^3 z\partial_t\partial_{x_1}z.$$

A new integration by parts leads to

$$I_{34} = -4s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1}z - s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2\partial_{x_1}^2 z\partial_t\partial_{x_1}^2 z$$
$$- 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)]\partial_{x_1}^2 z\partial_t z$$
$$= -4s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1}z + s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\partial_{x_1}\varphi_0)|\partial_{x_1}^2 z|^2$$
$$- 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)]\partial_{x_1}^2 z\partial_t z$$
$$= -4s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1}z + \mathcal{R}_{34}$$

with

$$\mathcal{R}_{34} = s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\partial_{x_1}\varphi_0)|\partial_{x_1}^2 z|^2$$
$$- 2\, s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)]\partial_{x_1}^2 z\partial_t z.$$

Moreover, from (3.2), (3.9) and (3.11) we deduce

$$\begin{aligned} |\mathcal{R}_{34}| &\lesssim \frac{\alpha}{s_0} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_{x_1}^2 z|^2 + \frac{\alpha}{s_0} s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_{x_1}^2 z||\partial_t z| \\ &\lesssim \frac{\alpha}{s_0} s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_{x_1}^2 z|^2 + \frac{\alpha}{s_0} s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^2|\partial_t z|^2. \end{aligned} \tag{5.21}$$

– Computation of I_{35} –

$$\begin{aligned} I_{35} =& - 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0) z\partial_{x_1}^4 z \\ =& 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)] z\partial_{x_1}^3 z \\ &+ 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)\partial_{x_1} z\partial_{x_1}^3 z \\ =& - 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)] z\partial_{x_1}^2 z \\ &- 12(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]\partial_{x_1} z\partial_{x_1}^2 z \\ &- 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2 \\ =& - 3(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^4[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|z|^2 \\ &+ 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|\partial_{x_1} z|^2 \\ &- 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2 \\ =& - 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2 + \mathcal{R}_{35}, \end{aligned}$$

where

$$\mathcal{R}_{35} = -3(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^4[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|z|^2$$
$$+ 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)]|\partial_{x_1} z|^2.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{35}| \lesssim \frac{1}{s_0^4} s^7 \mu^8 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2 + \frac{1}{s_0^2} s^5 \mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2. \tag{5.22}$$

– Computation of I_{41} –

$$\begin{aligned} I_{41} &= -8s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4 \partial_{x_1} z \partial_t \partial_{x_1} z \\ &= 16 s^4 \alpha \iint_{(0,T)\times\mathcal{I}} \partial_t \partial_{x_1}\varphi_0 (\partial_{x_1}\varphi_0)^3 |\partial_{x_1} z|^2 = \mathcal{R}_{41}. \end{aligned}$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{41}| \lesssim \frac{\alpha}{s_0} s^5 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2. \tag{5.23}$$

– Computation of I_{42} –

$$\begin{aligned} I_{42} &= -8s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 \partial_{x_1}^3 z \partial_t \partial_{x_1} z \\ &= 16 s^2 \alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0) \partial_{x_1}^2 z \partial_t \partial_{x_1} z + \mathcal{R}_{42}, \end{aligned}$$

with

$$\mathcal{R}_{42} = -8s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t \partial_{x_1}\varphi_0) |\partial_{x_1}^2 z|^2.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{42}| \lesssim \frac{\alpha}{s_0} s^3 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2. \tag{5.24}$$

– Computation of I_{43} –

$$I_{43} = -2s\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0) \partial_t \partial_{x_1} z \partial_t \partial_{x_1}^2 z = s\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0) |\partial_t \partial_{x_1} z|^2.$$

– Computation of I_{44} –

$$I_{44} = -2s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 \partial_t \partial_{x_1} z \partial_t z = 3s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 (\partial_{x_1}^2\varphi_0) |\partial_t z|^2.$$

– Computation of I_{45} –

$$\begin{aligned} I_{45} &= -12(1+\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_t\partial_{x_1}z z \\ &= 12(1+\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_{x_1}z\partial_t z + \mathcal{R}_{45}, \end{aligned}$$

with

$$\mathcal{R}_{45} = -6(1+\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t[(\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)]|z|^2.$$

Moreover, from (3.2), (3.9) and (3.11) we deduce

$$|\mathcal{R}_{45}| \lesssim \frac{\alpha}{s_0^3}s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2. \tag{5.25}$$

– Computation of I_{51} –

$$I_{51} = -4s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3\partial_{x_1}z\partial_t^2 z = -6s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_t z|^2 + \mathcal{R}_{51},$$

with

$$\mathcal{R}_{51} = 12s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\partial_{x_1}\varphi_0)\partial_{x_1}z\partial_t z.$$

Moreover, from (3.2) and (3.11) we deduce

$$\begin{aligned} |\mathcal{R}_{51}| &\lesssim \frac{1}{s_0}s^4\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^4|\partial_{x_1}z||\partial_t z| \lesssim \frac{1}{s_0}s^5\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2 \\ &\quad + \frac{1}{s_0}s^3\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t z|^2. \end{aligned} \tag{5.26}$$

– Computation of I_{52} –

$$\begin{aligned} I_{52} &= -4s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)\partial_{x_1}^3 z\partial_t^2 z \\ &= 4s \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)\partial_{x_1}^3 z\partial_t z + 4s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)\partial_t\partial_{x_1}^3 z\partial_t z \\ &= 4s \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)\partial_{x_1}^3 z\partial_t z - 4s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)\partial_t\partial_{x_1}^2 z\partial_t z \\ &\quad - 2s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)\partial_{x_1}|\partial_t\partial_{x_1}z|^2 \\ &= 4s \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)\partial_{x_1}^3 z\partial_t z + 4s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^3\varphi_0)\partial_t\partial_{x_1}z\partial_t z \\ &\quad + 6s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)|\partial_t\partial_{x_1}z|^2 \end{aligned}$$

$$= 6s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2 \varphi_0)|\partial_t \partial_{x_1} z|^2 + \mathcal{R}_{52},$$

with

$$\mathcal{R}_{52} = 4s \iint_{(0,T)\times\mathcal{I}} (\partial_t \partial_{x_1} \varphi_0)\partial_{x_1}^3 z \partial_t z - 2s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^4 \varphi_0)|\partial_t z|^2.$$

Moreover, from (3.2), (3.9) and (3.11) we deduce

$$\begin{aligned}
|\mathcal{R}_{52}| &\lesssim \frac{1}{s_0} s^2 \mu \iint_{(0,T)\times\mathcal{I}} \xi_0^2 |\partial_{x_1}^3 z||\partial_t z| + \frac{1}{s_0^2} s^3 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2 \\
&\lesssim \frac{1}{s_0} s\mu \iint_{(0,T)\times\mathcal{I}} \xi_0 |\partial_{x_1}^3 z|^2 \ + \frac{1}{s_0} s^3 \mu \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2 \\
&\quad + \frac{1}{s_0^2} s^3 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2.
\end{aligned} \tag{5.27}$$

– Computation of I_{53} –

$$I_{53} = -\alpha \iint_{(0,T)\times\mathcal{I}} \partial_t \partial_{x_1}^2 z \partial_t^2 z = \alpha \iint_{(0,T)\times\mathcal{I}} \partial_t \partial_{x_1} z \partial_t^2 \partial_{x_1} z = \frac{\alpha}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t |\partial_t \partial_{x_1} z|^2 = 0.$$

– Computation of I_{54} –

$$I_{54} = -s^2 \alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1} \varphi_0)^2 \partial_t z \partial_t^2 z = s^2 \alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1} \varphi_0)(\partial_t \partial_{x_1} \varphi_0)|\partial_t z|^2 = \mathcal{R}_{54}.$$

Moreover, from (3.2) and (3.11) we deduce

$$|\mathcal{R}_{54}| \lesssim \frac{\alpha}{s_0} s^3 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2. \tag{5.28}$$

– Computation of I_{55} –

$$\begin{aligned}
I_{55} &= -6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1} \varphi_0)^2 (\partial_{x_1}^2 \varphi_0) \partial_t^2 z z \\
&= 6(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1} \varphi_0)^2 (\partial_{x_1}^2 \varphi_0)|\partial_t z|^2 + \mathcal{R}_{55},
\end{aligned}$$

with

$$\mathcal{R}_{55} = -3(1+\beta)s^3 \iint_{(0,T)\times\mathcal{I}} \partial_t^2 [(\partial_{x_1} \varphi_0)^2 (\partial_{x_1}^2 \varphi_0)]|z|^2.$$

Moreover, from (3.2), (3.9), (3.10) and (3.11) we deduce

$$|\mathcal{R}_{55}| \lesssim \frac{1}{s_0^4} s^7 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2. \tag{5.29}$$

Finally, by combining the above expressions we obtain

$$\begin{aligned}
&\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z \cdot \mathcal{M}_{21}z = \sum_{i,j=1}^{i=5,j=5} I_{ij} \\
&= (8-6\beta)s^7 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^6 (\partial_{x_1}^2\varphi_0)|z|^2 \\
&\quad + (66+36\beta)s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4 (\partial_{x_1}^2\varphi_0)|\partial_{x_1}z|^2 \\
&\quad + (12-6\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 (\partial_{x_1}^2\varphi_0)|\partial_{x_1}^2 z|^2 \\
&\quad + (3\alpha^2+6\beta)s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 (\partial_{x_1}^2\varphi_0)|\partial_t z|^2 \\
&\quad + 2s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)|\partial_{x_1}^3 z|^2 + (\alpha^2+6)s \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)|\partial_t\partial_{x_1} z|^2 \\
&\quad + (32+12\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3 (\partial_{x_1}^2\varphi_0)\partial_t z \partial_{x_1} z \\
&\quad + 12s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z \partial_t\partial_{x_1} z + \mathcal{R}_2,
\end{aligned} \tag{5.30}$$

where $\mathcal{R}_2$ is the sum of the $\mathcal{R}_{ij}$ that are estimated in (5.14)-(5.29) and that satisfies

$$\begin{aligned}
|\mathcal{R}_2| \lesssim \left(\frac{1+\alpha}{s_0}\right) \iint_{(0,T)\times\mathcal{I}} &\Big(s^7\mu^8\xi_0^7|z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}z|^2 + s^3\mu^4\xi_0^3|\partial_{x_1}^2 z|^2 \\
&+ s^3\mu^4\xi_0^3|\partial_t z|^2 + s\mu^2\xi_0|\partial_{x_1}^3 z|^2 \Big).
\end{aligned} \tag{5.31}$$

For later use, we rewrite (5.30) as

$$\begin{aligned}
&\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z \cdot \mathcal{M}_{21}z = (8-6\beta)I_1 + (66+36\beta)I_2 + (12-6\beta)I_3 \\
&\quad + (3\alpha^2+6\beta)I_4 + 2I_5 + (\alpha^2+6)I_6 + (32+12\beta)\alpha I_7 + 12\alpha I_8 + \mathcal{R}_2,
\end{aligned} \tag{5.32}$$

where I_ℓ, $\ell = 1, \ldots, 8$, denote the integral terms.

5.2 Estimate of the cross product from below for arbitrary $\alpha \geq 0$—Proof of Theorem 1.11

In what follows, the value of the free parameter β plays no role, so for simplicity we fix $\beta = 1$. Then (5.32) gives

$$\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z\cdot\mathcal{M}_{21}z = 2I_1+102I_2+6I_3+(3\alpha^2+6)I_4+2I_5+(\alpha^2+6)I_6+44\alpha I_7+12\alpha I_8+\mathcal{R}_2. \tag{5.33}$$

Using the fact that $z = e^{s\varphi_0}\eta$ we observe that

$$\partial_t z = s\partial_t\varphi_0\, z + e^{s\varphi_0}\partial_t\eta. \tag{5.34}$$

Hence, we have

$$\begin{aligned}
44\alpha I_7 &= 44\alpha s^4 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_t z\partial_{x_1}z \\
&= 44\alpha s^4 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)e^{s\varphi_0}\partial_t\eta\partial_{x_1}z \\
&\quad +44\alpha s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)z\partial_{x_1}z \\
&= 44\alpha s^4 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)(e^{s\varphi_0}\partial_t\eta)\partial_{x_1}z + \mathcal{R}_3.
\end{aligned}$$

with

$$\mathcal{R}_3 = 22\alpha s^5 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)]|z|^2.$$

Moreover, we have

$$\begin{aligned}
12\alpha I_8 &= 12\alpha s^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1}z \\
&= -12\alpha s^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^3 z\partial_t z \\
&\quad -12\alpha s^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)]\partial_{x_1}^2 z\partial_t z \\
&= -12\alpha s^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)\partial_{x_1}^3 z(e^{s\varphi_0}\partial_t\eta) + \mathcal{R}_4,
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}_4 = &-12\alpha s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)\partial_{x_1}^3 z z \\
&- 12\alpha s^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}[(\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)]\partial_{x_1}^2 z\partial_t z.
\end{aligned}$$

Note that from (3.2), (3.9) and (3.11) we deduce

$$|\mathcal{R}_3| + |\mathcal{R}_4| \lesssim \frac{\alpha}{s_0} \iint_{(0,T)\times\mathcal{I}} (s^7\mu^6\xi_0^7|z|^2 + s^4\mu^3\xi_0^4|\partial_{x_1}^3 z|\,|z| + s^3\mu^4\xi_0^3|\partial_{x_1}^2 z|\,|\partial_t z|)$$
$$\lesssim \frac{\alpha}{s_0} \iint_{(0,T)\times\mathcal{I}} s^7\mu^6\xi_0^7|z|^2 + s\xi_0|\partial_{x_1}^3 z|^2 + s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2 + |\partial_t z|^2) \tag{5.35}$$

Moreover, from (3.12) and Cauchy–Schwarz inequality we deduce

$$\begin{aligned}
&\left|12\alpha s^2 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}\varphi_0)(\partial_{x_1}^2\varphi_0)(e^{s\varphi_0}\partial_t\eta)\partial_{x_1}^3 z\right| \\
&\quad + \left|44\alpha s^4 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)(e^{s\varphi_0}\partial_t\eta)\partial_{x_1} z\right| \\
&\quad \le 6\varepsilon s \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}^2\varphi_0)|\partial_{x_1}^3 z|^2 + 22\varepsilon s^5 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)^4|\partial_{x_1} z|^2 \\
&\quad + \frac{28}{\varepsilon}\alpha^2 s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2|\partial_{x_1}^2\varphi_0|e^{2s\varphi_0}|\partial_t\eta|^2 \\
&\quad = 6\varepsilon I_5 + 22\varepsilon I_2 - 6\varepsilon s \iint_{(0,T)\times J_0} (\partial_{x_1}^2\varphi_0)|\partial_{x_1}^3 z|^2 \\
&\quad -22\varepsilon s^5 \iint_{(0,T)\times J_0} (\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)^4|\partial_{x_1} z|^2 \\
&\quad + \frac{28}{\varepsilon}\alpha^2 s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2|\partial_{x_1}^2\varphi_0|e^{2s\varphi_0}|\partial_t\eta|^2
\end{aligned}$$

and then, with (5.33), by choosing ε small enough we get

$$\begin{aligned}
&\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z\cdot\mathcal{M}_{21}z + s\iint_{(0,T)\times J_0} |\partial_{x_1}^2\varphi_0||\partial_{x_1}^3 z|^2 + s^5\iint_{(0,T)\times J_0} |\partial_{x_1}^2\varphi_0|(\partial_{x_1}\varphi_0)^4|\partial_{x_1} z|^2 \\
&\quad +\alpha^2 s^3 \iint_{(0,T)\times\mathcal{I}} |\partial_{x_1}^2\varphi_0|(\partial_{x_1}\varphi_0)^2\, e^{2s\varphi_0}|(\partial_t\eta)|^2 - \mathcal{R}_2 - \mathcal{R}_3 - \mathcal{R}_4 \\
&\quad \gtrsim (I_1 + I_2 + I_3 + (1+\alpha^2)I_4 + I_5 + (1+\alpha^2)I_6).
\end{aligned}$$

Thus, with (5.13), using (3.12), (3.2) and (5.9), (5.12), (5.31), (5.35) we find that for $\lambda_0 \ge 1$ and $\widehat{s_0} = \frac{s_0}{1+\alpha} \ge 1$ large enough, for all $\lambda \ge \lambda_0$ and all $s \ge \widehat{s_0}(1+\alpha)(T^k + T^{k-1}) = s_0(T^k + T^{k-1})$,

$$\begin{aligned}
&\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})} + \|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})} \\
&\quad + \iint_{(0,T)\times\mathcal{I}} \Big(s^7\mu^8\xi_0^7\,|z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2 \\
&\quad +(1+\alpha^2)|\partial_t z|^2) + s\mu^2\xi_0(|\partial_{x_1}^3 z|^2 + (1+\alpha^2)|\partial_t\partial_{x_1} z|^2)\Big) \\
&\lesssim \Big(\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2 + \alpha^2 s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3\, e^{2s\varphi_0}|\partial_t\eta|^2
\end{aligned}$$

$$+\iint_{(0,T)\times J_0}\Big(s^7\mu^8\xi_0^7\,|z|^2+s^5\mu^6\xi_0^5|\partial_{x_1}z|^2+s^3\mu^4\xi_0^3(|\partial_{x_1}^2z|^2+(1+\alpha^2)|\partial_t z|^2)$$
$$+s\mu^2\xi_0(|\partial_{x_1}^3z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2)\Big) \tag{5.36}$$

From inequality (5.36), we directly obtain the Theorem 1.11.

5.3 The case $\alpha > 0$ –Proof of Theorem 1.12

From now on, we assume α to be positive. We will now prove our main Carleman estimates that is Theorem 1.12.

5.3.1 Estimate of the local terms in $\partial_{x_1}z$ and $\partial_{x_1}^2 z$

Let $J_0 \Subset J_1 \Subset J$ and $\chi_1 \in C_0^\infty(J_1)$ such that $\chi_1 \equiv 1$ in J_0 and let $\varepsilon_1 > 0$. By integrating by parts and by using (3.2), (3.11), we get

$$\begin{aligned}
s^3\mu^4\iint_{(0,T)\times J_0}\xi_0^3|\partial_{x_1}^2z|^2 &\le s^3\mu^4\iint_{(0,T)\times J_1}\chi_1\xi_0^3|\partial_{x_1}^2z|^2\\
&=-\iint_{(0,T)\times J_1}\partial_{x_1}(s^3\mu^4\chi_1\xi_0^3\partial_{x_1}^2z)\partial_{x_1}z\\
&\le C_1\iint_{(0,T)\times J_1}(s^4\mu^5\xi_0^4|\partial_{x_1}^2z||\partial_{x_1}z|+s^3\mu^4\xi_0^3|\partial_{x_1}^3z||\partial_{x_1}z|)\\
&\le C_1\varepsilon_1\iint_{(0,T)\times\mathcal{I}}(s\mu^2\xi_0|\partial_{x_1}^3z|^2+s^3\mu^4\xi_0^3|\partial_{x_1}^2z|^2)+\frac{C_1}{\varepsilon_1}\iint_{(0,T)\times J_1}s^5\mu^6\xi_0^5|\partial_{x_1}z|^2.
\end{aligned}$$

where $C_1 > 0$ is independent on ε_1. Then by choosing $\varepsilon_1 > 0$ small enough, we can remove the local term $s^3\mu^4\iint_{(0,T)\times J_0}\xi_0^3|\partial_{x_1}^2z|^2$ and replace J_0 by J_1 in the equality (5.36), up to a modification of the constant C there.

Next, let $J_1 \Subset J_2 \Subset J$ and let $\chi_2 \in C_0^\infty(J_2)$ such that $\chi_2 \equiv 1$ in J_1 and let $\varepsilon_2 > 0$. By integrating by parts and by using (3.2), (3.11), we get

$$\begin{aligned}
s^5\mu^6\iint_{(0,T)\times J_1}\xi_0^5|\partial_{x_1}z|^2 &\le s^5\mu^6\iint_{(0,T)\times J}\chi_2\xi_0^5|\partial_{x_1}z|^2\\
&=-\iint_{(0,T)\times J_2}\partial_{x_1}(s^5\mu^6\chi_2\xi_0^5\partial_{x_1}z)z\\
&\le C_2\iint_{(0,T)\times J_2}(s^5\mu^6\xi_0^5|\partial_{x_1}^2z||z|+s^6\mu^7\xi_0^6|\partial_{x_1}z||z|)\\
&\le C_2\varepsilon_2\iint_{(0,T)\times\mathcal{I}}(s^3\mu^4\xi_0^3|\partial_{x_1}^2z|^2+s^5\mu^6\xi_0^5|\partial_{x_1}z|^2)+\frac{C_2}{\varepsilon_2}\iint_{(0,T)\times J_2}s^7\mu^8\xi_0^7|z|^2,
\end{aligned}$$

where $C_2 > 0$ is independent on ε_2. Then, similarly as above, by choosing $\varepsilon_2 > 0$ small enough we can remove the local term $s^5\mu^6\iint_{(0,T)\times J_1}\xi_0^5|\partial_{x_1}z|^2$ and finally obtain

$$
\begin{aligned}
&\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})} + \|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})}\\
&\quad + \iint_{(0,T)\times\mathcal{I}} \Big(s^7\mu^8\xi_0^7\, |z|^2 + s^5\mu^6\xi_0^5 |\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3 (|\partial_{x_1}^2 z|^2 + (1+\alpha^2)|\partial_t z|^2)\\
&\quad + s\mu^2\xi_0 (|\partial_{x_1}^3 z|^2 + (1+\alpha^2)|\partial_t\partial_{x_1} z|^2)\Big)\\
&\le C\Big(\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2 + \alpha^2 s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3\, e^{2s\varphi_0}|\partial_t \eta|^2\\
&\quad + \iint_{(0,T)\times J_2} \Big(s^7\mu^8\xi_0^7\, |z|^2 + (1+\alpha^2) s^3\mu^4\xi_0^3|\partial_t z|^2 + s\mu^2\xi_0(|\partial_{x_1}^3 z|^2\\
&\quad + (1+\alpha^2)|\partial_t\partial_{x_1} z|^2\Big)
\end{aligned} \tag{5.37}
$$

for some $C > 0$ that can be chosen independent on α.

5.3.2 A Carleman estimate with high order terms in $\partial_t^2 z$, $\partial_t\partial_{x_1}^2 z$ and $\partial_{x_1}^4 z$

Now our goal is to estimate $\partial_t^2 z$, $\alpha\partial_t\partial_{x_1}^2 z$ and $\partial_{x_1}^4 z$.

First, from (5.2) and (5.5) we have

$$
\begin{aligned}
\alpha\partial_t\partial_{x_1}^2 z = &-4s^3(\partial_{x_1}\varphi_0)^3\partial_{x_1} z - 4s\partial_{x_1}\varphi_0\partial_{x_1}^3 z - s^2\alpha(\partial_{x_1}\varphi_0)^2\partial_t z\\
&- 12 s^3(\partial_{x_1}\varphi_0)^2\partial_{x_1}^2\varphi_0 z + \mathcal{M}_{22} - \mathcal{M}_2 z,
\end{aligned}
$$

and with (3.2) and (3.11) we deduce

$$
\begin{aligned}
&\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_t\partial_{x_1}^2 z|^2 \lesssim \iint_{(0,T)\times\mathcal{I}} (s^7\mu^8\xi_0^7|z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1} z|^2\\
&+\alpha^2 s^3\mu^4\xi_0^3|\partial_t z|^2 + s\mu^2\xi_0\, |\partial_{x_1}^3 z|^2)\\
&+ \iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}(|\mathcal{M}_{22}|^2 + |\mathcal{M}_2|^2).
\end{aligned} \tag{5.38}
$$

Moreover, from (5.6) and from (3.2) and (3.9) we deduce

$$
\frac{1}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}\, |\mathcal{M}_{22}|^2 \lesssim \left(\frac{1+\alpha^2}{s_0^2}\right) \iint_{(0,T)\times\mathcal{I}} \Big(s^5\mu^6\xi_0^5|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3|\partial_t z|^2\Big) \tag{5.39}
$$

and from (3.11) we deduce

$$
\iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}\, |\mathcal{M}_2|^2 \lesssim \frac{1}{s_0}\|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})}. \tag{5.40}
$$

Then by combining (5.37) with (5.38), (5.39) and (5.40) we obtain for $\widehat{s}_0 = \frac{s_0}{1+\alpha} \ge 1$,

$$\begin{aligned}
&\alpha^2 \iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0} |\partial_t \partial_{x_1}^2 z|^2 \\
&\quad + \iint_{(0,T)\times\mathcal{I}} (s^7\mu^8\xi_0^7\,|z|^2 + s^5\mu^6\xi_0^6\,|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3\,(|\partial_{x_1}^2 z|^2 + (1+\alpha^2)|\partial_t z|^2) \\
&\quad + s\mu^2\xi_0\,((1+\alpha^2)|\partial_t\partial_{x_1} z|^2 + |\partial_{x_1}^3 z|^2) \\
&\quad \lesssim \Big(\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2 + \alpha^2 s^3\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^3\, e^{2s\varphi_0}|\partial_t \eta|^2 \\
&\quad + \iint_{(0,T)\times J_2} \Big(s^7\mu^8\xi_0^7\,|z|^2 + s^3\mu^4\xi_0^3(1+\alpha^2)|\partial_t z|^2 + s\mu^2\xi_0(|\partial_{x_1}^3 z|^2 \\
&\quad + (1+\alpha^2)|\partial_t\partial_{x_1} z|^2\Big).
\end{aligned} \tag{5.41}$$

Next, by observing that

$$\frac{1}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_{x_1}^4 z + \partial_t^2 z|^2 = \frac{1}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}(|\partial_{x_1}^4 z|^2 + |\partial_t^2 z|^2 + 2\partial_{x_1}^4 z\partial_t^2 z)$$

and with

$$\begin{aligned}
&\frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}\partial_{x_1}^4 z\partial_t^2 z = -\frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}\partial_{x_1}^4 \partial_t z\partial_t z - \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_t(\xi_0^{-1})\partial_{x_1}^4 z\partial_t z \\
&\quad = \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}\partial_{x_1}^3\partial_t z\partial_{x_1}\partial_t z + \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}(\xi_0^{-1})\partial_{x_1}^3\partial_t z\partial_t z \\
&\quad + \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_t(\xi_0^{-1})\partial_{x_1}^3 z\partial_{x_1}\partial_t z + \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t(\xi_0^{-1})\partial_{x_1}^3 z\partial_t z \\
&\quad = -\frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_{x_1}^2\partial_t z|^2 - \frac{4}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}(\xi_0^{-1})\partial_{x_1}^2\partial_t z\partial_{x_1}\partial_t z \\
&\quad - \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2(\xi_0^{-1})\partial_{x_1}^2\partial_t z\partial_t z - \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_t(\xi_0^{-1})\partial_{x_1}^2\partial_t z\partial_{x_1}^2 z \\
&\quad - \frac{4}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t(\xi_0^{-1})\partial_{x_1}^2 z\partial_{x_1}\partial_t z - \frac{2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2\partial_t(\xi_0^{-1})\partial_{x_1}^2 z\partial_t z,
\end{aligned}$$

we deduce

$$\begin{aligned}
\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}(|\partial_{x_1}^4 z|^2 + |\partial_t^2 z|^2) &= \frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_{x_1}^4 z + \partial_t^2 z|^2 \\
&\quad + \frac{2\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_{x_1}^2\partial_t z|^2 + \mathcal{R}_6,
\end{aligned} \tag{5.42}$$

where

$$\begin{aligned}\mathcal{R}_6 &= -\frac{4\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2(\xi_0^{-1})|\partial_{x_1}\partial_t z|^2\\ &\quad+\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^4(\xi_0^{-1})|\partial_t z|^2-\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_t^2(\xi_0^{-1})|\partial_{x_1}^2 z|^2\\ &\quad+\frac{4\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\partial_t(\xi_0^{-1})\partial_{x_1}\partial_t z\partial_{x_1}^2 z+\frac{2\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2\partial_t(\xi_0^{-1})\partial_{x_1}^2 z\partial_t z.\end{aligned}$$

Moreover, from (3.2), (3.9), (3.10) and (3.11) we deduce

$$|\mathcal{R}_6| \lesssim \frac{\alpha^2}{s_0^2}\iint_{(0,T)\times\mathcal{I}} \left(s^3\mu^4\xi_0^3|\partial_{x_1}^2 z|^2+s^3\mu^4\xi_0^3|\partial_t z|^2+s\mu^2\xi_0|\partial_{x_1}\partial_t z|^2\right). \tag{5.43}$$

Thus, from (5.2) and (5.3) we deduce

$$\begin{aligned}\alpha(\partial_t^2 z+\partial_{x_1}^4 z) &= -\alpha s^4(\partial_{x_1}\varphi_0)^4 z-6\alpha s^2(\partial_{x_1}\varphi_0)^2\partial_{x_1}^2 z-2\alpha^2 s(\partial_{x_1}\varphi_0)\partial_t\partial_{x_1} z\\ &\quad+\alpha\mathcal{M}_1 z-\alpha\mathcal{M}_{12} z,\end{aligned}$$

and with (3.2) we deduce

$$\begin{aligned}&\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\partial_{x_1}^4 z+\partial_t^2 z|^2 \lesssim \alpha^2\left(\iint_{(0,T)\times\mathcal{I}} s^7\mu^8\xi_0^7\,|z|^2+s^3\mu^4\xi_0^3\,|\partial_{x_1}^2 z|^2\right.\\ &\quad\left.+\alpha^2 s\mu^2\xi_0\,|\partial_t\partial_{x_1} z|^2\right)+\alpha^2\iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}\,(|\mathcal{M}_{12}|^2+|\mathcal{M}_1|^2).\end{aligned} \tag{5.44}$$

Moreover, from (5.4) and from (3.2) and (3.9) we deduce

$$\frac{\alpha^2}{s}\iint_{(0,T)\times\mathcal{I}} \frac{1}{\xi_0}|\mathcal{M}_{12}|^2 \lesssim \left(\frac{\alpha^2}{s_0^4}+\frac{\alpha^4}{s_0^2}\right)\iint_{(0,T)\times\mathcal{I}} \left(s^7\mu^8\xi_0^7|z|^2+s^3\mu^4\xi_0^3|\partial_{x_1}^2 z|^2\right). \tag{5.45}$$

Moreover, from (3.11) we deduce

$$\alpha^2\iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}\,|\mathcal{M}_1|^2 \lesssim \frac{\alpha^2}{s_0}\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})}. \tag{5.46}$$

Then by combining (5.42), (5.43), (5.44), (5.45), (5.46), (5.38), (5.39) and (5.40) with $\widehat{s_0}=\frac{s_0}{1+\alpha}\geq 1$ we deduce

$$\begin{aligned}&\iint_{(0,T)\times\mathcal{I}} \frac{\alpha^2}{s\xi_0}(|\partial_{x_1}^4 z|^2+|\partial_t^2 z|^2) \lesssim (1+\alpha^2)\iint_{(0,T)\times\mathcal{I}}(|\mathcal{M}_1|^2+|\mathcal{M}_2|^2)\\ &\quad+(1+\alpha^2)\iint_{(0,T)\times\mathcal{I}}(s^7\mu^8\xi_0^7\,|z|^2+s^5\mu^6\xi_0^6\,|\partial_{x_1} z|^2+s^3\mu^4\xi_0^3\,(|\partial_{x_1}^2 z|^2\\ &\quad+(1+\alpha^2)|\partial_t^2 z|^2)+s\mu^2\xi_0\,(\alpha^2|\partial_t\partial_{x_1} z|^2+|\partial_{x_1}^3 z|^2).\end{aligned}$$

Then with (5.37) and (5.41), it follows that

$$\frac{\alpha^2}{(1+\alpha^2)}\iint_{(0,T)\times\mathcal{I}}\frac{1}{s\xi_0}(|\partial_{x_1}^4 z|^2+|\partial_t^2 z|^2)+\alpha^2\iint_{(0,T)\times\mathcal{I}}\frac{1}{s\xi_0}|\partial_t\partial_{x_1}^2 z|^2$$
$$+\iint_{(0,T)\times\mathcal{I}}(s^7\mu^8\xi_0^7\,|z|^2+s^5\mu^6\xi_0^6\,|\partial_{x_1}z|^2+s^3\mu^4\xi_0^3\,(|\partial_{x_1}^2 z|^2+(1+\alpha^2)|\partial_t z|^2)$$
$$+s\mu^2\xi_0\,((1+\alpha^2)|\partial_t\partial_{x_1}z|^2+|\partial_{x_1}^3 z|^2)\lesssim\Big(\iint_{(0,T)\times\mathcal{I}}e^{2s\varphi_0}|f_\eta|^2$$
$$+\alpha^2 s^3\mu^4\iint_{(0,T)\times\mathcal{I}}\xi_0^3\,e^{2s\varphi_0}|\partial_t\eta|^2+\iint_{(0,T)\times J_2}\Big(s^7\mu^8\xi_0^7\,|z|^2+s^3\mu^4\xi_0^3(1+\alpha^2)|\partial_t z|^2$$
$$+s\mu^2\xi_0(|\partial_{x_1}^3 z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2\Big).\tag{5.47}$$

5.3.3 Estimate of the local terms in $\partial_{x_1}^3 z$ and $\partial_t z$ and $\partial_t\partial_{x_1} z$

Let $J_2 \Subset J_3 \Subset J$ and $\chi_3 \in C_0^\infty(J_3)$ such that $\chi_3 \equiv 1$ in J_2 and let $\varepsilon_3 > 0$. By integrating by parts and by using (3.2), (3.11) we get

$$s\mu^2\iint_{(0,T)\times J_2}\xi_0(|\partial_{x_1}^3 z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2)\le s\mu^2\iint_{(0,T)\times J_3}\chi_3\xi_0(|\partial_{x_1}^3 z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2)$$
$$=-\iint_{(0,T)\times J_3}\partial_{x_1}(s\mu^2\chi_3\xi_0\partial_{x_1}^3 z)\partial_{x_1}^2 z-(1+\alpha^2)\iint_{(0,T)\times J_3}\partial_{x_1}(s\mu^2\chi_3\xi_0\partial_t\partial_{x_1}z)\partial_t z$$
$$\le C_3\iint_{(0,T)\times J_3}(s\mu^2\xi_0(|\partial_{x_1}^4 z||\partial_{x_1}^2 z|+(1+\alpha^2)|\partial_t\partial_{x_1}^2 z||\partial_t z|)+s\mu^3\xi_0(|\partial_{x_1}^3 z||\partial_{x_1}^2 z|+(1+\alpha^2)|\partial_t\partial_{x_1}z||\partial_t z|))$$
$$\le C_3\varepsilon_3\iint_{(0,T)\times\mathcal{I}}\frac{1}{s\xi_0}\left(\frac{\alpha^2}{1+\alpha^2}|\partial_{x_1}^4 z|^2+\alpha^2|\partial_t\partial_{x_1}^2 z|^2\right)+s\mu^2\xi_0(|\partial_{x_1}^3 z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2))$$
$$+\left(\frac{1+\alpha^2}{\alpha^2}\right)\frac{C_3}{\varepsilon_3}\iint_{(0,T)\times J_3}s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2+(1+\alpha^2)|\partial_t z|^2),$$

where $C_3 > 0$ is independent on ε_3. Then by choosing $\varepsilon_3 > 0$ small enough, in (5.47) we can replace the local term

$$\iint_{(0,T)\times J_2}\Big(s\mu^2\xi_0(|\partial_{x_1}^3 z|^2+(1+\alpha^2)|\partial_t\partial_{x_1}z|^2\Big)$$

by

$$\left(\frac{1+\alpha^2}{\alpha^2}\right)\iint_{(0,T)\times J_3}s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2+(1+\alpha^2)|\partial_t z|^2).$$

Next, to remove the local term $\iint_{(0,T)\times J_3}s^3\mu^4\xi_0^3|\partial_t z|^2$ we introduce $J_3 \Subset J_4 \Subset J$ and $\chi_4 \in C_0^\infty(J_4)$ such that $\chi_4 \equiv 1$ in J_3. Let $\varepsilon_4 > 0$. By integrating by parts and by using (3.2), (3.9), (3.11) we get

$$\frac{(1+\alpha^2)^2}{\alpha^2}s^3\mu^4\iint_{(0,T)\times J_3}\xi_0^3|\partial_t z|^2\le\frac{(1+\alpha^2)^2}{\alpha^2}s^3\mu^4\iint_{(0,T)\times J_4}\chi_4\xi_0^3|\partial_t z|^2$$
$$=-\frac{(1+\alpha^2)^2}{\alpha^2}\iint_{(0,T)\times J_4}\partial_t(s^3\mu^4\chi_4\xi_0^3\partial_t z)z$$

$$
\begin{aligned}
&\le \frac{(1+\alpha^2)^2}{\alpha^2} C_4 \iint_{(0,T)\times J_4} (s^3\mu^4\xi_0^3|\partial_t^2 z||z| + s^4\mu^4\xi_0^4|\partial_t z||z|) \\
&\le C_4\varepsilon_4 \iint_{(0,T)\times\mathcal{I}} \left(\frac{\alpha^2}{1+\alpha^2}\frac{1}{s\xi_0}|\partial_t^2 z|^2\right. \\
&\quad \left.+(1+\alpha^2)s^3\mu^4\xi_0^4|\partial_t z|^2\right) + \left(\frac{(1+\alpha^5)^2}{\alpha^6}\right)\frac{C_4}{\varepsilon_4}\iint_{(0,T)\times J_4} s^7\mu^8\xi_0^7|z|^2.
\end{aligned}
$$

where $C_4 > 0$ is independent on ε_4. Then by choosing $\varepsilon_4 > 0$ small enough, we can remove the desired local term.

Finally, to remove the local term $\frac{1+\alpha^2}{\alpha^2}\iint_{(0,T)\times J_3} s^3\mu^4\xi_0^3|\partial_{x_1}^2 z|^2$ we proceed as in Sect. 5.3.1. Then finally, we obtain:

$$
\begin{aligned}
&\frac{\alpha^2}{(1+\alpha^2)}\iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}(|\partial_{x_1}^4 z|^2 + |\partial_t^2 z|^2) + \alpha^2 \iint_{(0,T)\times\mathcal{I}} \frac{1}{s\xi_0}|\partial_t\partial_{x_1}^2 z|^2 \\
&+\iint_{(0,T)\times\mathcal{I}} (s^7\mu^8\xi_0^7\,|z|^2 + s^5\mu^6\xi_0^6\,|\partial_{x_1} z|^2 + s^3\mu^4\xi_0^3\,(|\partial_{x_1}^2 z|^2 + (1+\alpha^2)|\partial_t z|^2) \\
&+s\mu^2\xi_0\,((1+\alpha^2)|\partial_t\partial_{x_1} z|^2 + |\partial_{x_1}^3 z|^2) \lesssim \Big(\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2 \\
&+\alpha^2 s^3\mu^4\iint_{(0,T)\times\mathcal{I}} \xi_0^3\, e^{2s\varphi_0}|\partial_t\eta|^2 + \left(\frac{1}{\alpha^8}+\alpha^4\right)\iint_{(0,T)\times J}\left(s^7\mu^8\xi_0^7\,|z|^2\right).
\end{aligned} \tag{5.48}
$$

Going back to the initial variable $\eta = ze^{-s\varphi_0}$, the estimate (5.48) gives immediately Theorem 1.12.

5.4 Estimate of the cross product from below in the case $\alpha < \alpha^*$—Proof of Theorem 1.13

We recall that, for any nonnegative real parameters α and β, the equation (5.32) reads

$$
\begin{aligned}
\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z\cdot\mathcal{M}_{21}z &= (8-6\beta)I_1 + (66+36\beta)I_2 + (12-6\beta)I_3 \\
&+(3\alpha^2+6\beta)I_4 + 2I_5 + (\alpha^2+6)I_6 + (32+12\beta)\alpha I_7 + 12\alpha I_8 + \mathcal{R}_2.
\end{aligned}
$$

To have positive coefficients in front of I_ℓ, for $\ell = 1,\dots,6$, we must impose $\beta \in \left[0, \frac{4}{3}\right)$.

It remains to estimate the terms $(32+12\beta)\alpha I_7$ and $12\alpha I_8$. For that, we follow the strategy of [7] which consists in using Cauchy–Schwarz inequality to bound those terms by I_2, I_3, I_4, I_8. We will observe that with such an approach the optimal value of β is β^*, and it allows to obtain a lower bound for the cross product for $\alpha < \alpha^*$, with β^* and α^* defined in (1.18).

First, recall that

$$(32+12\beta)\alpha I_7 = (32+12\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_t z\partial_{x_1} z.$$

For any $\theta_1 > 0$, we have by (3.12) and Young's inequality

$$\begin{aligned}
&\left|(32+12\beta)s^4\alpha \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}\varphi_0)^3(\partial_{x_1}^2\varphi_0)\partial_t z\partial_{x_1} z\right| \\
&\quad\le \frac{(16+6\beta)\alpha^2}{(3\alpha^2+6\beta)\theta_1}s^5 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}\varphi_0)^4(\partial_{x_1}^2\varphi_0)|\partial_{x_1} z|^2 \\
&\qquad+(16+6\beta)\theta_1(3\alpha^2+6\beta)s^3 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)|\partial_t z|^2. \quad (5.49)
\end{aligned}$$

Then to bound $|(32+12\beta)\alpha I_7|$ by $(66+36\beta)I_2$ and $(3\alpha^2+6\beta)I_4$, and some corresponding local terms in J_0, we need $\frac{(16+6\beta)\alpha^2}{(3\alpha^2+6\beta)\theta_1} < 66+36\beta$ and $(16+6\beta)\theta_1 < 1$, or equivalently

$$\frac{(8+3\beta)\alpha^2}{(3\alpha^2+6\beta)(33+18\beta)} < \theta_1 < \frac{1}{(16+6\beta)}.$$

The limiting case corresponds to $\frac{(8+3\beta)\alpha^2}{(3\alpha^2+6\beta)(33+18\beta)} = \frac{1}{(16+6\beta)}$ or equivalently $\alpha^2 = \frac{6\beta(33+18\beta)}{18\beta^2+42\beta+29}$. As a consequence, we control the term $|(32+12\beta)\alpha I_7|$ for any $\alpha < \alpha_1^*(\beta) = \sqrt{\frac{6\beta(33+18\beta)}{18\beta^2+42\beta+29}}$.

Similarly, we recall that

$$12\alpha I_8 = 12\, s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1} z$$

and for $\theta_2 > 0$ we have

$$\begin{aligned}
&\left|12s^2\alpha \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)\partial_{x_1}^2 z\partial_t\partial_{x_1} z\right| \\
&\quad\le \frac{6\alpha^2}{(\alpha^2+6)\theta_2}s^3 \iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}^2\varphi_0)(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2 z)^2 + 6\theta_2(\alpha^2+6)s \\
&\quad\iint_{(0,T)\times\mathcal{I}\setminus J_0} (\partial_{x_1}^2\varphi_0)(\partial_t\partial_{x_1} z)^2. \quad (5.50)
\end{aligned}$$

Then to bound $|12\alpha I_8|$ by $(12-6\beta)I_3$ and $(\alpha^2+6)I_6$, and some corresponding local terms in J_0, we need $\frac{6\alpha^2}{(\alpha^2+6)\theta_2} < 12-6\beta$ and $6\theta_2 < 1$ which leads to

$$\frac{\alpha^2}{(\alpha^2+6)(2-\beta)} < \theta_2 < \frac{1}{6}.$$

The limiting case corresponds to $\frac{\alpha^2}{(\alpha^2+6)(2-\beta)} = \frac{1}{6}$ or equivalently $\alpha^2 = \frac{6(2-\beta)}{4+\beta}$. As a consequence, we control the term $|12\alpha I_8|$ for any $\alpha < \alpha_2^*(\beta) = \sqrt{\dfrac{6(2-\beta)}{4+\beta}}$.

To conclude, we obtain a lower bound on the cross product for α less than $\alpha^*(\beta) = \min\{\alpha_1^*(\beta), \alpha_2^*(\beta)\}$. It is easily seen that the value $\alpha^*(\beta)$ is maximal for β such that $\alpha_1^*(\beta) = \alpha_2^*(\beta)$ which is to say

$$\frac{6(2-\beta)}{4+\beta} = \frac{6\beta(33+18\beta)}{18\beta^2+42\beta+29}$$

or equivalently

$$P(\beta) = 36\beta^3 + 111\beta^2 + 77\beta - 58 = 0.$$

It turns out that P admits a unique positive roots β_* which verifies $\beta_* < \frac{4}{3}$. For $\beta = \beta^*$ and $\alpha < \alpha^* = \sqrt{\frac{6(2-\beta^*)}{4+\beta^*}}$, we have

$$\iint_{(0,T)\times\mathcal{I}} \mathcal{M}_{11}z \cdot \mathcal{M}_{21}z + \iint_{(0,T)\times J_0} \Big(s^5\mu^6\xi_0^5|\partial_{x_1}z|^2 + s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2 + |\partial_t z|^2) + |\partial_t\partial_{x_1}z|^2)\Big) - \mathcal{R}_2 \geq C(I_1+I_2+I_3+I_4+I_5+I_6), \tag{5.51}$$

for some constant $C > 0$ that may depend on $\alpha \in (0, \alpha^*)$. Combining (5.51) with (5.9), (5.12) and (5.13), we find that for $\lambda_0 \geq 1$ and $s_0 \geq 1$ large enough, for all $\lambda \geq \lambda_0$ and all $s \geq s_0(T^k + T^{k-1}) = s_0(T^k + T^{k-1})$,

$$\begin{aligned}
&\|\mathcal{M}_1 z\|^2_{L^2((0,T)\times\mathcal{I})} + \|\mathcal{M}_2 z\|^2_{L^2((0,T)\times\mathcal{I})} + \iint_{(0,T)\times\mathcal{I}} \Big(s^7\mu^8\xi_0^7\ |z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}z|^2 \\
&\quad + s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2 + |\partial_t z|^2) + s\mu^2\xi_0(|\partial_{x_1}^3 z|^2 + |\partial_t\partial_{x_1}z|^2)\Big) \\
&\leq C\Big(\iint_{(0,T)\times J_0} \Big(s^7\mu^8\xi_0^7\ |z|^2 + s^5\mu^6\xi_0^5|\partial_{x_1}z|^2 + s^3\mu^4\xi_0^3(|\partial_{x_1}^2 z|^2 + |\partial_t z|^2) \\
&\quad + s\mu^2\xi_0(|\partial_{x_1}^3 z|^2 + |\partial_t\partial_{x_1}z|^2)\iint_{(0,T)\times\mathcal{I}} e^{2s\varphi_0}|f_\eta|^2\Big)
\end{aligned} \tag{5.52}$$

for some constant $C > 0$ that may depend on $\alpha \in (0, \alpha^*)$. Finally, we proceed as in Sect. 5.3 to end the proof of Theorem 1.13.

5.5 Bounding $\mathcal{R}_1$—Proof of the estimate (5.12)

To complete our study, it remains to prove that $\mathcal{R}_1$, defined in (5.11), satisfies (5.12). This is the purpose of the present section.

For each $(i, j) \in \mathbb{I}$, where the set $\mathbb{I}$ is defined in (5.11), we first give an adequate expressions of I_{ij} by using integration by parts, and then an estimate of $|I_{ij}|$ obtained by using (3.2), (3.9), (3.10) and (3.11). The estimate (5.12) follows by combining all these estimates.

– Computation of I_{16} –

$$I_{16} = -2 \iint_{(0,T)\times\mathcal{I}} s^5 (\partial_{x_1}\varphi_0)^4 (\partial_t \varphi_0) z \partial_t z = s^5 \iint_{(0,T)\times\mathcal{I}} \partial_t [(\partial_{x_1}\varphi_0)^4 (\partial_t \varphi_0)] |z|^2,$$

which yields to the following estimate:

$$|I_{16}| \lesssim \frac{1}{s_0^2} s^7 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2.$$

– Computation of I_{17} –

$$I_{17} = -2\alpha s^6 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5 (\partial_t \varphi_0) z \partial_{x_1} z = \alpha s^6 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1} \Big[(\partial_{x_1}\varphi_0)^5 (\partial_t \varphi_0) \Big] |z|^2,$$

which implies $|I_{17}| \lesssim \dfrac{\alpha}{s_0} s^7 \mu^6 \displaystyle\iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2$.

– Computation of I_{26} –

$$\begin{aligned} I_{26} &= -12\alpha s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2 (\partial_t \varphi_0) \partial_t z \partial_{x_1}^2 z \\ &= 12\alpha s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1} \Big[(\partial_{x_1}\varphi_0)^2 (\partial_t \varphi_0) \Big] \partial_{x_1} z \partial_t z \\ &\quad - 6\alpha s^3 \iint_{(0,T)\times\mathcal{I}} \partial_t \Big[(\partial_{x_1}\varphi_0)^2 (\partial_t \varphi_0) \Big] |\partial_{x_1} z|^2, \end{aligned}$$

which yields the estimate

$$\begin{aligned} |I_{26}| &\lesssim \frac{\alpha}{s_0} s^4 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^4 |\partial_{x_1} z| |\partial_t z| + \frac{\alpha}{s_0^2} s^5 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2 \\ &\lesssim \frac{\alpha}{s_0} s^3 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2 + \left(\frac{\alpha}{s_0} + \frac{\alpha}{s_0^2} \right) s^5 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2. \end{aligned}$$

– Computation of I_{36} –

$$\begin{aligned} I_{36} &= -2s \iint_{(0,T)\times\mathcal{I}} (\partial_t \varphi_0) \partial_t z \partial_{x_1}^4 z \\ &= 2s \iint_{(0,T)\times\mathcal{I}} (\partial_t \partial_{x_1} \varphi_0) \partial_t z \partial_{x_1}^3 z + 2s \iint_{(0,T)\times\mathcal{I}} (\partial_t \varphi_0) \partial_t \partial_{x_1} z \partial_{x_1}^3 z \\ &= -2s \iint_{(0,T)\times\mathcal{I}} (\partial_t \partial_{x_1}^2 \varphi_0) \partial_t z \partial_{x_1}^2 z - 4s \iint_{(0,T)\times\mathcal{I}} (\partial_t \partial_{x_1} \varphi_0) \partial_t \partial_{x_1} z \partial_{x_1}^2 z \\ &\quad + s \iint_{(0,T)\times\mathcal{I}} (\partial_t^2 \varphi_0) |\partial_{x_1}^2 z|^2. \end{aligned}$$

This gives the following estimate

$$
\begin{aligned}
|I_{36}| &\lesssim \frac{1}{s_0^2} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z||\partial_{x_1}^2 z| + \frac{1}{s_0} s^2\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^2 |\partial_t\partial_{x_1} z||\partial_{x_1}^2 z| \\
&\quad + \frac{1}{s_0^2} s^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2 \lesssim \frac{1}{s_0^2} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 (|\partial_t z|^2 + |\partial_{x_1}^2 z|^2) \\
&\quad + \frac{1}{s_0} s\mu \iint_{(0,T)\times\mathcal{I}} \xi_0 |\partial_t\partial_{x_1} z|^2.
\end{aligned}
$$

– **Computation of** I_{37} –

$$
\begin{aligned}
I_{37} &= -2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\partial_{x_1} z \partial_{x_1}^4 z \\
&= 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]\partial_{x_1} z \partial_{x_1}^3 z \\
&\quad - s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_{x_1}^2 z|^2 \\
&= s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^3\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_{x_1} z|^2 \\
&\quad - 3\alpha s^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_{x_1}^2 z|^2.
\end{aligned}
$$

As a consequence, we obtain $|I_{37}| \lesssim \dfrac{\alpha}{s_0^3} s^5\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2 + \dfrac{\alpha}{s_0} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2$.

– **Computation of** I_{46} –

$$
\begin{aligned}
I_{46} &= -4\, s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\partial_t\partial_{x_1} z \partial_t z \\
&= 2\, s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_t z|^2.
\end{aligned}
$$

From this, we obtain the estimate $|I_{46}| \lesssim \dfrac{\alpha}{s_0} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2$.

– **Computation of** I_{47} –

$$
\begin{aligned}
I_{47} &= -4\, s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\partial_{x_1} z \partial_t\partial_{x_1} z \\
&= 2\, s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big]|\partial_{x_1} z|^2.
\end{aligned}
$$

As a consequence, we obtain $|I_{47}| \lesssim \dfrac{\alpha^2}{s_0^2} s^5\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2$.

– **Computation of** I_{56} –

$$I_{56} = -2\,s \iint_{(0,T)\times\mathcal{I}} (\partial_t \varphi_0)\partial_t z \partial_t^2 z = s \iint_{(0,T)\times\mathcal{I}} (\partial_t^2 \varphi_0)|\partial_t z|^2,$$

which implies $|I_{56}| \lesssim \frac{1}{s_0^2} s^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2$.

– **Computation of** I_{57} –

$$\begin{aligned} I_{57} &= -2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\partial_{x_1} z \partial_t^2 z \\ &= 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]\partial_{x_1} z \partial_t z - s^2\alpha \\ &\quad \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_t z|^2, \end{aligned}$$

and this gives

$$\begin{aligned} |I_{57}| &\lesssim \frac{\alpha}{s_0^2} s^4\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^4 |\partial_{x_1} z||\partial_t z| + \frac{\alpha}{s_0} s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2 \\ &\lesssim \frac{\alpha}{s_0^2} s^5\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2 + \left(\frac{\alpha}{s_0^2} + \frac{\alpha}{s_0}\right) s^3\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_t z|^2. \end{aligned}$$

– **Computation of** I_{61} –

$$I_{61} = -4s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)^2 z \partial_{x_1} z = 2s^5 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)^2\big]|z|^2,$$

which leads to $|I_{61}| \lesssim \frac{1}{s_0^2} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2$.

– **Computation of** I_{62} –

$$\begin{aligned} I_{62} &= -4s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2 z \partial_{x_1}^3 z \\ &= 4s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big] z \partial_{x_1}^2 z - 2s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big]|\partial_{x_1} z|^2 \\ &= -4s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big] z \partial_{x_1} z - 6s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big]|\partial_{x_1} z|^2 \\ &= 2s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^3\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big]|z|^2 - 6s^3 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big]|\partial_{x_1} z|^2. \end{aligned}$$

As a result, we obtain $|I_{62}| \lesssim \frac{1}{s_0^4} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2 + \frac{1}{s_0^2} s^5\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2$.

– **Computation of** I_{63} –

$$
\begin{aligned}
I_{63} &= -s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)^2 z\partial_t\partial_{x_1}^2 z\\
&= 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)(\partial_t\varphi_0)z\partial_t\partial_{x_1} z - s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t^2\varphi_0)(\partial_t\varphi_0)|\partial_{x_1} z|^2\\
&= s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_t\partial_{x_1}\big[(\partial_t\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|z|^2 - 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\partial_t z\partial_{x_1} z\\
&\quad - s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t^2\varphi_0)(\partial_t\varphi_0)|\partial_{x_1} z|^2.
\end{aligned}
$$

From this, we get the following estimate:

$$
\begin{aligned}
|I_{63}| &\lesssim \frac{\alpha}{s_0^5}s^7\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2 + \frac{\alpha}{s_0^2}s^4\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^4|\partial_t z||\partial_{x_1} z|\\
&\quad + \frac{\alpha}{s_0^3}s^5 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1} z|^2 \lesssim \frac{\alpha}{s_0^5}s^7\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2\\
&\quad + \frac{\alpha}{s_0^2}s^3\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t z|^2 + \left(\frac{\alpha}{s_0^2}+\frac{\alpha}{s_0^3}\right)s^5\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1} z|^2.
\end{aligned}
$$

– Computation of I_{64} –

$$
I_{64} = -s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)^2 z\partial_t z = \frac{s^4\alpha}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)^2\big]|z|^2,
$$

It follows that $|I_{64}| \lesssim \dfrac{\alpha}{s_0^3}s^7\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$.

– Computation of I_{65} –

As we directly have $I_{65} = -12\, s^5 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)^2|z|^2$, we obtain immediately $|I_{65}| \lesssim \dfrac{1}{s_0^2}s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$.

– Computation of I_{66} –

$$
I_{66} = -2\, s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)^3 z\partial_t z = 3\, s^3 \iint_{(0,T)\times\mathcal{I}} (\partial_t^2\varphi_0)(\partial_t\varphi_0)^2|z|^2
$$

which leads to $|I_{66}| \lesssim \dfrac{1}{s_0^4}s^7 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$.

– Computation of I_{67} –

$$
I_{67} = -2s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^3 z\partial_{x_1} z = s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^3\big]|z|^2.
$$

Accordingly, we obtain $|I_{67}| \lesssim \dfrac{\alpha}{s_0^3} s^7 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^7 |z|^2.$

– Computation of I_{71} –

$$
\begin{aligned}
I_{71} &= -4s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\partial_{x_1} z \partial_{x_1}^2 z \\
&= 2s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\big]|\partial_{x_1} z|^2,
\end{aligned}
$$

and so $|I_{71}| \lesssim \dfrac{\alpha}{s_0} s^5 \mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2.$

– Computation of I_{72} –

$$
\begin{aligned}
I_{72} &= -4s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\partial_{x_1}^2 z \partial_{x_1}^3 z \\
&= 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)\big]|\partial_{x_1}^2 z|^2,
\end{aligned}
$$

and it follows that $|I_{72}| \lesssim \dfrac{\alpha}{s_0} s^3 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2.$

– Computation of I_{73} –

$$
I_{73} = -s\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)\partial_{x_1}^2 z \partial_t \partial_{x_1}^2 z = \frac{s\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} (\partial_t^2\varphi_0)|\partial_{x_1}^2 z|^2,
$$

from which we deduce $|I_{73}| \lesssim \dfrac{\alpha^2}{s_0^2} s^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 |\partial_{x_1}^2 z|^2.$

– Computation of I_{74} –

$$
\begin{aligned}
I_{74} &= -s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\partial_t z \partial_{x_1}^2 z \\
&= s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big]\partial_t z \partial_{x_1} z \\
&\quad - \frac{s^3\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big]|\partial_{x_1} z|^2,
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
|I_{74}| &\lesssim \frac{\alpha}{s_0} s^4 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^4 |\partial_{x_1} z|(\alpha|\partial_t z|) + \frac{\alpha^2}{s_0^2} s^5 \mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2 \\
&\lesssim \frac{\alpha}{s_0} s^3 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3 (\alpha^2|\partial_t z|^2) + \left(\frac{\alpha}{s_0} + \frac{\alpha^2}{s_0^2}\right) s^5 \mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^5 |\partial_{x_1} z|^2.
\end{aligned}
$$

– Computation of I_{75} –

$$\begin{aligned}I_{75} &= -12s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)z\partial_{x_1}^2 z\\ &= -6s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^2\big[(\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)\big]|z|^2\\ &\quad + 12s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)|\partial_{x_1}z|^2.\end{aligned}$$

Thereby, it follows that $|I_{75}| \lesssim \frac{\alpha}{s_0^3}s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2+\frac{\alpha}{s_0}s^5\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2$.

– Computation of I_{76} –

$$\begin{aligned}I_{76} &= -2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)^2\partial_t z\partial_{x_1}^2 z\\ &= 4s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)(\partial_t\partial_{x_1}\varphi_0)\partial_t z\partial_{x_1} z - 2s^2\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_t\varphi_0)(\partial_t^2\varphi_0)|\partial_{x_1}z|^2.\end{aligned}$$

A direct estimation of these two terms gives

$$\begin{aligned}|I_{76}| &\lesssim \frac{\alpha}{s_0^2}s^4\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^4|\partial_t z||\partial_{x_1}z| + \frac{\alpha}{s_0^3}s^5 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2\\ &\lesssim \frac{\alpha}{s_0^2}s^3\mu \iint_{(0,T)\times\mathcal{I}} \xi_0^3|\partial_t z|^2 + \left(\frac{\alpha}{s_0^3}+\frac{\alpha}{s_0^2}\right)s^5 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2.\end{aligned}$$

– Computation of I_{77} –

$$\begin{aligned}I_{77} &= -2s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\partial_{x_1}^2 z\partial_{x_1} z\\ &= s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)(\partial_t\varphi_0)^2\big]|\partial_{x_1}z|^2,\end{aligned}$$

therefore $|I_{77}| \lesssim \frac{\alpha^2}{s_0^2}s^5\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1}z|^2$.

– Computation of I_{81} –

$$\begin{aligned}I_{81} &= -4s^6\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^5(\partial_t\varphi_0)z\partial_{x_1} z\\ &= 2s^6\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^5(\partial_t\varphi_0)\big]|z|^2,\end{aligned}$$

from which we deduce $|I_{81}| \lesssim \frac{\alpha}{s_0}s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$.

– Computation of I_{82} –

$$\begin{aligned}
I_{82} &= -4s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0) z\partial_{x_1}^3 z \\
&= 4s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\big] z\partial_{x_1}^2 z - 2s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\big] |\partial_{x_1} z|^2 \\
&= 2s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}^3\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\big] |z|^2 - 6s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)\big] |\partial_{x_1} z|^2,
\end{aligned}$$

and as a consequence $|I_{82}| \lesssim \dfrac{\alpha}{s_0^3} s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2 + \dfrac{\alpha}{s_0} s^5\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1} z|^2.$

– Computation of I_{83} –

$$\begin{aligned}
I_{83} &= -s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0) z\partial_t\partial_{x_1}^2 z \\
&= s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big] z\partial_t\partial_{x_1} z - \frac{s^3\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big] |\partial_{x_1} z|^2 \\
&= \frac{s^3\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\partial_{x_1}^2\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big] |z|^2 - s^3\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big] \partial_t z\, \partial_{x_1} z \\
&\quad - \frac{s^3\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)\big] |\partial_{x_1} z|^2.
\end{aligned}$$

$$\begin{aligned}
|I_{83}| &\lesssim \frac{\alpha^2}{s_0^4} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2 + \frac{\alpha}{s_0} s^4\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^4(\alpha|\partial_t z|)|\partial_{x_1} z| \\
&\quad + \frac{\alpha^2}{s_0^2} s^5\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1} z|^2 \lesssim \frac{\alpha^2}{s_0^4} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2 \\
&\quad + \frac{\alpha}{s_0} s^3\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^3(\alpha^2|\partial_t z|^2) + \left(\frac{\alpha}{s_0} + \frac{\alpha^2}{s_0^2}\right) s^5\mu^3 \iint_{(0,T)\times\mathcal{I}} \xi_0^5|\partial_{x_1} z|^2.
\end{aligned}$$

– Computation of I_{84} –

$$I_{84} = -s^5\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial_t\varphi_0) z\partial_t z = \frac{s^5\alpha^2}{2} \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^4(\partial_t\varphi_0)\big]|z|^2,$$

and therefore $|I_{84}| \lesssim \dfrac{\alpha^2}{s_0^2} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2.$

– Computation of I_{85} –

As we have $I_{85} = -12\, s^6\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^4(\partial_{x_1}^2\varphi_0)(\partial_t\varphi_0)|z|^2$, we directly obtain

$$|I_{85}| \lesssim \frac{\alpha}{s_0} s^7\mu^6 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2.$$

– Computation of I_{86} –

$$I_{86} = -2s^4\alpha \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)^2 z\partial_t z = s^4\alpha \iint_{(0,T)\times\mathcal{I}} \partial_t\big[(\partial_{x_1}\varphi_0)^2(\partial_t\varphi_0)^2\big]|z|^2,$$

which directly gives $|I_{86}| \lesssim \dfrac{\alpha}{s_0^3} s^7\mu^2 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$.

– Computation of I_{87} –

$$\begin{aligned} I_{87} &= -2s^5\alpha^2 \iint_{(0,T)\times\mathcal{I}} (\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)^2 z\partial_{x_1} z \\ &= s^5\alpha^2 \iint_{(0,T)\times\mathcal{I}} \partial_{x_1}\big[(\partial_{x_1}\varphi_0)^3(\partial_t\varphi_0)^2\big]|z|^2, \end{aligned}$$

which in turn implies $|I_{87}| \lesssim \dfrac{\alpha^2}{s_0^2} s^7\mu^4 \iint_{(0,T)\times\mathcal{I}} \xi_0^7|z|^2$, and gives the last estimate needed to obtain (5.12).

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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