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THE DIRICHLET-TO-NEUMANN OPERATOR ON ROUGH DOMAINS WITH FINITE VOLUME

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Abstract. Using a variational formulation we consider the Dirichlet-to-Neumann operator on a connected open set $\Omega \subset \mathbb{R}^d$ of finite volume, assuming only that the surface measure is locally finite on the boundary. Then the boundary may have infinite measure and trace properties become delicate. We show that this has consequences for the kernel of the Dirichlet-to-Neumann operator and characterise the situation in which a trace on Ω both exists and is unique.

Keywords: Dirichlet-to-Neumann operator; trace; form method; rough boundary

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1. Introduction

With the use of form methods we investigate trace properties of elements in $H^1(\Omega)$ for a fairly general open set $\Omega \subset \mathbb{R}^d$. We assume only that Ω is connected, has finite volume and that the natural surface measure is locally finite on the boundary Γ. The possible irregularity of Γ may cause either nonexistence or nonuniqueness of the trace on Ω. We obtain characterisations of existence and uniqueness of the trace on Ω in terms of the Dirichlet-to-Neumann operator on Γ.

In [1] a variational Dirichlet-to-Neumann operator was used to study trace properties on bounded domains with rough boundary. In [6] similar domains were investigated via the principal eigenvalue of the related Robin problem. We seek to extend the results of [1] to the case in which Ω is unbounded with finite volume. It is hence possible in our setting for the boundary to have infinite surface measure. The treatment of this case constitutes our principal endeavour. When the boundary has finite surface measure, every constant function always has a trace. In

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Lemma 3.1 we characterise the situation in which the constant function $\mathbb{1}$ has a trace in case the boundary has infinite surface measure. We subsequently present an example of a domain with infinite surface measure such that the trace exists but is not unique.

In order to formulate our main results, we introduce some notation. Throughout this paper we denote by $\Omega \subset \mathbb{R}^d$ a nonempty connected open set with finite volume, where $d \geqslant 2$. We write $\Gamma = \partial\Omega$ and provide Γ with the $(d-1)$-dimensional Hausdorff measure σ. We assume that σ is locally finite on Γ. By a compactness argument one obtains that $\sigma(\Gamma \cap B_r(x)) < \infty$ for all $x \in \mathbb{R}^d$ and $r > 0$. Therefore, the Lebesgue measure $|\Gamma \cap B_r(x)| = 0$ for all $x \in \mathbb{R}^d$ and $r > 0$, and $|\Gamma| = 0$.

If Ω is bounded with Lipschitz boundary Γ, it is well known that there exists a unique trace operator on $H^1(\Omega)$. Under our hypotheses, however, such a mapping cannot be defined. Indeed, for a function $u \in H^1(\Omega)$ both nonexistence and nonuniqueness of the trace are possible, see [3], Examples 4.2 and 4.3, or [10], Example 6.5. We therefore adopt the following generalised notions for the existence and uniqueness of a trace.

Let $u \in H^1(\Omega)$. We call $\varphi \in L_2(\Gamma)$ a *trace* of u if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. Since we require that σ is locally finite on Γ, one deduces via truncation, as for example in the proof of Theorem 4.5, that φ is a trace of u if and only if φ is an approximate $L_2(\Gamma)$ trace of u in the sense of Sauter, see [10].

We say that *the trace on Ω is unique* if $0 \in L_2(\Gamma)$ is the only trace of $0 \in H^1(\Omega)$. By [2], Lemma 4.14, there exists a Borel set $\Gamma_0 \subset \Gamma$ such that

$$\{\varphi \in L_2(\Gamma)\colon\ \varphi \text{ is a trace of } 0\} = L_2(\Gamma_0).$$

Hence, the trace on Ω is unique if and only if $\sigma(\Gamma_0) = 0$. Since Γ_0 may prevent uniqueness of the trace, we call its complement $\Gamma_r = \Gamma \setminus \Gamma_0$ the *regular part of the boundary.*

In order to define the Dirichlet-to-Neumann operator on Γ, we adopt the following generalised notion of the normal derivative. If $u \in L_{1,\mathrm{loc}}(\Omega)$, we denote by $\Delta u \in \mathcal{D}(\Omega)'$ the distributional Laplacian applied to u. Let $u \in H^1(\Omega)$. We say that u has a *normal derivative* if $\Delta u \in L_2(\Omega)$ and there exists an element $\psi \in L_2(\Gamma)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\overline{v} = \int_\Gamma \psi \overline{v}$$

for all $v \in H^1(\Omega) \cap C_c(\overline{\Omega})$. Then the Stone-Weierstraß theorem provides that ψ is unique. We write $\partial_\nu u = \psi$.

Finally we introduce the *Dirichlet-to-Neumann operator* $\mathcal{N}$ in $L_2(\Gamma)$. Let $\varphi, \psi \in L_2(\Gamma)$. We write $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if there exists an element $u \in H^1(\Omega)$ such that $\Delta u = 0$, φ is a trace of u and u has a normal derivative with $\partial_\nu u = \psi$. In Theorem 2.3 we shall show that $\mathcal{N}$ is well-defined using a form method presented in [2]. As a consequence, $\mathcal{N}$ is a positive self-adjoint operator in $L_2(\Gamma)$.

If Ω is bounded, then $\ker \mathcal{N} = \mathbb{C}\mathbb{1}_\Gamma + L_2(\Gamma_0)$ by [1], Proposition 4.1. In the general case, however, the kernel of $\mathcal{N}$ depends on the measure of Γ_r. Throughout this paper the field is the complex numbers $\mathbb{C}$.

Theorem 1.1.
(a) *If* $\sigma(\Gamma_r) < \infty$, *then* $\ker \mathcal{N} = \mathbb{C}\mathbb{1}_{\Gamma_r} + L_2(\Gamma_0)$.
(b) *If* $\sigma(\Gamma_r) = \infty$, *then* $\ker \mathcal{N} = L_2(\Gamma_0)$.

Consequently, we obtain the following characterisation of uniqueness of the trace.

Corollary 1.2. *The following are equivalent.*
(i) *The trace on* Ω *is unique.*
(ii) $\dim \ker \mathcal{N} < \infty$.
(iii) $\dim \ker \mathcal{N} \leqslant 1$.

We prove Theorem 1.1 and Corollary 1.2 in Section 3. In Example 3.3 we construct a domain for which $\sigma(\Gamma) = \infty$ whilst $\sigma(\Gamma_r) < \infty$.

If an element of $H^1(\Omega)$ has a trace, then it must belong to the closure of $H^1(\Omega) \cap C_c(\overline{\Omega})$. We define

$$\widetilde{H}^1(\Omega) = \overline{H^1(\Omega) \cap C_c(\overline{\Omega})}^{H^1(\Omega)}.$$

In Section 4 we establish the following characterisation of the case in which each element of $\widetilde{H}^1(\Omega)$ has a unique trace in $L_2(\Gamma)$.

Theorem 1.3. *The following are equivalent.*
(i) *Every element of* $\widetilde{H}^1(\Omega)$ *has a unique trace.*
(ii) *There exists a constant* $c > 0$ *such that*

$$\int_\Gamma |u|^2 \leqslant c\Big(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Big)$$

for all $u \in H^1(\Omega) \cap C_c(\overline{\Omega})$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\int_\Gamma |u|^2 \leqslant c \int_\Omega |\nabla u|^2$$

for all $u \in H^1(\Omega) \cap C_c(\overline{\Omega})$ *with* $\int_\Gamma u = 0$.
(iv) $0 \notin \sigma_{\mathrm{ess}}(\mathcal{N})$.

The above extends [1], Theorem 1.3, where a similar characterisation was provided for the case when Ω is bounded. The major difference between the proof of the above and that of [1], Theorem 1.3, is the treatment of the case $\sigma(\Gamma)=\infty$.

This paper is organised as follows. In Section 2 we characterise the Dirichlet-to-Neumann operator $\mathcal{N}$ in terms of a j-sectorial sesquilinear form. In Section 3 we consider the situation in which the trace is unique. We prove Theorem 1.1 and Corollary 1.2 and obtain that uniqueness of the trace has consequences for the kernel and range of $\mathcal{N}$. In Section 4 we introduce a trace map on Ω and examine the case in which every element of $\widetilde{H}^1(\Omega)$ has a trace before proving Theorem 1.3. Finally, we show that if $\sigma(\Gamma)=\infty$ and $\mathcal{N}$ has compact resolvent, then the operator $\mathcal{N}$ is bijective.

2. The Dirichlet-to-Neumann operator

In this section we describe the Dirichlet-to-Neumann operator $\mathcal{N}$ using form methods. As a consequence, one deduces that $\mathcal{N}$ is single-valued and hence well-defined.

Let H be a Hilbert space. Let $\mathfrak{a}\colon D(\mathfrak{a})\times D(\mathfrak{a})\to\mathbb{C}$ be a sesquilinear form and let $j\colon D(\mathfrak{a})\to H$ be a linear map. The form $\mathfrak{a}$ is called *j-sectorial* if there exist $\gamma\in\mathbb{R}$ and $\theta\in[0,\frac{1}{2}\pi)$ such that

$$\mathfrak{a}(u)-\gamma\|j(u)\|_H^2\in\Sigma_\theta$$

for all $u\in D(\mathfrak{a})$, where $\Sigma_\theta=\{r\mathrm{e}^{\mathrm{i}\alpha}\colon r\in[0,\infty)\text{ and }\alpha\in[-\theta,\theta]\}$.

We use the following version of [2], Theorem 3.2, to characterise $\mathcal{N}$.

Theorem 2.1. *Let H be a Hilbert space. Let $\mathfrak{a}\colon D(\mathfrak{a})\times D(\mathfrak{a})\to\mathbb{C}$ be a sesquilinear form and $j\colon D(\mathfrak{a})\to H$ a linear map. Suppose that j has dense range and $\mathfrak{a}$ is j-sectorial. Then there exists an operator A in H such that for all $\varphi,\psi\in H$, one has that $\varphi\in D(A)$ and $A\varphi=\psi$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{a})$ such that*

(I) $\lim\limits_{n\to\infty} j(u_n)=\varphi$ *in H,*

(II) $\lim\limits_{n,m\to\infty}\operatorname{Re}\mathfrak{a}(u_n-u_m)=0$ *and*

(III) $\lim\limits_{n\to\infty}\mathfrak{a}(u_n,v)=(\psi,j(v))_H$ *for all $v\in D(\mathfrak{a})$.*

Moreover, the operator A is m-sectorial.

Proof. See [2], Theorem 3.2 (b) and (c). □

In the situation of Theorem 2.1, we call A the *operator associated with* $(\mathfrak{a},j)$. Define the sesquilinear form $\mathfrak{l}\colon D(\mathfrak{l})\times D(\mathfrak{l})\to\mathbb{C}$ by

$$\mathfrak{l}(u,v)=\int_\Omega\nabla u\cdot\overline{\nabla v},$$

where $D(\mathfrak{l})=H^1(\Omega)\cap C_{\rm c}(\overline{\Omega})$. Define $j\colon D(\mathfrak{l})\to L_2(\Gamma)$ by $j(u)=u|_\Gamma$.

In Theorem 2.3 we shall prove that $\mathcal{N}$ is equal to the operator in $L_2(\Gamma)$ associated with $(\mathfrak{l}, j)$. We first verify that $\overline{j(D(\mathfrak{l}))} = L_2(\Gamma)$.

Lemma 2.2. *The map j has dense range.*

Proof. Let $f \in L_2(\Gamma)$ and $\varepsilon > 0$. Since Γ is locally compact and σ is regular by [7], Theorem 2.1, there exists a function $\varphi \in C_c(\Gamma)$ such that $\|f - \varphi\|_{L_2(\Gamma)} < \varepsilon$. There exists a bounded open set $U \subset \mathbb{R}^d$ such that $\operatorname{supp} \varphi \subset U$. By the Stone-Weierstrass theorem there exists a function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\varphi - \chi|_{\Gamma\cap\overline{U}}\|_{C(\Gamma\cap\overline{U})} < \varepsilon\sigma(\Gamma \cap \overline{U})^{-1/2}$. Further, there exists a function $\tau \in C_c(\mathbb{R}^d)$ such that $0 \leqslant \tau \leqslant \mathbb{1}$, $\tau|_{\operatorname{supp}\varphi} = \mathbb{1}$ and $\operatorname{supp}\tau \subset U$. Then $\|\varphi - (\tau\chi)|_{\Gamma\cap\overline{U}}\|_{C(\Gamma\cap\overline{U})} < \varepsilon\sigma(\Gamma \cap \overline{U})^{-1/2}$. Therefore, $\|\varphi - (\tau\chi)|_\Gamma\|_{L_2(\Gamma)} < \varepsilon$. Now $(\tau\chi)|_{\overline{\Omega}} \in D(\mathfrak{l})$ and $\|f - j((\tau\chi)|_{\overline{\Omega}})\|_{L_2(\Gamma)} \leqslant \|f - \varphi\|_{L_2(\Gamma)} + \|\varphi - (\tau\chi)|_\Gamma\|_{L_2(\Gamma)} < 2\varepsilon$. ☐

In what follows we frequently use the following estimate due to Maz'ya. One deduces from Example 3.6.2/1 and Theorem 3.6.3 in [9] and (24) in [4] that there exists a constant $c_M > 0$ such that

$$\int_\Omega |u|^2 \leqslant c_M\Big(\int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2\Big) \tag{2.1}$$

for all $u \in H^1(\Omega) \cap C_c(\overline{\Omega})$. Here we use that Ω has finite volume and that the boundary Γ is equipped with the $(d-1)$-dimensional Hausdorff measure σ.

Theorem 2.3. *The form $\mathfrak{l}$ is j-sectorial. Let A be the operator associated with $(\mathfrak{l}, j)$. Let $\varphi, \psi \in L_2(\Gamma)$. Then the following are equivalent.*

(i) *$\varphi \in D(A)$ and $A\varphi = \psi$.*

(ii) *There exists an element $u \in H^1(\Omega)$ such that $\Delta u = 0$, φ is a trace of u and u has a normal derivative with $\partial_\nu u = \psi$.*

In particular, A is equal to the Dirichlet-to-Neumann operator $\mathcal{N}$ alluded to in the introduction. Moreover, $\mathcal{N}$ is positive and self-adjoint.

Proof. The proof proceeds as in [1], Theorem 3.3. For the sake of self-containment, we include the argument. Note that by Lemma 2.2 the map j has dense range. Moreover, since $\mathfrak{l}(u) = \int_\Omega |\nabla u|^2 \in [0, \infty)$ for all $u \in D(\mathfrak{l})$, the form $\mathfrak{l}$ is j-sectorial. By [2], Remark 3.5, it follows from the positivity and symmetry of the form $\mathfrak{l}$ that the operator A is positive and self-adjoint.

Let $\varphi, \psi \in L_2(\Gamma)$. Then it remains to show the equivalence.

(i) $\Rightarrow$ (ii). By definition there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$, $\lim\limits_{n,m\to\infty} \int_\Omega |\nabla(u_n - u_m)|^2 = 0$ and $\lim \int_\Omega \nabla u_n \cdot \overline{\nabla v} = \int_\Gamma \psi\overline{v}$

for all $v \in H^1(\Omega) \cap C_{\mathrm{c}}(\overline{\Omega})$. Then $\lim_{n,m\to\infty} \int_\Gamma |u_n - u_m|^2 = 0$. Let $c_M > 0$ be as in (2.1). Then

$$\int_\Omega |u_n - u_m|^2 \leqslant c_M \Big(\int_\Omega |\nabla(u_n - u_m)|^2 + \int_\Gamma |u_n - u_m|^2 \Big)$$

for all $n, m \in \mathbb{N}$, so $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Hence, there exists an element $u \in H^1(\Omega)$ such that $\lim u_n = u$ in $H^1(\Omega)$. So φ is a trace of u. Moreover,

$$\int_\Omega \nabla u \cdot \overline{\nabla v} = \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \overline{\nabla v} = \int_\Gamma \psi\, \overline{v}$$

for all $v \in H^1(\Omega) \cap C_{\mathrm{c}}(\overline{\Omega})$. Choosing $v \in C_{\mathrm{c}}^\infty(\Omega)$ in the above then yields that $\Delta u = 0$. Therefore,

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_\Gamma \psi \overline{v}$$

for all $v \in H^1(\Omega) \cap C_{\mathrm{c}}(\overline{\Omega})$. Then u has a normal derivative in $L_2(\Gamma)$ and $\partial_\nu u = \psi$. Hence, $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$, so $A \subset \mathcal{N}$.

(ii) $\Rightarrow$ (i). Since φ is a trace of u, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{l})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. Then $\lim_{n,m\to\infty} \mathfrak{l}(u_n - u_m) = 0$. Since $\Delta u = 0$, it follows from the definition of the normal derivative that

$$\lim_{n\to\infty} \mathfrak{l}(u_n, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_\Gamma \psi\, \overline{v}$$

for all $v \in D(\mathfrak{l})$. So $\varphi \in D(A)$ and $A\varphi = \psi$. ☐

It is a remarkable consequence of Theorem 2.1 that although an element of $H^1(\Omega)$ may have more than one trace, the operator $\mathcal{N}$ remains well-defined.

3. UNIQUENESS OF THE TRACE

In this section we consider the situation in which the elements of $H^1(\Omega)$ each have at most one trace. Many characterisations and sufficient conditions have been obtained by Sauter, see [10]. For example, if Ω has a continuous boundary, then the trace on Ω is unique. We shall obtain a characterisation in terms of the dimension of the kernel of the Dirichlet-to-Neumann operator, see Corollary 1.2.

We begin with the observation that the measure of Γ_r determines whether or not the constant function $\mathbb{1}_\Omega$ has a trace.

Lemma 3.1.
(a) *If* $\sigma(\Gamma_r) < \infty$, *then* $\mathbb{1}_{\Gamma_r}$ *is a trace of* $\mathbb{1}_\Omega$.
(b) *If* $\sigma(\Gamma_r) = \infty$, *then* $\mathbb{1}_\Omega$ *does not have a trace.*

Proof. (a) Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leqslant \chi \leqslant \mathbb{1}$ and $\chi|_{B_1(0)} = \mathbb{1}$. For each $n \in \mathbb{N}$ define $\chi_n \in C_c^\infty(\mathbb{R}^d)$ by $\chi_n(x) = \chi(n^{-1}x)$. Let $\varepsilon > 0$. Then there exists a number $n \in \mathbb{N}$ such that $\|\chi_n|_\Omega - \mathbb{1}_\Omega\|_{H^1(\Omega)} < \varepsilon$ and $\|\chi_n|_{\Gamma_r} - \mathbb{1}_{\Gamma_r}\|_{L_2(\Gamma)} < \varepsilon$. Moreover, since $\chi_n|_{\Gamma_0} \in L_2(\Gamma_0)$, it follows that $\chi_n|_{\Gamma_0}$ is a trace of 0. So there exists a funtion $v \in H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\|\chi_n|_{\Gamma_0} - v|_\Gamma\|_{L_2(\Gamma)} < \varepsilon$ and $\|v\|_{H^1(\Omega)} < \varepsilon$. Write $w = \chi_n|_{\overline{\Omega}} - v$. Then $w \in H^1(\Omega) \cap C_c(\overline{\Omega})$ and

$$\|w - \mathbb{1}_\Omega\|_{H^1(\Omega)} \leqslant \|\chi_n|_\Omega - \mathbb{1}_\Omega\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} < 2\varepsilon.$$

Moreover,

$$\|w|_\Gamma - \mathbb{1}_{\Gamma_r}\|_{L_2(\Gamma)} \leqslant \|\chi_n|_{\Gamma_r} - \mathbb{1}_{\Gamma_r}\|_{L_2(\Gamma)} + \|\chi_n|_{\Gamma_0} - v|_\Gamma\|_{L_2(\Gamma)} < 2\varepsilon.$$

Hence, $\mathbb{1}_{\Gamma_r}$ is a trace of $\mathbb{1}_\Omega$.

(b) Suppose to the contrary that there exists an element $\varphi \in L_2(\Gamma)$ such that φ is a trace of $\mathbb{1}_\Omega$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim u_n = \mathbb{1}_\Omega$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$. Then $(\chi u_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega) \cap C_c(\overline{\Omega})$ with $\lim \chi u_n = \chi|_\Omega$ in $H^1(\Omega)$ and $\lim (\chi u_n)|_\Gamma = \chi|_\Gamma \varphi$ in $L_2(\Gamma)$. So $\chi|_\Gamma\varphi$ is a trace of $\chi|_\Omega$. Alternatively, clearly $\chi|_\Gamma$ is a trace of $\chi|_{\overline{\Omega}} = \chi|_\Omega$ in $H^1(\Omega)$. Hence, $(\chi|_\Gamma\varphi)|_{\Gamma_r} = (\chi|_\Gamma)|_{\Gamma_r}$. So $\varphi = \mathbb{1}$ on $\Gamma_r \cap [\chi \neq 0]$. By a covering argument one deduces that $\varphi|_{\Gamma_r} = \mathbb{1}_{\Gamma_r}$ σ-a.e. Then $\sigma(\Gamma_r) \leqslant \int_\Gamma |\varphi|^2 < \infty$. This is a contradiction. □

We are now able to characterise the kernel of the Dirichlet-to-Neumann operator.

Proof of Theorem 1.1. Note first that for all $\varphi \in L_2(\Gamma_0)$, the choice $u = 0 \in H^1(\Omega)$ yields that $\Delta u = 0$, φ is a trace of u and $\partial_\nu u = 0$. So $\varphi \in \ker \mathcal{N}$ and it follows that $L_2(\Gamma_0) \subset \ker \mathcal{N}$.

(a), $\supset$. Since $\Delta \mathbb{1}_\Omega = 0$ and $\partial_\nu \mathbb{1}_\Omega = 0$, it follows from Lemma 3.1 (a) that $\mathbb{1}_{\Gamma_r} \in D(\mathcal{N})$ and $\mathcal{N}\mathbb{1}_{\Gamma_r} = 0$. Then $\mathbb{C}\mathbb{1}_{\Gamma_r} + L_2(\Gamma_0) \subset \ker \mathcal{N}$ by the linearity of $\mathcal{N}$.

(a), $\subset$. Let $\varphi \in \ker \mathcal{N}$. Without loss of generality we may assume that $\varphi \in L_2(\Gamma_r)$. Since $\mathcal{N}\varphi = 0$, there exists an element $u \in H^1(\Omega)$ such that $\Delta u = 0$, φ is a trace of u and $\partial_\nu u = 0$. So

$$\int_\Omega \nabla u \cdot \overline{\nabla v} = -\int_\Omega (\Delta u)\overline{v} + \int_\Gamma (\partial_\nu u)\overline{v} = 0$$

for all $v \in H^1(\Omega) \cap C_c(\overline{\Omega})$. Moreover, $u \in \widetilde{H}^1(\Omega)$ since u has a trace. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$. Then

$$\int_\Omega |\nabla u|^2 = \lim_{n\to\infty} \int_\Omega \nabla u \cdot \overline{\nabla u_n} = 0.$$

So u is constant, since Ω is connected. Without loss of generality we may assume that $u = \mathbb{1}_\Omega$. Then $\varphi = \mathbb{1}_{\Gamma_r}$ by Lemma 3.1 (a). Hence, $\ker \mathcal{N} \subset \mathbb{C}\mathbb{1}_{\Gamma_r} + L_2(\Gamma_0)$.

(b) Let $\varphi \in \ker \mathcal{N}$. Without loss of generality we may again assume that $\varphi \in L_2(\Gamma_r)$. Then there exists a $u \in H^1(\Omega)$ such that $\Delta u = 0$, φ is a trace of u and $\partial_\nu u = 0$. Arguing as in the above, one deduces that u is constant. If $u \neq 0$, we may assume without loss of generality that $u = \mathbb{1}_\Omega$. Then u does not have a trace by Lemma 3.1 (b). Hence, $u = 0$ and $\varphi = 0 \in L_2(\Gamma_0)$. ☐

We use a measure-theoretic property to characterise uniqueness of the trace.

Proof of Corollary 1.2. If $\sigma(\Gamma_0) > 0$, then by [8], Exercise 264Y (g), the space Γ_0 is atomless, because $d \geqslant 2$ by assumption. So $\dim L_2(\Gamma_0) = \infty$ if $\sigma(\Gamma_0) > 0$. Now Corollary 1.2 follows from Theorem 1.1. ☐

Corollary 3.2. *Suppose that the trace on* Ω *is unique.*
(a) *If* $\sigma(\Gamma) < \infty$, *then* $\dim \ker \mathcal{N} = 1$.
(b) *If* $\sigma(\Gamma) = \infty$, *then* $\mathcal{N}$ *is injective.*

It is well known that there exist unbounded domains in $\mathbb{R}^3$ with finite surface measure. We conclude this section with an example of an unbounded domain with finite volume such that $\sigma(\Gamma_0) = \infty$ whilst $\sigma(\Gamma_r) < \infty$. The following is a modification of [1], Example 4.4, see also Figure 1 in [1].

Example 3.3. For all $(a, b) \in \mathbb{R}^2$ and $r > 0$ denote by

$$C(a, b\,; r) = \bigl\{(x, y, z) \in \mathbb{R}^3 \colon \sqrt{(x-a)^2 + (y-b)^2} \leqslant r \quad \text{and} \quad z \in [0, 1]\bigr\}$$

the closed cylinder centred at $(a, b, 0) \in \mathbb{R}^3$ with radius r and height 1. Let $n \in \mathbb{N}$. Write $B_n = [0, 2^{-n}] \times [2n, 2n+1] \times [-2^{-n}, 0]$ and consider the set

$$\Omega_n = B_n \cup \bigcup_{k \in \mathbb{N}} \bigcup_{i=1}^{k} C\Bigl(2^{-(k+n)}, 2n + \frac{i}{k+1}; 4^{-kn}\Bigr).$$

Then $|\Omega_n| = 2^{-2n} + \sum\limits_{k \in \mathbb{N}} k\pi 4^{-2kn} < \infty$. We may connect B_n to B_{n+1} via a tiny tube T_n such that $\Omega_n \cup T_n \cup \Omega_{n+1}$ is connected, $\sum\limits_{n \in \mathbb{N}} |T_n| < \infty$ and $\sum\limits_{n \in \mathbb{N}} \sigma(\partial T_n) < \infty$. Define

$$\Omega = \operatorname{Int}\Bigl(\bigcup_{n \in \mathbb{N}} \Omega_n \cup T_n\Bigr).$$

Then

$$|\Omega| = \sum_{n \in \mathbb{N}} 2^{-2n} + \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} k\pi 4^{-2kn} + \sum_{n \in \mathbb{N}} |T_n| < \infty.$$

Hence, Ω is a connected open set with finite volume.

We first consider Γ_0. Fix $n \in \mathbb{N}$. We shall show that $\{0\} \times [2n, 2n+1] \times [0,1] \subset \Gamma_0$, up to a set of zero σ-measure. For all $m \in \mathbb{N}$ define the function $u_m \in H^1(\Omega) \cap C_c(\overline{\Omega})$ by $u_m(x,y,z) = (0 \vee 3^m\, z \wedge 1) \cdot \mathbb{1}_{[0, 2^{-m} + 4^{-m}]}(x) \cdot \mathbb{1}_{\overline{\Omega_n}}(x,y,z)$. Then

$$\|u_m\|_{H^1(\Omega)}^2 \leqslant \sum_{\substack{k \in \mathbb{N} \\ k+n \geqslant m}} k\pi 4^{-2kn} + \sum_{\substack{k \in \mathbb{N} \\ k+n \geqslant m}} k\pi 3^{2m} 4^{-2kn} 3^{-m} \leqslant \sum_{\substack{k \in \mathbb{N} \\ k+n \geqslant m}} k\pi 4^{-2k} (1 + 3^m)$$

for all $m \in \mathbb{N}$. It follows that $\lim u_m = 0$ in $H^1(\Omega)$. Since

$$\sigma(\partial\Omega_n) = 4 \cdot 2^{-n} + 2 \cdot 2^{-2n} + \sum_{k \in \mathbb{N}} k(2\pi 4^{-kn} + \pi 4^{-2kn}) < \infty,$$

the dominated convergence theorem provides that $\lim u_m|_\Gamma = \mathbb{1}_{\{0\} \times [2n, 2n+1] \times [0,1]}$ in $L_2(\Gamma)$. Then $\mathbb{1}_{\{0\} \times [2n, 2n+1] \times [0,1]} \in L_2(\Gamma)$ is a trace of 0 and therefore the set $\{0\} \times [2n, 2n+1] \times [0,1] \subset \Gamma_0$, up to a σ-nullset. Hence,

$$\bigcup_{n \in \mathbb{N}} \{0\} \times [2n, 2n+1] \times [0,1] \subset \Gamma_0$$

and $\sigma(\Gamma_0) = \infty$. Moreover,

$$\begin{aligned} \sigma(\Gamma_r) &\leqslant \sum_{n \in \mathbb{N}} \sigma(\Gamma_r \cap \partial\Omega_n) + \sum_{n \in \mathbb{N}} \sigma(\partial T_n) \\ &\leqslant \sum_{n \in \mathbb{N}} (4 \cdot 2^{-n} + 2 \cdot 2^{-2n}) + \pi \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} k\,(2 \cdot 4^{-kn} + 4^{-2kn}) + \sum_{n \in \mathbb{N}} \sigma(\partial T_n) < \infty, \end{aligned}$$

as required.

4. Existence of the trace

We now investigate the situation in which the elements of $\widetilde{H}^1(\Omega)$ each have at least one trace. This section culminates in the proof of Theorem 1.3, which characterises the situation in which every element of $\widetilde{H}^1(\Omega)$ has a unique trace.

We begin by introducing a trace map on $H^1(\Omega)$. Define

$$H^1_\sigma(\Omega) = \{u \in H^1(\Omega) \colon \text{ there exists a } \varphi \in L_2(\Gamma) \text{ such that } \varphi \text{ is a trace of } u\}.$$

Then there exists a unique well-defined map

$$\mathrm{Tr} \colon H^1_\sigma(\Omega) \to L_2(\Gamma_r)$$

such that $\operatorname{Tr} u$ is a trace of $u \in H^1_\sigma(\Omega)$. Clearly $H^1(\Omega) \cap C_{\mathrm{c}}(\overline{\Omega}) \subset H^1_\sigma(\Omega)$ and $\operatorname{Tr} u = u|_{\Gamma_r}$ for all $u \in H^1(\Omega) \cap C_{\mathrm{c}}(\overline{\Omega})$. We identify $L_2(\Gamma_r)$ with the subspace of $L_2(\Gamma)$ consisting of functions that vanish σ-a.e. on Γ_0. It follows from the Maz'ya inequality (2.1) that

$$\int_\Omega |u|^2 \leqslant c_M \biggl(\int_\Omega |\nabla u|^2 + \int_{\Gamma_r} |\operatorname{Tr} u|^2 \biggr) \tag{4.1}$$

for all $u \in H^1_\sigma(\Omega)$. Define the norm

$$\|u\|_{H^1_\sigma(\Omega)} = \biggl(\int_\Omega |\nabla u|^2 + \int_{\Gamma_r} |\operatorname{Tr} u|^2 \biggr)^{1/2}$$

on $H^1_\sigma(\Omega)$. Then Tr is continuous with respect to the norm $\|\cdot\|_{H^1_\sigma(\Omega)}$ on $H^1_\sigma(\Omega)$. It is, however, worth noting that the map $\operatorname{Tr}\colon (H^1_\sigma(\Omega), \|\cdot\|_{H^1(\Omega)}) \to L_2(\Gamma_r)$ is not continuous in general. A counterexample can be found in [5], Remark 3.5 (f).

By (4.1) the norm

$$u \mapsto (\|u\|^2_{H^1(\Omega)} + \|\operatorname{Tr} u\|^2_{L_2(\Gamma_r)})^{1/2}$$

is equivalent to $\|\cdot\|_{H^1_\sigma(\Omega)}$ on $H^1_\sigma(\Omega)$. Moreover, the inclusion $H^1_\sigma(\Omega) \hookrightarrow L_2(\Omega)$ is continuous.

As in [4], Proposition 5.5, the Maz'ya inequality (4.1) implies that Γ_r is not trivial.

Proposition 4.1. *We have* $\sigma(\Gamma_r) > 0$.

Proof. Suppose $\sigma(\Gamma_r) = 0$. Then $\mathbb{1}_{\Gamma_r}$ is a trace of $\mathbb{1}_\Omega$ by Lemma 3.1 (a). Applying (4.1) to $u = \mathbb{1}_\Omega$ provides that $|\Omega| = 0$, which is a contradiction. □

Corollary 4.2. *If the* C_0*-semigroup generated by* $-\mathcal{N}$ *is irreducible, then the trace on* Ω *is unique.*

Proof. Let S be the C_0-semigroup generated by $-\mathcal{N}$. Suppose S is irreducible. If $\varphi \in L_2(\Gamma_0)$, then $\varphi \in \ker \mathcal{N}$ by Theorem 1.1, so $\mathcal{N}\varphi = 0$ and $S_t\varphi = \varphi \in L_2(\Gamma_0)$ for all $t > 0$. Hence, $S_t L_2(\Gamma_0) \subset L_2(\Gamma_0)$ for all $t > 0$. Since S is irreducible, this implies that either $\sigma(\Gamma_0) = 0$ or $\sigma(\Gamma_r) = 0$. But $\sigma(\Gamma_r) > 0$. □

Next we extend the form $\mathfrak{l}$ and map j to the completion of the form domain $D(\mathfrak{l})$, yielding an alternate characterisation of $\mathcal{N}$. First define the map $\Phi\colon (D(\mathfrak{l}), \|\cdot\|_{\mathfrak{l}}) \to H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$ by

$$\Phi(u) = (u, u|_{\Gamma_0}),$$

where $\|u\|_{\mathfrak{l}} = (\mathfrak{l}(u) + \|j(u)\|^2_{L_2(\Gamma)})^{1/2}$ for all $u \in D(\mathfrak{l})$. Then Φ is an isometry with dense range and it follows that the space $H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$ corresponds to the completion of the pre-Hilbert space $(D(\mathfrak{l}), \|\cdot\|_{\mathfrak{l}})$. We identify $D(\mathfrak{l})$ with $\Phi\big(D(\mathfrak{l})\big)$ in the natural manner.

Define the sesquilinear form $\tilde{\mathfrak{l}}\colon (H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)) \times (H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)) \to \mathbb{C}$ by

$$\tilde{\mathfrak{l}}((u,\zeta),(v,\eta)) = \int_\Omega \nabla u \cdot \overline{\nabla v}.$$

Then $\tilde{\mathfrak{l}}$ is continuous and $\tilde{\mathfrak{l}}(\Phi(u),\Phi(v)) = \mathfrak{l}(u,v)$ for all $u, v \in D(\mathfrak{l})$. Moreover, the map $\tilde{\jmath}\colon H^1_\sigma(\Omega) \oplus L_2(\Gamma_0) \to L_2(\Gamma)$ defined by

$$\tilde{\jmath}((u,\zeta)) = \operatorname{Tr} u + \zeta$$

is the continuous extension of j to $H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$ and $\tilde{\mathfrak{l}}$ is $\tilde{\jmath}$-elliptic, that is, there are $\mu, \omega > 0$ such that

$$\mu\|(u,\zeta)\|^2_{H^1_\sigma(\Omega)\oplus L_2(\Gamma_0)} \leqslant \operatorname{Re}\tilde{\mathfrak{l}}(u,\zeta) + \omega\|\tilde{\jmath}((u,\zeta))\|^2_{L_2(\Gamma)}$$

for all $(u,\zeta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$. Actually, one can choose $\mu = \omega = 1$. The next proposition states that the Dirichlet-to-Neumann operator $\mathcal{N}$ is equal to the operator associated with $(\tilde{\mathfrak{l}}, \tilde{\jmath})$.

Proposition 4.3. *Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if and only if there exists a pair $(u,\zeta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$ such that $\tilde{\jmath}((u,\zeta)) = \varphi$ and*

$$\tilde{\mathfrak{l}}((u,\zeta),(v,\eta)) = (\psi, \tilde{\jmath}((v,\eta)))_{L_2(\Gamma)}$$

for all $(v,\eta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$.

Proof. The claim follows from Theorem 2.3 together with [2], Proposition 3.3. □

We denote by $\widehat{\mathcal{N}}$ the part of the operator $\mathcal{N}$ in $L_2(\Gamma_r)$. We next show that $\widehat{\mathcal{N}}$ is equal to the operator in $L_2(\Gamma_r)$ associated with $(\tilde{\mathfrak{l}}|_{H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)}, \operatorname{Tr})$. As a consequence $\widehat{\mathcal{N}}$, is positive and self-adjoint.

Proposition 4.4. *Let $\varphi, \psi \in L_2(\Gamma_r)$. Then $\varphi \in D(\widehat{\mathcal{N}})$ and $\widehat{\mathcal{N}}\varphi = \psi$ if and only if there exists an element $u \in H^1_\sigma(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and*

$$\int_\Omega \nabla u \cdot \overline{\nabla v} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1_\sigma(\Omega)$.

Proof. $\Rightarrow$ Suppose first that $\varphi \in D(\widehat{\mathcal{N}})$ and $\widehat{\mathcal{N}}\varphi = \psi$. Then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$. By Proposition 4.3 there exists a pair $(u,\zeta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$ such that $\tilde{\jmath}((u,\zeta)) = \varphi$ and

$$\tilde{\mathfrak{l}}((u,\zeta),(v,\eta)) = (\psi, \tilde{\jmath}((v,\eta)))_{L_2(\Gamma)}$$

for all $(v,\eta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$. Then $\operatorname{Tr} u = \varphi$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v} = \tilde{\mathfrak{l}}((u,\zeta),(v,0)) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1_\sigma(\Omega)$.

$\Leftarrow$ Suppose that $u \in H^1_\sigma(\Omega)$ is such that $\operatorname{Tr} u = \varphi$ and $\int_\Omega \nabla u \cdot \overline{\nabla v} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in H^1_\sigma(\Omega)$. Then

$$\begin{aligned}\tilde{\mathfrak{l}}((u,0),(v,\eta)) &= \int_\Omega \nabla u \cdot \overline{\nabla v} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)} \\ &= (\psi, \operatorname{Tr} v)_{L_2(\Gamma)} + (\psi, \eta)_{L_2(\Gamma)} = (\psi, \tilde{\jmath}((v,\eta)))_{L_2(\Gamma)}\end{aligned}$$

for all $(v,\eta) \in H^1_\sigma(\Omega) \oplus L_2(\Gamma_0)$. So $\varphi = \tilde{\jmath}((u,0)) \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$, by Proposition 4.3. Since $\varphi, \psi \in L_2(\Gamma_r)$, the claim follows. $\square$

We next characterise when every element of $\widetilde{H}^1(\Omega)$ has a trace.

Theorem 4.5. *The following are equivalent.*

(i) *Every element of $\widetilde{H}^1(\Omega)$ has a trace.*

(ii) *There exists a constant $c > 0$ such that*

$$\int_{\Gamma_r} |u|^2 \leqslant c\Big(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Big) \tag{4.2}$$

for all $u \in H^1(\Omega) \cap C_c(\overline{\Omega})$.

(iii) *There exists a constant $c > 0$ such that*

$$\int_{\Gamma_r} |u|^2 \leqslant c \int_\Omega |\nabla u|^2$$

for all $u \in H^1(\Omega) \cap C_c(\overline{\Omega})$ with $\int_{\Gamma_r} u = 0$.

(iv) $0 \notin \sigma_{\mathrm{ess}}(\widehat{\mathcal{N}})$.

In fact, if $\sigma(\Gamma_r) = \infty$, then each of the statements (i)–(iv) *is false.*

Proof. If $\sigma(\Gamma_r) < \infty$, then the proof is similar to the proof of [1], Theorem 6.1. From now on suppose that $\sigma(\Gamma_r) = \infty$. We shall show that each statement is false.

(i) Let $\chi \in C_{\rm c}^1(\mathbb{R}^d)$ be such that $0 \leqslant \chi \leqslant \mathbb{1}$ and $\chi|_{B_1(0)} = \mathbb{1}$. For all $n \in \mathbb{N}$ define $\chi_n \in C_{\rm c}^1(\mathbb{R}^d)$ by $\chi_n(x) = \chi(n^{-1}x)$. Then $\chi_n|_{\overline{\Omega}} \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ and $\lim\limits_{n\to\infty} \chi_n|_\Omega = \mathbb{1}_\Omega$ in $H^1(\Omega)$. So $\mathbb{1}_\Omega \in \widetilde{H}^1(\Omega)$. But $\mathbb{1}_\Omega$ does not have a trace by Lemma 3.1 (b). So (i) is false.

(ii) Suppose to the contrary that there exists a constant $c > 0$ such that (4.2) is valid for all $u \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$. Let $\chi \in C_{\rm c}^1(\mathbb{R}^d)$ be such that $0 \leqslant \chi \leqslant \mathbb{1}$ and $\chi|_{B_1(0)} = \mathbb{1}$. For each $n \in \mathbb{N}$ define $\chi_n \in C_{\rm c}^1(\mathbb{R}^d)$ by $\chi_n(x) = \chi(n^{-1}x)$. Then $\chi_n|_{\overline{\Omega}} \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ and $|(\nabla\chi_n)(x)| = n^{-1}|(\nabla\chi)(n^{-1}x)| \leqslant \|\nabla\chi\|_\infty/n$ for all $x \in \mathbb{R}^d$. Hence,

$$\int_{\Gamma_r} |\chi_n|^2 \leqslant c\Big(\int_\Omega \frac{\|\nabla\chi\|_\infty^2}{n^2} + \int_\Omega |\chi_n|^2\Big) \leqslant c\Big(\frac{\|\nabla\chi\|_\infty^2}{n^2}|\Omega| + |\Omega|\Big)$$

for all $n \in \mathbb{N}$. Then by Fatou's lemma one deduces that

$$\sigma(\Gamma_r) = \int_{\Gamma_r} \liminf_{n\to\infty} \chi_n^2 \leqslant c|\Omega| < \infty,$$

a contradiction.

(iii) Suppose that (iii) is valid. We shall prove that (ii) is then valid. Again let $\chi \in C_{\rm c}^1(\mathbb{R}^d)$ be such that $0 \leqslant \chi \leqslant \mathbb{1}$ and $\chi|_{B_1(0)} = \mathbb{1}$. For all $n \in \mathbb{N}$ define $\chi_n \in C_{\rm c}^1(\mathbb{R}^d)$ by $\chi_n(x) = \chi(n^{-1}x)$. Then $\chi_n|_{\overline{\Omega}} \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$. By Proposition 4.1 we may assume without loss of generality that $\int_{\Gamma_r} \chi_n \neq 0$. Let $u \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ and write $u_n = u - \alpha_n \chi_n|_{\overline{\Omega}}$, where $\alpha_n = \int_{\Gamma_r} u / \int_{\Gamma_r} \chi_n$ for all $n \in \mathbb{N}$. Then $\lim \alpha_n = 0$ and $\lim \alpha_n \nabla\chi_n = 0$ in $L_2(\Omega)$. Since $u_n \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ and $\int_{\Gamma_r} u_n = 0$, it follows that

$$\int_{\Gamma_r} |u|^2 \leqslant 2\int_{\Gamma_r} |u_n|^2 + 2\int_{\Gamma_r} |\alpha_n\,\chi_n|^2 \leqslant 2c\int_\Omega |\nabla u - \alpha_n\,\nabla\chi_n|^2 + 2\int_{\Gamma_r} |\alpha_n\,\chi_n|^2$$

for all $n \in \mathbb{N}$. Moreover,

$$\int_{\Gamma_r} |\alpha_n\,\chi_n|^2 = |\alpha_n|^2 \int_{\Gamma_r} \chi_n^2 \leqslant |\alpha_n|^2 \int_{\Gamma_r} \chi_n = \overline{\alpha_n} \int_{\Gamma_r} u$$

for all $n \in \mathbb{N}$. Hence, taking the limit $n \to \infty$ yields that

$$\int_{\Gamma_r} |u|^2 \leqslant 2c\int_\Omega |\nabla u|^2.$$

Hence, (ii) is valid. This is a contradiction.

(iv) Suppose that (iv) is valid. We shall prove that (ii) is valid. Since $\ker \widehat{\mathcal{N}} = \{0\}$ by Theorem 1.1 (b) and $0 \notin \sigma_{\rm ess}(\widehat{\mathcal{N}})$ by assumption, there exists a positive $\delta > 0$ such that $(\widehat{\mathcal{N}}\varphi, \varphi)_{L_2(\Gamma_r)} \geqslant \delta \|\varphi\|^2_{L_2(\Gamma_r)}$ for all $\varphi \in D(\widehat{\mathcal{N}})$. We use Proposition 4.4. Write $\hat{\mathfrak{l}} = \mathfrak{l}|_{H^1_\sigma(\Omega)\times H^1_\sigma(\Omega)}$. Let $u \in H^1_\sigma(\Omega)$. Suppose $\operatorname{Tr} u \in D(\widehat{\mathcal{N}})$ and $\hat{\mathfrak{l}}(u,v) = (\widehat{\mathcal{N}} \operatorname{Tr} u, \operatorname{Tr} v)_{L_2(\Gamma_r)}$ for all $v \in H^1_\sigma(\Omega)$. Then $\delta \|\operatorname{Tr} u\|^2_{L_2(\Gamma_r)} \leqslant (\widehat{\mathcal{N}} \operatorname{Tr} u, \operatorname{Tr} u)_{L_2(\Gamma_r)} = \hat{\mathfrak{l}}(u,u)$. Hence, by density, [2], Proposition 2.3 (ii), one deduces that $\delta \|\operatorname{Tr} u\|^2_{L_2(\Gamma_r)} \leqslant \hat{\mathfrak{l}}(u,u)$ for all $u \in V(\hat{\mathfrak{l}}) = \{u \in H^1_\sigma(\Omega) \colon \hat{\mathfrak{l}}(u,v) = 0 \text{ for all } v \in \ker \operatorname{Tr}\}$. Finally, let $u \in H^1_\sigma(\Omega)$. By [2], Theorem 2.5, there are $u_1 \in \ker \operatorname{Tr}$ and $u_2 \in V(\hat{\mathfrak{l}})$ such that $u = u_1 + u_2$. Then symmetry of $\hat{\mathfrak{l}}$ gives

$$\begin{aligned}\delta \|\operatorname{Tr} u\|^2_{L_2(\Gamma_r)} &= \delta \|\operatorname{Tr} u_2\|^2_{L_2(\Gamma_r)} \leqslant \hat{\mathfrak{l}}(u_2,u_2)\\ &= \hat{\mathfrak{l}}(u,u) - 2\operatorname{Re}\hat{\mathfrak{l}}(u_2,u_1) - \hat{\mathfrak{l}}(u_1,u_1) \leqslant \hat{\mathfrak{l}}(u,u)\end{aligned}$$

and (ii) is valid. This is a contradiction. □

Finally, we characterise the case in which every element of $\widetilde{H}^1(\Omega)$ has a unique trace.

Proof of Theorem 1.3. (i)⇒(iv). Suppose that every element of $\widetilde{H}^1(\Omega)$ has a unique trace. Then $0 \notin \sigma_{\rm ess}(\widehat{\mathcal{N}})$ by Theorem 4.5 (i) ⇒ (iv). Moreover, $\sigma(\Gamma_0) = 0$, so $\mathcal{N} = \widehat{\mathcal{N}}$ and it follows that (iv) is valid.

(iv) ⇒ (iii). Since $0 \notin \sigma_{\rm ess}(\mathcal{N})$, one has that $\dim \ker \mathcal{N} < \infty$. So $\sigma(\Gamma_0) = 0$ by Corollary 1.2. Hence, $\mathcal{N} = \widehat{\mathcal{N}}$ and $\Gamma = \Gamma_r$. Then the claim follows from Theorem 4.5 (iv) ⇒ (iii).

(iii) ⇒ (ii). If $\sigma(\Gamma) = \infty$, then the implication follows from an argument similar to the proof that condition (iii) in Theorem 4.5 is false in case $\sigma(\Gamma_r) = \infty$, by replacing Γ_r with Γ. Hence, we may assume that $\sigma(\Gamma) < \infty$. We shall show that $L_2(\Gamma_0) = \{0\}$. Then $\Gamma = \Gamma_r$ and the claim follows from Theorem 4.5 (iii) ⇒ (ii).

Let $c > 0$ be as in (iii) and let $\varphi \in L_2(\Gamma_0)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ such that $\lim u_n = 0$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. Moreover, by Lemma 3.1 (a) one has that $\mathbb{1}_{\Gamma_r}$ is a trace of $\mathbb{1}_\Omega$, so there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ such that $\lim v_n = \mathbb{1}_\Omega$ in $H^1(\Omega)$ and $\lim v_n|_\Gamma = \mathbb{1}_{\Gamma_r}$ in $L_2(\Gamma)$. By Proposition 4.1 we may assume without loss of generality that $\int_\Gamma v_n \neq 0$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ write $\alpha_n = \int_\Gamma u_n / \int_\Gamma v_n$ and $w_n = u_n - \alpha_n v_n$. Then $w_n \in H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ and $\int_\Gamma w_n = 0$. Therefore,

$$\int_\Gamma |w_n|^2 \leqslant c \int_\Omega |\nabla w_n|^2 \tag{4.3}$$

for all $n \in \mathbb{N}$. Moreover, $\lim \alpha_n = \alpha$, where $\alpha = \int_\Gamma u/\sigma(\Gamma_r)$. So $\lim w_n = -\alpha \mathbb{1}_\Omega$ in $H^1(\Omega)$ and $\lim w_n|_\Gamma = \varphi - \alpha \mathbb{1}_{\Gamma_r}$ in $L_2(\Gamma)$. Taking the limit $n \to \infty$ in (4.3) then yields that

$$\int_{\Gamma_0} |\varphi - \alpha \mathbb{1}_{\Gamma_r}|^2 \leqslant c \cdot 0 = 0.$$

Hence, $\varphi - \alpha \mathbb{1}_{\Gamma_r} = 0$ and $\varphi = 0$. Therefore, $L_2(\Gamma_0) = \{0\}$.

(ii) $\Rightarrow$ (i). We first show that the trace on Ω is unique. Let $\varphi \in L_2(\Gamma_0)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C_{\rm c}(\overline{\Omega})$ such that $\lim u_n = 0$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. By assumption there exists a constant $c > 0$ such that

$$\int_\Gamma |u_n|^2 \leqslant c\Big(\int_\Omega |\nabla u_n|^2 + \int_\Omega |u_n|^2\Big)$$

for all $n \in \mathbb{N}$. Then $\varphi = \lim u_n|_\Gamma = 0$ and the trace on Ω is unique. Hence, $\Gamma = \Gamma_r$ and it follows from Theorem 4.5 (ii) $\Rightarrow$ (i) that every element of $\widetilde{H}^1(\Omega)$ has a trace. □

Corollary 4.6. *If $\sigma(\Gamma) = \infty$ and $\mathcal{N}$ has compact resolvent, then $\mathcal{N}$ is bijective.*

Proof. Since $(I + \mathcal{N})^{-1}$ is compact, it follows that $0 \notin \sigma_{\rm ess}(\mathcal{N})$. So by Theorem 1.3 (iv)$\Rightarrow$(i), the trace on Ω is unique. Hence, $\Gamma = \Gamma_r$ and Theorem 1.1 (b) provides that $\mathcal{N}$ is injective. Next, the Fredholm operator $I - (I + \mathcal{N})^{-1}$ has closed range. Since

$$\mathcal{N}(I + \mathcal{N})^{-1} = (I + \mathcal{N} - I)(I + \mathcal{N})^{-1} = I - (I + \mathcal{N})^{-1},$$

one deduces that $R(\mathcal{N}) = R(\mathcal{N}(I + \mathcal{N})^{-1})$ is closed. Then $R(\mathcal{N}) = \overline{R(\mathcal{N})} = (\ker \mathcal{N})^\perp = L_2(\Gamma)$. Therefore, $\mathcal{N}$ is surjective. □

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