Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 559–577

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4509

NEW GENERALIZED FOURIER TRANSFORMS AND THEIR APPLICATIONS TO ORDINARY, PARTIAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

ENES ATA AND İ. ONUR KIYMAZ

Received 02 December, 2022

Abstract. This article presents new generalized definitions of Fourier, Fourier sine, Fourier cosine, inverse Fourier, inverse Fourier sine and inverse Fourier cosine transforms, which encompass various studies on the generalized Fourier transforms in the existing literature. We also give some fundamental properties such that linearity, shifting, differentiability and convolution. Moreover, the solutions to the ordinary electric current differential equation and the fractional motion differential equation are obtained through the use of the generalized Fourier and inverse Fourier transforms. Subsequently, the solutions to the partial diffusion differential equation are obtained through the use of the generalized Fourier sine, inverse Fourier sine, Fourier cosine, and inverse Fourier cosine transforms. Furthermore, we illustrate the relations of the new generalized Fourier transforms with other the generalized Fourier transforms available in the literature. Finally, we provide tables of the new generalized Fourier transforms, and then graphs of the approximate behaviours of the solution of the ordinary electric current differential equation.

2010 *Mathematics Subject Classification:* 42A38; 26A33; 34A12; 35E05; 34A08

Keywords: Fourier transform, fractional derivatives and integrals, ordinary differential equations, partial differential equations, fractional differential equations

1. INTRODUCTION

The Fourier transform is an integral transformation that is popular in many fields such as medicine, astronomy, physics, chemistry, biology and engineering sciences. In the most general sense, the Fourier and inverse Fourier transforms in [12] for the function u belonging to the Schwartz space, respectively, are defined by

$$\mathfrak{F}[u(t)](w)=\hat{u}(w)=\int_{\mathbb{R}}\exp(iwt)u(t)\mathrm{d}t,\quad (w\in\mathbb{R})$$

and

$$\mathfrak{F}^{-1}[\hat{u}(w)](t)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp(-iwt)\hat{u}(w)\mathrm{d}w,\quad (t\in\mathbb{R}).$$

The Fourier transform has become a powerful technique due to its multi-purpose use in various branches of science. For example, each person's voice can be expressed as a sum of sine and cosine. Since the frequency spectrum of each sound is different, the sum of the sines and cosines will have a different frequency. Thus, with the Fourier transform, we can find out who the sound belongs to. Similarly, the Fourier transform can be used in phenomena such as light waves, tides in the ocean, solar motion, electric current and thermal conductivity. Again, the Fourier transform can also be used for solutions of ordinary, partial and fractional differential equations.

For studies on the generalization of the Fourier transform in the literature (see for example [4, 5, 8–12, 14] and reference therein).

The objective and importance of this article is to introduce new generalized Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms, covering the existing literature on various generalized Fourier transform studies. Subsequently, solutions of miscellaneous ordinary, partial and fractional differential equations are obtained with the help of the newly defined generalized Fourier transforms, thereby extending the application areas of integral transforms.

We set the rest of this article as follows: Section 2 presents the fundamental concepts that are essential to the article. In Section 3, the new generalized Fourier, Fourier sine, and Fourier cosine transforms with and their inverse transforms are defined, and subsequently some of their properties are presented. In Section 4, solutions to the ordinary electric current differential equation, fractional motion differential equation, and partial diffusion differential equation are obtained through the use of the new generalized Fourier transforms. Section 5 elucidates the relationships between the generalized Fourier transforms that have been previously identified in the literature and the new generalized Fourier transforms that are introduced in this article. Additionally, at the end of the article, tables of the new generalized Fourier transforms and approximate graphs of the solutions to the ordinary electric current differential equation are presented.

2. PRELIMINARIES

This section presents the fundamental concepts that is required throughout the article. This includes the following: the Schwartz space, Lizorkin space, convolution, Dirac delta function and fractional integral and derivative operators.

The Schwartz space [15] is defined by

$$S(\mathbb{R}) = \left\{ u \in C^{\infty} : \sup \left| t^m \frac{d^n}{dt^n} u(t) \right| < \infty, \quad \forall m, n \in \mathbb{N}_0 \right\}.$$

The $V(\mathbb{R})$ set be as follows:

$$V(\mathbb{R}) = \left\{ v \in S(\mathbb{R}) : v^{(n)}(0) = 0, \quad n \in \mathbb{N}_0 \right\}.$$

Then the Lizorkin space [13] is given by

$$\phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) \colon \mathfrak{F}[\varphi] \in V(\mathbb{R})\}.$$

The convolution [2] of functions u and v for $t \in \mathbb{R}$ is defined by

$$u(t) * v(t) = \int_{\mathbb{R}} u(t-\tau)v(\tau)\mathrm{d}\tau$$

or

$$v(t) * u(t) = \int_{\mathbb{R}} v(t-\tau)u(\tau)\mathrm{d}\tau.$$

The Dirac delta function [3] is given by

$$\delta(t) = \begin{cases} 0, & \text{for } t \neq 0, \\ \infty, & \text{for } t = 0. \end{cases}$$

Also the following equation [3] holds true:

$$\int_{\mathbb{R}} u(t)\delta(t)\mathrm{d}t = u(0). \tag{2.1}$$

The Riemann-Liouville fractional integral (RLFI) [7] of order β is defined by

$${}_{-\infty}I_t^{\beta}u(t) = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-\tau)^{\beta-1}u(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

Let $m-1 < \Re(\beta) < m$ for $m \in \mathbb{N}$. The Riemann-Liouville fractional derivative (RLFD) [7] of order β is given by

$${}_{-\infty}D_t^{\beta}u(t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_{-\infty}^{t}(t-\tau)^{m-\beta-1}u(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

Let $m-1 < \Re(\beta) < m$ for $m \in \mathbb{N}$. The Caputo fractional derivative (CFD) [7] of order β is defined by

$${}_{-\infty}^{\ c}D_t^{\beta}u(t) = \frac{1}{\Gamma(m-\beta)}\int_{-\infty}^{t}(t-\tau)^{m-\beta-1}u^{(m)}(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

3. NEW GENERALIZED FOURIER TRANSFORMS AND THEIR PROPERTIES

This section introduces new generalisations of the Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms, and presents some of their fundamental properties.

Definition 1. Let $u \in \phi(\mathbb{R})$, $p \colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q \colon \mathbb{R} \to \mathbb{R}^+$ and $w, t \in \mathbb{R}$. Then, the generalized Fourier and inverse Fourier transforms, respectively, are defined as:

$${}_p\mathfrak{F}_q[u(t)](w) = {}_p\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right)u(t)\mathrm{d}t \tag{3.1}$$

and

$$ {}_p\mathfrak{F}_q^{-1}\left[{}_p\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(-iw^{q(\alpha)}t\right) {}_p\hat{u}_q(w) w^{q(\alpha)-1}\mathrm{d}w, \tag{3.2} $$

where

$$ \exp\left(iw^{q(\alpha)}t\right) := \begin{cases} \exp\left(-i|w|^{q(\alpha)}t\right), & \text{for } w \le 0, \\ \exp\left(i|w|^{q(\alpha)}t\right), & \text{for } w \ge 0. \end{cases} $$

Definition 2. Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then, the generalized Fourier sine and inverse Fourier sine transforms, respectively, are defined as:

$$ {}_p^s\mathfrak{F}_q[u(t)](w) = {}_p^s\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t \tag{3.3} $$

and

$$ {}_p^s\mathfrak{F}_q^{-1}\left[{}_p^s\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) {}_p^s\hat{u}_q(w) w^{q(\alpha)-1}\mathrm{d}w. \tag{3.4} $$

Definition 3. Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then, the generalized Fourier cosine and inverse Fourier cosine transforms, respectively, are defined as:

$$ {}_p^c\mathfrak{F}_q[u(t)](w) = {}_p^c\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t \tag{3.5} $$

and

$$ {}_p^c\mathfrak{F}_q^{-1}\left[{}_p^c\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) {}_p^c\hat{u}_q(w) w^{q(\alpha)-1}\mathrm{d}w. \tag{3.6} $$

Remark 1. If we take $p(\alpha) = q(\alpha) = 1$ in the equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6), we obtain classic Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms in [3], respectively.

It should also be noted that throughout this article the generalized Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms will be denoted by the symbols ${}_p\mathfrak{F}_q$, ${}_p\mathfrak{F}_q^{-1}$, ${}_p^s\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q^{-1}$, ${}_p^c\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q^{-1}$ respectively.

Theorem 1. *Let $u,v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and λ_1,λ_2, $w,t \in \mathbb{R}$. Then,*

$$ {}_p\mathfrak{F}_q\left[\lambda_1 u(t) + \lambda_2 v(t)\right](w) = \lambda_1\, {}_p\mathfrak{F}_q[u(t)](w) + \lambda_2\, {}_p\mathfrak{F}_q[v(t)](w). $$

Proof. Using the ${}_p\mathfrak{F}_q$ transform, we have

$$ {}_p\mathfrak{F}_q\left[\lambda_1 u(t) + \lambda_2 v(t)\right](w) = \lambda_1\left(p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) u(t)\mathrm{d}t\right) $$

$$+\lambda_2\left(p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)v(t)\mathrm{d}t\right)$$
$$=\lambda_1\ {}_p\mathfrak{F}_q[u(t)](w)+\lambda_2\ {}_p\mathfrak{F}_q[v(t)](w). \qquad \square$$

Theorem 2. *Let $u\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}$. Then,*

$${}_p\mathfrak{F}_q[u(t-x)](w)=\exp\left(iw^{q(\alpha)}x\right){}_p\mathfrak{F}_q[u(t)](w).$$

Proof. Using the ${}_p\mathfrak{F}_q$ transform, we have

$$\begin{aligned}{}_p\mathfrak{F}_q[u(t-x)](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u(t-x)\mathrm{d}t,\quad (t-x=\tau)\\&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}(x+\tau)\right)u(\tau)\mathrm{d}\tau\\&=\exp\left(iw^{q(\alpha)}x\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}\tau\right)u(\tau)\mathrm{d}\tau\\&=\exp\left(iw^{q(\alpha)}x\right){}_p\mathfrak{F}_q[u(t)](w).\end{aligned} \qquad \square$$

Theorem 3. *Let $u\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}$. Then,*

$${}_p\mathfrak{F}_q\left[u^{(n)}(t)\right](w)=\left(-iw^{q(\alpha)}\right)^n{}_p\mathfrak{F}_q[u(t)](w). \tag{3.7}$$

Proof. Let us prove the result by contradiction. For $n=1$, we have

$$\begin{aligned}{}_p\mathfrak{F}_q\left[u'(t)\right](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u'(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right){}_p\mathfrak{F}_q[u(t)](w).\end{aligned}$$

For $n=k$, let the equation be true:

$${}_p\mathfrak{F}_q\left[u^{(k)}(t)\right](w)=\left(-iw^{q(\alpha)}\right)^k{}_p\mathfrak{F}_q[u(t)](w). \tag{3.8}$$

For $n=k+1$, considering the equation (3.8), we obtain

$$\begin{aligned}{}_p\mathfrak{F}_q\left[u^{(k+1)}(t)\right](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u^{(k+1)}(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u^{(k)}(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)\left(-iw^{q(\alpha)}\right)^k{}_p\mathfrak{F}_q[u(t)](w)\\&=\left(-iw^{q(\alpha)}\right)^{k+1}{}_p\mathfrak{F}_q[u(t)](w).\end{aligned} \qquad \square$$

Theorem 4. *Let $u, v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w, t \in \mathbb{R}$. Then,*

$$ {}_p\mathfrak{F}_q[u(t) * v(t)](w) = \frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}. \tag{3.9}$$

Proof. Using the convolution and ${}_p\mathfrak{F}_q$ transform, we have

$$ \begin{aligned} {}_p\mathfrak{F}_q[u(t) * v(t)](w) &= p(\alpha) \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) \left(u(t) * v(t)\right) \mathrm{d}t \\ &= p(\alpha) \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) \int_{\mathbb{R}} u(t-x)v(x)\mathrm{d}x\mathrm{d}t. \end{aligned} $$

Considering Fubini's theorem and taking $\tau = t - x$, we get

$$ {}_p\mathfrak{F}_q[u(t) * v(t)](w) = p(\alpha) \int_{\mathbb{R}} v(x) \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}(x+\tau)\right) u(\tau)\mathrm{d}\tau\mathrm{d}x. \tag{3.10}$$

Multiplying and dividing the right side of (3.10) by function $p(\alpha)$, we obtain

$$ \begin{aligned} {}_p\mathfrak{F}_q[u(t) * v(t)](w) &= p(\alpha) \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}x\right) v(x)\mathrm{d}x\, \frac{p(\alpha)}{p(\alpha)} \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}\tau\right) u(\tau)\mathrm{d}\tau \\ &= \frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}. \end{aligned} $$

□

Theorem 5. *Let $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w, t \in \mathbb{R}$. Then,*

$$ {}_p\mathfrak{F}_q\left[\delta(t)\right](w) = p(\alpha). \tag{3.11}$$

Proof. Considering equation (2.1) in the ${}_p\mathfrak{F}_q$ transform, we have

$$ \begin{aligned} {}_p\mathfrak{F}_q[\delta(t)](w) &= p(\alpha) \int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) \delta(t)\mathrm{d}t \\ &= p(\alpha). \end{aligned} $$

□

Theorem 6. *Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w, t \in \mathbb{R}_0^+$. Then,*

$$ \begin{aligned} &{}_p^s\mathfrak{F}_q[u'(t)](w) = -w^{q(\alpha)}\ {}_p^c\mathfrak{F}_q[u(t)](w), \\ &{}_p^s\mathfrak{F}_q[u''(t)](w) = -w^{2q(\alpha)}\ {}_p^s\mathfrak{F}_q[u(t)](w) + w^{q(\alpha)}p(\alpha)u(0). \end{aligned} \tag{3.12}$$

Proof. Using the function $u'(t)$ in the ${}_p^s\mathfrak{F}_q$ transform, we have

$$ \begin{aligned} {}_p^s\mathfrak{F}_q[u'(t)](w) &= p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) u'(t)\mathrm{d}t \\ &= -w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t \\ &= -w^{q(\alpha)}\ {}_p^c\mathfrak{F}_q[u(t)](w). \end{aligned} $$

Using the function $u''(t)$ in the ${}_p^s\mathfrak{F}_q$ transform, we get

$$ {}_p^s\mathfrak{F}_q[u''(t)](w) = p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) u''(t)\mathrm{d}t $$

$$= -w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t$$
$$= -w^{2q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t + w^{q(\alpha)} p(\alpha) u(0)$$
$$= -w^{2q(\alpha)}\ {}_p^s\mathfrak{F}_q[u(t)](w) + w^{q(\alpha)} p(\alpha) u(0). \qquad \square$$

Theorem 7. *Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then,*

$$\begin{aligned} {}_p^c\mathfrak{F}_q[u'(t)](w) &= w^{q(\alpha)}\ {}_p^s\mathfrak{F}_q[u(t)](w) - p(\alpha)u(0), \\ {}_p^c\mathfrak{F}_q[u''(t)](w) &= -w^{2q(\alpha)}\ {}_p^c\mathfrak{F}_q[u(t)](w) - p(\alpha)u'(0). \end{aligned} \tag{3.13}$$

Proof. Using the function $u'(t)$ in the ${}_p^c\mathfrak{F}_q$ transform, we have

$$\begin{aligned} {}_p^c\mathfrak{F}_q[u'(t)](w) &= p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t \\ &= w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t - p(\alpha)u(0) \\ &= w^{q(\alpha)}\ {}_p^s\mathfrak{F}_q[u(t)](w) - p(\alpha)u(0). \end{aligned}$$

Using the function $u''(t)$ in the ${}_p^c\mathfrak{F}_q$ transform, we get

$$\begin{aligned} {}_p^c\mathfrak{F}_q[u''(t)](w) &= p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u''(t) \mathrm{d}t \\ &= w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t - p(\alpha)u'(0) \\ &= -w^{2q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t - p(\alpha)u'(0) \\ &= -w^{2q(\alpha)}\ {}_p^c\mathfrak{F}_q[u(t)](w) - p(\alpha)u'(0). \qquad \square \end{aligned}$$

Theorem 8. *Let $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then,*

$${}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) = \frac{1}{p(\alpha)} \frac{1}{\sqrt{a\pi}} \exp\left(\frac{-t^2}{4a}\right). \tag{3.14}$$

Proof. Using the function $\exp\left(-aw^{2q(\alpha)}\right)$ in the ${}_p^c\mathfrak{F}_q^{-1}$ transform, we have

$${}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) = \frac{q(\alpha)}{p(\alpha)} \frac{2}{\pi} \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) w^{q(\alpha)-1} \exp\left(-aw^{2q(\alpha)}\right) \mathrm{d}w.$$

Using the formula $\cos\left(w^{q(\alpha)} t\right) = \frac{1}{2}\left[\exp\left(iw^{q(\alpha)} t\right) + \exp\left(-iw^{q(\alpha)} t\right)\right]$, we get

$$\begin{aligned} &{}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) \\ &\quad = \frac{q(\alpha)}{p(\alpha)} \frac{1}{\pi} \int_{\mathbb{R}_0^+} \exp\left(-aw^{2q(\alpha)}\right) \left(\exp\left(iw^{q(\alpha)} t\right) + \exp\left(-iw^{q(\alpha)} t\right)\right) w^{q(\alpha)-1} \mathrm{d}w. \end{aligned}$$

By making the necessary arrangements, we obtain

$$
\begin{aligned}
{}_{p}^{c}\mathfrak{F}_{q}^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) &= \frac{q(\alpha)}{p(\alpha)}\frac{1}{\pi}\exp\left(-\frac{t^{2}}{4a}\right) \\
&\quad\times\left[\int_{\mathbb{R}_0^+}\exp\left(-a\left(w^{q(\alpha)}-\frac{it}{2a}\right)^{2}\right)w^{q(\alpha)-1}\mathrm{d}w\right. \\
&\quad\left.+\int_{\mathbb{R}_0^+}\exp\left(-a\left(w^{q(\alpha)}+\frac{it}{2a}\right)^{2}\right)w^{q(\alpha)-1}\mathrm{d}w\right].
\end{aligned}
$$

Taking $y=\left(w^{q(\alpha)}-\frac{it}{2a}\right)$ and $z=\left(w^{q(\alpha)}+\frac{it}{2a}\right)$, we have

$$
\begin{aligned}
&{}_{p}^{c}\mathfrak{F}_{q}^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) \\
&\quad=\frac{1}{p(\alpha)}\frac{1}{\pi}\exp\left(-\frac{t^{2}}{4a}\right)\left[\int_{\mathbb{R}_0^+}\exp\left(-ay^{2}\right)dy+\int_{\mathbb{R}_0^+}\exp\left(-az^{2}\right)\mathrm{d}z\right].
\end{aligned}
$$

Considering the formula $\int_{\mathbb{R}}\exp\left(-ax^{2}\right)dx=\sqrt{\frac{\pi}{a}}$, we get

$$
{}_{p}^{c}\mathfrak{F}_{q}^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{1}{\sqrt{a\pi}}\exp\left(\frac{-t^{2}}{4a}\right). \qquad \square
$$

Theorem 9. *Let $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^{+}$ and $w,t\in\mathbb{R}_0^{+}$. Then,*

$$
{}_{p}^{s}\mathfrak{F}_{q}^{-1}\left[w^{q(\alpha)}\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{t}{2\sqrt{\pi}}\frac{1}{a^{3/2}}\exp\left(\frac{-t^{2}}{4a}\right). \tag{3.15}
$$

Proof. Using the function $\exp\left(-aw^{2q(\alpha)}\right)$ in the ${}_{p}^{c}\mathfrak{F}_{q}^{-1}$ transform, we have

$$
{}_{p}^{c}\mathfrak{F}_{q}^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)w^{q(\alpha)-1}\exp\left(-aw^{2q(\alpha)}\right)\mathrm{d}w.
$$

Considering equation (3.14), we get

$$
\frac{1}{p(\alpha)}\frac{1}{\sqrt{a\pi}}\exp\left(\frac{-t^{2}}{4a}\right)=\frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)w^{q(\alpha)-1}\exp\left(-aw^{2q(\alpha)}\right)\mathrm{d}w.
$$

Differentiating both sides of the last equation according to the parameter t, we obtain

$$
{}_{p}^{s}\mathfrak{F}_{q}^{-1}\left[w^{q(\alpha)}\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{t}{2\sqrt{\pi}}\frac{1}{a^{3/2}}\exp\left(\frac{-t^{2}}{4a}\right). \qquad \square
$$

Theorem 10. *Let $v\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^{+}$ and $w,t\in\mathbb{R}$. Then,*

$$
{}_{p}\mathfrak{F}_{q}\left[{}_{-\infty}I_{t}^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{-\beta}{}_{p}\hat{v}_{q}(w),\quad\left(0<\Re(\beta)<1\right). \tag{3.16}
$$

Proof. There is a relationship between the convolution of the functions $u_+(t)$ and $v(t)$ and the RLFI as follows:

$$ {}_{-\infty}I_t^{\beta} v(t) = u_+(t) * v(t). \tag{3.17} $$

Here the function $u_+(t)$ is as follows:

$$ u_+(t) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases} $$

Applying the ${}_p\mathfrak{F}_q$ transform to the equation (3.17) and considering the equation (3.9), we have

$$ \begin{aligned} {}_p\mathfrak{F}_q\left[{}_{-\infty}I_t^{\beta} v(t)\right](w) &= {}_p\mathfrak{F}_q\left[u_+(t) * v(t)\right](w) \\ &= \frac{{}_p\mathfrak{F}_q\left[u_+(t)\right](w)\; {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)} \\ &= \left(-iw^{q(\alpha)}\right)^{-\beta} {}_p\hat{v}_q(w). \end{aligned} $$

□

Theorem 11. *Let $v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}$, $m \in \mathbb{N}$, $m-1 < \Re(\beta) < m$. Then,*

$$ {}_p\mathfrak{F}_q\left[{}_{-\infty}D_t^{\beta} v(t)\right](w) = \left(-iw^{q(\alpha)}\right)^{\beta} {}_p\hat{v}_q(w), \quad (\Re(\beta) > 0). $$

Proof. The relationship between the RLFD and the RLFI is as follows:

$$ {}_{-\infty}D_t^{\beta} v(t) = \frac{d^n}{dt^n}\left({}_{-\infty}I_t^{(n-\beta)} v(t)\right). $$

By choosing ${}_{-\infty}I_t^{(n-\beta)} v(t) = g(t)$, we have

$$ {}_{-\infty}D_t^{\beta} v(t) = g^{(n)}(t), $$

and then applying the ${}_p\mathfrak{F}_q$ transform and considering equation (3.7), we get

$$ \begin{aligned} {}_p\mathfrak{F}_q\left[{}_{-\infty}D_t^{\beta} v(t)\right](w) &= {}_p\mathfrak{F}_q\left[g^{(n)}(t)\right](w) \\ &= \left(-iw^{q(\alpha)}\right)^{n} {}_p\hat{g}_q(w). \end{aligned} \tag{3.18} $$

Applying the ${}_p\mathfrak{F}_q$ transform to the equation $g(t) = {}_{-\infty}I_t^{(n-\beta)} v(t)$, we obtain

$$ {}_p\mathfrak{F}_q\left[g(t)\right](w) = {}_p\mathfrak{F}_q\left[{}_{-\infty}I_t^{(n-\beta)} v(t)\right](w), $$

and then considering equation (3.16), we have

$$ {}_p\hat{g}_q(w) = \left(-iw^{q(\alpha)}\right)^{-(n-\beta)} {}_p\hat{v}_q(w). \tag{3.19} $$

Using equation (3.19) in equation (3.18), we get

$$ {}_p\mathfrak{F}_q\left[{}_{-\infty}D_t^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w). \qquad \square $$

Theorem 12. *Let $v\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^{+}$ and $w,t\in\mathbb{R}$, $m\in\mathbb{N}$, $m-1<\Re(\beta)<m$. Then,*

$$ {}_p\mathfrak{F}_q\left[{}_{-\infty}^{\ \ c}D_t^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w), \quad (\Re(\beta)>0). $$

Proof. The relationship between the CFD and the RLFI is as follows:

$$ {}_{-\infty}^{\ \ c}D_t^{\beta}v(t)={}_{-\infty}I_t^{(n-\beta)}v^{(n)}(t). $$

By choosing $v^{(n)}(t)=g(t)$, we have

$$ {}_{-\infty}^{\ \ c}D_t^{\beta}v(t)={}_{-\infty}I_t^{(n-\beta)}g(t). $$

Applying the ${}_p\mathfrak{F}_q$ transform and considering equations (3.7) and (3.16), we get

$$ \begin{aligned} {}_p\mathfrak{F}_q\left[{}_{-\infty}^{\ \ c}D_t^{\beta}v(t)\right](w) &= \left(-iw^{q(\alpha)}\right)^{-(n-\beta)}{}_p\hat{g}_q(w) \\ &= \left(-iw^{q(\alpha)}\right)^{-(n-\beta)}\left(-iw^{q(\alpha)}\right)^{n}{}_p\hat{v}_q(w) \\ &= \left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w). \qquad \square \end{aligned} $$

Corollary 1. *The ${}_p\mathfrak{F}_q$ transform of the RLFD and CFD operators overlap.*

4. Applications

In this section, solutions to ordinary electric current differential equation, partial diffusion differential equation and fractional motion differential equation are obtained by means of the ${}_p\mathfrak{F}_q$, ${}_p\mathfrak{F}_q^{-1}$, ${}_p^s\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q^{-1}$, ${}_p^c\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q^{-1}$ transforms.

Problem 1. *Let the ordinary electric current differential equation be given as:*

$$ L\frac{\mathrm{d}I(t)}{\mathrm{d}t}+RI(t)=E(t), $$

where L is inductance, I is current, R is resistance and E is applied electromagnetic force.

Taking $E(t)=\delta(t)$ and applying the ${}_p\mathfrak{F}_q$ transform to the ordinary electric current differential equation and considering the equation (3.11), we have

$$ \begin{aligned} L\,{}_p\mathfrak{F}_q\left[\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right](w)+R\,{}_p\mathfrak{F}_q\left[I(t)\right](w) &= {}_p\mathfrak{F}_q\left[\delta(t)\right](w) \\ -iLw^{q(\alpha)}{}_p\mathfrak{F}_q\left[I(t)\right](w)+R\,{}_p\mathfrak{F}_q\left[I(t)\right](w) &= p(\alpha). \end{aligned} $$

That is,

$$ {}_{p}\mathfrak{F}_{q}\left[I(t)\right](w) = \frac{p(\alpha)}{R - iLw^{q(\alpha)}}. \tag{4.1} $$

Applying the ${}_{p}\mathfrak{F}_{q}^{-1}$ transform to equation (4.1), we get

$$ I(t) = \frac{q(\alpha)}{L}\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\exp\left(-iw^{q(\alpha)}t\right) w^{q(\alpha)-1}}{\left(\frac{R}{iL} - w^{q(\alpha)}\right)} dw. \tag{4.2} $$

Since $w = \left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}$ pole point in equation (4.2), using the Cauchy's residue theorem [16], we obtain

$$ \begin{aligned} I(t) &= \frac{q(\alpha)}{L}\frac{2\pi i}{2\pi i}\operatorname{Res}\left(w = \left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}\right) \\ &= \frac{q(\alpha)}{L}\lim_{w \to \left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}} \frac{\left(\frac{R}{iL} - w^{q(\alpha)}\right)\exp\left(-iw^{q(\alpha)}t\right) w^{q(\alpha)-1}}{\left(\frac{R}{iL} - w^{q(\alpha)}\right)} \\ &= \frac{q(\alpha)}{L}\exp\left(\frac{-Rt}{L}\right)\left(\frac{R}{iL}\right)^{\frac{q(\alpha)-1}{q(\alpha)}}. \end{aligned} $$

Problem 2. *Let the fractional motion differential equation be given as:*

$$ y''(t) + \lambda_1\left({}_{-\infty}D_t^{\beta}y(t)\right) + \lambda_2 y(t) = f(t), $$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $0 < \Re(\beta) < 1$.

Applying the ${}_{p}\mathfrak{F}_{q}$ transform to the fractional motion differential equation, we have

$$ \begin{aligned} {}_{p}\mathfrak{F}_{q}\left[f(t)\right](w) &= {}_{p}\mathfrak{F}_{q}\left[y''(t)\right](w) + \lambda_1\, {}_{p}\mathfrak{F}_{q}\left[{}_{-\infty}D_t^{\beta}y(t)\right](w) + \lambda_2\, {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) \\ &= \left(-iw^{q(\alpha)}\right)^2 {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) \\ &\quad + \lambda_2\, {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w). \end{aligned} $$

That is,

$$ {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) = \frac{{}_{p}\mathfrak{F}_{q}\left[f(t)\right](w)}{\left(-iw^{q(\alpha)}\right)^2 + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} + \lambda_2}. $$

By choosing $A(w) = \left(-iw^{q(\alpha)}\right)^2 + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} + \lambda_2$, we get

$$ {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) = \frac{{}_{p}\mathfrak{F}_{q}\left[f(t)\right](w)}{A(w)}. $$

Taking ${}_{p}\mathfrak{F}_{q}\left[g(t)\right] = \frac{1}{A(w)}$, we obtain

$$ {}_{p}\mathfrak{F}_{q}\left[y(t)\right](w) = {}_{p}\mathfrak{F}_{q}\left[f(t)\right](w)\, {}_{p}\mathfrak{F}_{q}\left[g(t)\right](w), $$

and then applying the ${}_p\mathfrak{F}_q^{-1}$ transform, we have

$$y(t) = {}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q\left[f(t)\right](w)\,{}_p\mathfrak{F}_q\left[g(t)\right](w)\right](t). \tag{4.3}$$

Applying the ${}_p\mathfrak{F}_q^{-1}$ transform to equation (3.9), we get

$$f(t) * g(t) = \frac{{}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q[f(t)](w)\ {}_p\mathfrak{F}_q[g(t)](w)\right](t)}{p(\alpha)}. \tag{4.4}$$

Multiplying and dividing the right side of (4.3) by function $p(\alpha)$, we obtain

$$y(t) = \frac{p(\alpha)}{p(\alpha)}\,{}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q\left[f(t)\right](w)\ {}_p\mathfrak{F}_q\left[g(t)\right](w)\right](t). \tag{4.5}$$

Using equation (4.4) in equation (4.5), we have

$$y(t) = p(\alpha)\big(f(t) * g(t)\big),$$

and then using the convolution, we get

$$y(t) = p(\alpha)\int_{\mathbb{R}} f(t-\tau)g(\tau)\mathrm{d}\tau. \tag{4.6}$$

Applying the ${}_p\mathfrak{F}_q^{-1}$ transform to the equation ${}_p\mathfrak{F}_q\left[g(t)\right](w) = \frac{1}{A(w)}$, we obtain

$$g(t) = \frac{q(\alpha)}{p(\alpha)}\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(-iw^{q(\alpha)}t\right)\frac{w^{q(\alpha)-1}}{A(w)}\mathrm{d}w. \tag{4.7}$$

Using equation (4.7) in equation (4.6), we have

$$y(t) = \frac{q(\alpha)}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t-\tau)\exp\left(-iw^{q(\alpha)}\tau\right)\frac{w^{q(\alpha)-1}}{\left(-iw^{q(\alpha)}\right)^2 + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} + \lambda_2}\mathrm{d}w\mathrm{d}\tau.$$

Problem 3. *The initial and boundary conditions*

(a) $u(t,0) = 0, \quad 0 < t < \infty,$
(b) $u(0,\xi) = f(\xi), \quad \xi \geq 0, \quad u(t,\xi) \to 0 \quad \text{as} \quad t \to \infty,$
(c) $u_t(0,\xi) = f(\xi), \quad \xi \geq 0, \quad u(t,\xi) \to 0 \quad \text{as} \quad t \to \infty,$

and partial diffusion differential equation

$$\frac{\partial u}{\partial \xi} = \kappa\frac{\partial^2 u}{\partial t^2}, \quad (0 < t < \infty,\ \xi > 0)$$

are given. Here κ *is constant.*

The ${}_p^s\mathfrak{F}_q$ transform of a function $u(t,\xi)$ is given by

$${}_p^s\mathfrak{F}_q[u(t,\xi)](w) = {}_p^s\hat{u}_q(w,\xi) = p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right)u(t,\xi)\mathrm{d}t.$$

Considering equation (3.12) and applying the ${}^{s}_{p}\mathfrak{F}_q$ transform to the partial diffusion differential equation and the initial condition (a), we have

$$\frac{d}{d\xi}\left\{{}^{s}_{p}\hat{u}_q(w,\xi)\right\} = \kappa\left(w^{q(\alpha)}p(\alpha)u(0,\xi) - w^{2q(\alpha)}\ {}^{s}_{p}\hat{u}_q(w,\xi)\right)$$
$${}^{s}_{p}\hat{u}_q(w,0) = 0.$$

Using the boundary condition (b), we get

$$\frac{d}{d\xi}\left\{{}^{s}_{p}\hat{u}_q(w,\xi)\right\} = \kappa w^{q(\alpha)}p(\alpha)f(\xi) - \kappa w^{2q(\alpha)}\ {}^{s}_{p}\hat{u}_q(w,\xi).$$

The solution of the last equation is obtained by

$${}^{s}_{p}\hat{u}_q(w,\xi) = \kappa w^{q(\alpha)}p(\alpha)\int_0^{\xi} f(\tau)\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\mathrm{d}\tau.$$

Applying the ${}^{s}_{p}\mathfrak{F}_q^{-1}$ transform, we have

$$u(t,\xi) = \kappa p(\alpha)\int_0^{\xi} f(\tau)\ {}^{s}_{p}\mathfrak{F}_q^{-1}\left[w^{q(\alpha)}\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\right](t)\mathrm{d}\tau.$$

By choosing $a=\kappa(\xi-\tau)$ in equation (3.15) and using in the last equation above, the solution of the given partial diffusion differential equation with the help of the ${}^{s}_{p}\mathfrak{F}_q$ transform is obtained by

$$u(t,\xi) = \frac{t}{2\sqrt{\kappa\pi}}\int_0^{\xi}\frac{f(\tau)}{(\xi-\tau)^{3/2}}\exp\left(\frac{-t^2}{4\kappa(\xi-\tau)}\right)\mathrm{d}\tau.$$

The ${}^{c}_{p}\mathfrak{F}_q$ transform of a function $u(t,\xi)$ is given by

$${}^{c}_{p}\mathfrak{F}_q[u(t,\xi)](w) = {}^{c}_{p}\hat{u}_q(w,\xi) = p(\alpha)\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)u(t,\xi)\mathrm{d}t.$$

Considering equation (3.13) and applying the ${}^{c}_{p}\mathfrak{F}_q$ transform to the partial diffusion differential equation and the initial condition (a), we have

$$\frac{d}{d\xi}\left\{{}^{c}_{p}\hat{u}_q(w,\xi)\right\} = \kappa\left(-p(\alpha)u'(0,\xi) - w^{2q(\alpha)}\ {}^{c}_{p}\hat{u}_q(w,\xi)\right)$$
$${}^{c}_{p}\hat{u}_q(w,0) = 0.$$

Using the boundary condition (c), we get

$$\frac{d}{d\xi}\left\{{}^{c}_{p}\hat{u}_q(w,\xi)\right\} = -\kappa p(\alpha)f(\xi) - \kappa w^{2q(\alpha)}\ {}^{c}_{p}\hat{u}_q(w,\xi).$$

The solution of the last equation is obtained by

$${}^{c}_{p}\hat{u}_q(w,\xi) = -\kappa p(\alpha)\int_0^{\xi} f(\tau)\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\mathrm{d}\tau.$$

Applying the ${}^{c}_{p}\mathfrak{F}_q^{-1}$ transform, we have

$$u(t,\xi) = -\kappa p(\alpha)\int_0^{\xi} f(\tau)\ {}^{c}_{p}\mathfrak{F}_q^{-1}\left[\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\right](t)\mathrm{d}\tau.$$

By choosing $a = \kappa(\xi-\tau)$ in equation (3.14) and using in the last equation above, the solution of the given partial diffusion differential equation with the help of the ${}^{c}_{p}\mathfrak{F}_q$ transform is obtained by

$$u(t,\xi) = -\sqrt{\frac{\kappa}{\pi}}\int_0^{\xi}\frac{f(\tau)}{\sqrt{(\xi-\tau)}}\exp\left(\frac{-t^2}{4\kappa(\xi-\tau)}\right)\mathrm{d}\tau.$$

5. Conclusion

In this article, we introduced the generalized Fourier (${}_{p}\mathfrak{F}_q$, ${}^{s}_{p}\mathfrak{F}_q$ and ${}^{c}_{p}\mathfrak{F}_q$) and inverse Fourier (${}_{p}\mathfrak{F}_q^{-1}$, ${}^{s}_{p}\mathfrak{F}_q^{-1}$ and ${}^{c}_{p}\mathfrak{F}_q^{-1}$) transforms and examine their certain properties. Our motivation in doing this was to define integral transforms that has a more general form than many Fourier-like integral transforms found in the literature. We also presented tables of ${}_{p}\mathfrak{F}_q$, ${}^{s}_{p}\mathfrak{F}_q$ and ${}^{c}_{p}\mathfrak{F}_q$ transformations in Table 1, Table 2 and Table 3, respectively. Then, we use generalized Fourier, Fourier sine, Fourier cosine and inverse Fourier, Fourier sine, Fourier cosine transforms to arrive the analytical solutions of ordinary electric current, partial diffusion and fractional motion problems. Finally, we examined the behavior for different $q(\alpha)$ values of the approximate solutions of the ordinary electric current problem in Figure 1.

We should say that the solutions of the application problems we obtained with the generalized Fourier and inverse Fourier transforms are in full agreement with the results obtained in the literature for $q(\alpha) = 1$ (see for example [1, 3, 6]). As a result of these applications, we found that the newly defined generalized Fourier transforms are much more general and are quite compatible with both ordinary, partial and fractional problems. We would like to note that in this research generalized Fourier transforms play a very important role in finding analytical solutions for both ordinary, partial and fractional problems, and therefore the results presented in this article are very important for their implementation.

We conclude this research by presenting some generalized Fourier transforms that have been recently presented to the literature and their relations with new generalized Fourier transforms (${}_{p}\mathfrak{F}_q$, ${}^{s}_{p}\mathfrak{F}_q$ and ${}^{c}_{p}\mathfrak{F}_q$) below.

- The generalized Fourier transform defined by Luchko et al. [9] is as follows:

$$(\mathcal{F}_\alpha u)(w) = \int_{\mathbb{R}} \exp_\alpha(w,t)\, u(t)\mathrm{d}t, \quad (u \in \phi(\mathbb{R}),\ w \in \mathbb{R},\ 0 < \alpha \leq 1)$$

  and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{1}{\alpha}}[u](w) = (\mathcal{F}_\alpha u)(w).$$

- The generalized Fourier transform defined by Romero and Luque [12] is as follows:

$$\mathfrak{F}_\alpha[u](w) = \int_{\mathbb{R}} \exp\left(iw^{\frac{1}{\alpha}}t\right) u(t)\mathrm{d}t, \quad (u \in \phi(\mathbb{R}),\ w \in \mathbb{R},\ 0 < \alpha \leq 1)$$

  and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{1}{\alpha}}[u](w) = \mathfrak{F}_\alpha[u](w).$$

- The generalized Fourier transform defined by Kumar [8] is as follows:

$$\Omega_\mu[u](w) = \int_{\mathbb{R}} \exp\left(iw^{\frac{\rho}{\mu}}t\right) u(t)\mathrm{d}t, \quad \left(u \in \phi(\mathbb{R}),\ w > 0,\ \mu \leq \rho;\ \mu,\rho \in \mathbb{R}^+\right)$$

  and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{\rho}{\mu}}[u](w) = \Omega_\mu[u](w).$$

- The generalized Fourier sine and Fourier cosine transforms defined by Mahor et al. [10] are as follows:

$$\hat{f}_s^\alpha(w) = \int_{\mathbb{R}_0^+} \sin_\alpha(wt)^\alpha u(t)(\mathrm{d}t)^\alpha, \quad (w > 0,\ 0 < \alpha \leq 1),$$
$$\hat{f}_c^\alpha(w) = \int_{\mathbb{R}_0^+} \cos_\alpha(wt)^\alpha u(t)(\mathrm{d}t)^\alpha, \quad (w > 0,\ 0 < \alpha \leq 1)$$

  and their relationship with the ${}_p^s\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q$ transforms are as follows:

$${}_1^s\mathfrak{F}_1[u(t)](w) = \hat{f}_s^1(w),$$
$${}_1^c\mathfrak{F}_1[u(t)](w) = \hat{f}_c^1(w).$$

ACKNOWLEDGEMENTS

The authors would like to thank to referees for their valuable comments and suggestions that helped to improve this article.

TABLE 1. ${}_p\mathfrak{F}_q$ transform table

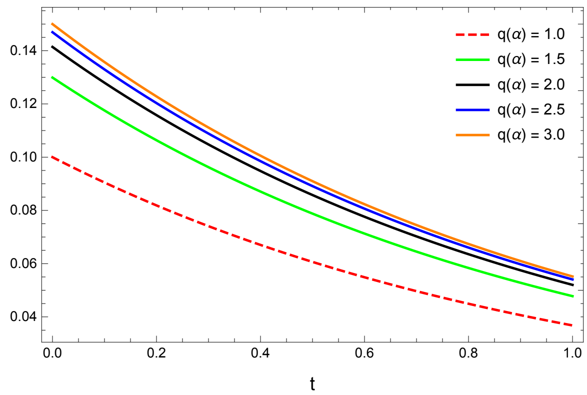
$u(t)$	${}_p\mathfrak{F}_q[u(t)](w) = p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) u(t)\mathrm{d}t$	Conditions
$\exp\left(-at^2\right)$	$p(\alpha)\sqrt{\frac{\pi}{a}}\exp\left(\frac{-w^{2q(\alpha)}}{4a}\right)$	$a > 0$
$\exp(-a\lvert t\rvert)$	$p(\alpha)\left(\frac{2a}{a^2+w^{2q(\alpha)}}\right)$	$a > 0$
$H(a-\lvert t\rvert)$	$2p(\alpha)\left(\frac{\sin\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	
$\exp(-at)H(t)$	$\frac{p(\alpha)}{a-iw^{q(\alpha)}}$	$a > 0$
$\delta(t)$	$p(\alpha)$	
$\sin(at)$	$-p(\alpha)i\pi\left[\delta\left(-w^{q(\alpha)}-a\right)-\delta\left(-w^{q(\alpha)}+a\right)\right]$	
$\cos(at)$	$p(\alpha)\pi\left[\delta\left(-w^{q(\alpha)}-a\right)+\delta\left(-w^{q(\alpha)}+a\right)\right]$	
c : constant	$2p(\alpha)c\pi\delta\left(w^{q(\alpha)}\right)$	
$u(t) * v(t)$	$\frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}$	
$u'(t)$	$\left(-iw^{q(\alpha)}\right) {}_p\mathfrak{F}_q[u(t)](w)$	
$u''(t)$	$\left(-iw^{q(\alpha)}\right)^2 {}_p\mathfrak{F}_q[u(t)](w)$	
$u^{(n)}(t)$	$\left(-iw^{q(\alpha)}\right)^n {}_p\mathfrak{F}_q[u(t)](w)$	
${}_{-\infty}I_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{-\beta} {}_p\mathfrak{F}_q[u(t)](w)$	$0 < \Re(\beta) < 1$
${}_{-\infty}D_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{\beta} {}_p\mathfrak{F}_q[u(t)](w)$	$\Re(\beta) > 0$
${}_{-\infty}^{\ c}D_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{\beta} {}_p\mathfrak{F}_q[u(t)](w)$	$\Re(\beta) > 0$

TABLE 2. ${}^{s}_{p}\mathfrak{F}_q$ transform table

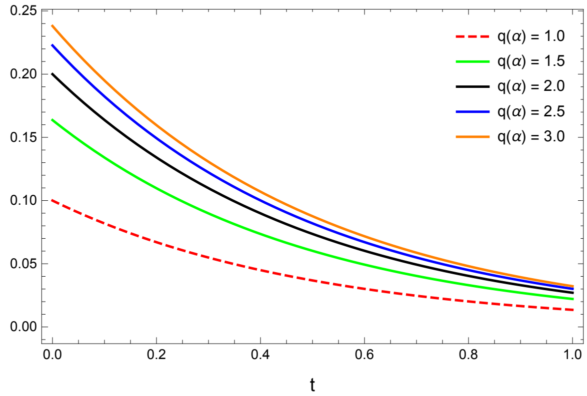
$u(t)$	${}^{s}_{p}\mathfrak{F}_q[u(t)](w) = p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t$	Conditions
$\exp(-at)$	$p(\alpha)\left(\frac{w^{q(\alpha)}}{a^2+w^{2q(\alpha)}}\right)$	$a>0$
$t\exp(-at)$	$p(\alpha)\left(\frac{2aw^{q(\alpha)}}{\left(a^2+w^{2q(\alpha)}\right)^2}\right)$	$a>0$
$\frac{t}{a^2+t^2}$	$\frac{p(\alpha)\pi\exp\left(-aw^{q(\alpha)}\right)}{2}$	$a>0$
$H(a-t)$	$p(\alpha)\left(\frac{1-\cos\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	$a>0$
$u'(t)$	$-w^{q(\alpha)}\ {}^{c}_{p}\mathfrak{F}_q[u(t)](w)$	
$u''(t)$	$-w^{2q(\alpha)}\ {}^{s}_{p}\mathfrak{F}_q[u(t)](w)+w^{q(\alpha)}p(\alpha)u(0)$	

TABLE 3. ${}^{c}_{p}\mathfrak{F}_q$ transform table

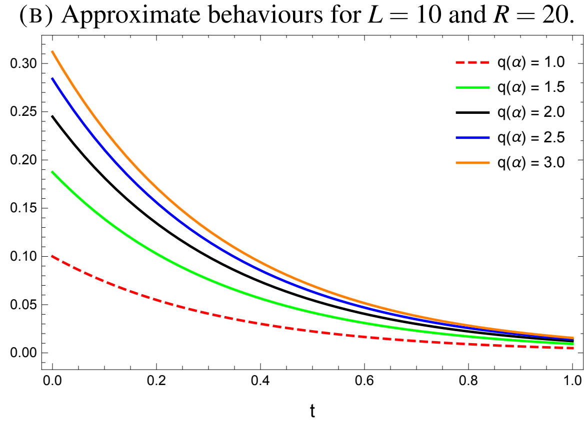
$u(t)$	${}^{c}_{p}\mathfrak{F}_q[u(t)](w) = p(\alpha)\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t$	Conditions
$\exp(-at)$	$p(\alpha)\left(\frac{a}{a^2+w^{2q(\alpha)}}\right)$	$a>0$
$t\exp(-at)$	$p(\alpha)\left(\frac{a^2-w^{2q(\alpha)}}{\left(a^2+w^{2q(\alpha)}\right)^2}\right)$	$a>0$
$\frac{1}{a^2+t^2}$	$\frac{p(\alpha)\pi\exp\left(-aw^{q(\alpha)}\right)}{2a}$	$a>0$
$H(a-t)$	$p(\alpha)\left(\frac{\sin\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	$a>0$
$u'(t)$	$w^{q(\alpha)}\ {}^{s}_{p}\mathfrak{F}_q[u(t)](w)-p(\alpha)u(0)$	
$u''(t)$	$-w^{2q(\alpha)}\ {}^{c}_{p}\mathfrak{F}_q[u(t)](w)-p(\alpha)u'(0)$	



(A) Approximate behaviours for $L = 10$ and $R = 10$.



(B) Approximate behaviours for $L = 10$ and $R = 20$.



(C) Approximate behaviours for $L = 10$ and $R = 30$.

FIGURE 1. Approximate behaviours corresponding to the different values of the function $q(\alpha)$ for solving the ordinary electric current differential equation.

REFERENCES

[1] H. Beyer and S. Kempfle, "Definition of physically consistent damping laws with fractional derivatives," *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 75, no. 8, pp. 623–635, 1995, doi: 10.1002/zamm.19950750820.

[2] R. N. Bracewell, *The Fourier transform and its applications*. McGraw-Hill New York, 2000.

[3] L. Debnath and D. Bhatta, *Integral transforms and their applications*. CRC press, 2014.

[4] M. El Hammaa, R. Dahera, and H. El Harraka, "On estimates for the Fourier-Bessel transform in the space Lp," *Filomat*, vol. 36, no. 2, pp. 591–597, 2022, doi: 10.2298/FIL2202591H.

[5] G. Jumarie, "Fourier's transform of fractional order via Mittag-Leffler function and modified Riemann-Liouville derivative," *Journal of Applied Mathematics & Informatics*, vol. 26, no. 5_6, pp. 1101–1121, 2008.

[6] S. Kempfle and L. Gaul, "Global solutions of fractional linear differential equations," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 76, pp. 571–572, 1996.

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier, 2006, vol. 204.

[8] V. Kumar, "On a generalized fractional Fourier transform," *Palestine J. Maths*, vol. 9, no. 2, pp. 903–907, 2020.

[9] Y. F. Luchko, H. Martinez, and J. J. Trujillo, "Fractional Fourier transform and some of its applications," *Fract. Calc. Appl. Anal*, vol. 11, no. 4, pp. 457–470, 2008.

[10] T. C. Mahor, R. Mishra, and R. Jain, "Fractionalization of Fourier sine and Fourier cosine transforms and their applications," *International Journal of Scientific & Technology Research*, vol. 9, no. 4, pp. 1720–1725, 2020.

[11] M. A. Ragusa and V. Shakhmurov, "Embedding of vector-valued Morrey spaces and separable differential operators," *Bulletin of Mathematical Sciences*, vol. 9, no. 2, pp. 1–23, 2018, doi: 10.1142/S1664360719500139.

[12] L. G. Romero, R. A. Cerutti, and L. L. Luque, "A new fractional Fourier transform and convolution products," *International Journal of Pure and Applied Mathematics*, vol. 66, no. 4, pp. 397–408, 2011.

[13] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, 1993, vol. 1.

[14] A. Selvam, S. Sabarinathan, S. Noeiaghdam, and V. Govindan, "Fractional Fourier transform and ulam stability of fractional differential equation with fractional Caputo-type derivative," *Journal of Function Spaces*, vol. 2022, pp. 1–5, 2022, doi: 10.1155/2022/3777566.

[15] E. M. Stein and R. Shakarchi, *Fourier analysis: an introduction*. Princeton University Press, 2007, vol. 1.

[16] D. G. Zill and P. D. Shanahan, *A first course in complex analysis with applications*. Jones & Bartlett Learning, 2003.

Authors' addresses

Enes Ata

(**Corresponding author**) Kırşehir Ahi Evran University, Department of Mathematics, Faculty of Arts and Science, 40100 Kırşehir, Turkey

E-mail address: enesata.tr@gmail.com

İ. Onur Kıymaz

Kırşehir Ahi Evran University, Department of Mathematics, Faculty of Arts and Science, 40100 Kırşehir, Turkey

E-mail address: iokiymaz@ahievran.edu.tr