



SS-PALT for Logistic-Exponential Distribution with Adaptive Type-I Progressively Hybrid Censored Data

Faten A. Momenkhan¹

Received: 29 November 2024 / Accepted: 25 November 2025
© The Author(s) 2026

Abstract

This study investigates the application of adaptive Type-I progressive hybrid censoring techniques for the estimation of failure times in step-stress partially accelerated life tests. We employ a Logistic-Exponential distribution to model the lifetimes of the test units and derive the maximum likelihood estimators for both the model parameters and the acceleration factor. Furthermore, we introduce an adaptive Type-I hybrid progressive censoring scheme to enhance the robustness of our analysis. To assess and compare the efficacy of the estimators under various hybrid censoring scenarios, we conduct a comprehensive Monte Carlo simulation study. Performance metrics, including mean squared error and bias, are utilized to facilitate a rigorous comparison of the estimators' effectiveness.

Keywords Adaptive Type-I progressive hybrid censoring · Life testing · Logistic-exponential distribution · Type-I progressive hybrid censoring · Simulation study

Mathematics Subject Classification 60E05 · 62E15

1 Introduction

In the context of accelerated life testing, Type-I and Type-II censoring schemes represent two prevalent methodologies for managing censored data. The conventional Type-I censoring scheme operates by terminating the experiment at a predetermined time point, resulting in a fixed duration while the number of observed failures varies. In contrast, the Type-II censoring scheme is characterized by its continuation until a specified number of failures is observed, which yields a fixed count of failures while the experimental duration becomes variable. A significant limitation inherent in these

✉ Faten A. Momenkhan
famomenkhan8@uj.edu.sa

¹ Department of Mathematics and Statistics, University of Jeddah, Jeddah, Saudi Arabia

traditional censoring schemes is the inability of researchers to ascertain both the time endpoint and the failure count in advance. To mitigate this limitation, hybrid censoring schemes have been introduced, integrating aspects of both Type-I and Type-II censoring. This hybrid approach enhances flexibility in experimental design and facilitates more robust data analysis. For further insights, refer to the works of Balakrishnan [1], Balakrishnan and Aggarwala [2], and Balakrishnan and Iliopoulos [3].

Consider a situation where n denotes the number of units under evaluation. We assume that the lifetimes of these sampled units are independent and identically distributed (iid) random variables. Let $x_{1:n} \leq x_{2:n} \leq \dots \leq x_{n:n}$ represent the ordered lifetimes of these units. The life test is terminated when either a predetermined number m (where $m < n$) of the n units have failed, or when a specified time T_0 is reached. Consequently, the conclusion of the life test occurs at a random time denoted by $\min(x_{m:n}, T_0)$. It is generally assumed that the failed units are not replaced during the course of the experiment, thereby maintaining the integrity of the test conditions.

Under a progressive Type-I hybrid censoring scheme, the experimenter establishes predefined parameters for both the testing duration and the allowable number of failures among the n units participating in the life testing experiment. The experiment is concluded at time $T = \min(x_{m:n}, T_0)$. Upon the occurrence of the first failure, R_1 units are randomly selected from the remaining $n - 1$ units and subsequently removed from the experiment. Following the failure of the second unit, R_2 units are randomly chosen from the remaining $n - 2 - R_1$ units for removal. This sequential process persists until the termination condition $T = \min(x_{m:n}, T_0)$ is satisfied. At this juncture, the number of surviving units is represented as $R_j^* = n - j - R_1 - R_2 - \dots - R_j$, where j denotes the total number of failures recorded up to the time $T = \min(x_{m:n}, T_0)$, and R_1, R_2, \dots, R_j are predetermined integers signifying the units removed at each instance of failure.

Under progressive hybrid censoring schemes, the experimenter often lacks prior knowledge or certainty regarding the precise termination time of the test. To address this challenge, Liang and Yang [4] proposed an alternative approach known as the adaptive Type-I progressive hybrid censoring scheme, which they utilized for estimating parameters within the exponential distribution framework. This adaptive scheme guarantees that the life testing experiment concludes at a predetermined fixed time T_0 and has been demonstrated to produce more efficient estimators compared to the traditional Type-I hybrid censoring scheme.

Lin et al. [5] investigated maximum likelihood and Bayesian estimation methods for the two-parameter Weibull distribution utilizing the adaptive Type-I progressive hybrid censoring scheme. While there has been some research addressing step-stress partially accelerated life tests (SS-PALT) within the context of this adaptive scheme, a comprehensive study specifically focusing on this application remains limited. The present study aims to estimate the failure times of test units in SS-PALT by employing the adaptive Type-I progressive hybrid censoring scheme. In this framework, we assume that the failure times of the test units are governed by a Logistic-Exponential distribution. Also, see Shakoor and Khan [6].

Using the Logistic-Exponential distribution under adaptive Type-I progressive hybrid censoring in SS-PAL allows for flexible modeling of time-to-event data, particularly in scenarios where data may be censored due to the nature of the test-

ing process. The Logistic-Exponential distribution is advantageous for its capacity to accommodate datasets exhibiting both extreme values and varying hazard rates, making it particularly suitable for practical applications in reliability engineering and survival analysis. By integrating adaptive Type-I progressive hybrid censoring, researchers can dynamically adjust testing conditions based on interim results, optimizing resource allocation while maintaining robust statistical power. Moreover, further applications can be seen in the area of censored samples data from a half-normal distribution which are detailed by Khawar Kiani et al. [7]. For nonresponses cases one can see Basit and Bhatti [8] and Basit et al. [9].

This paper is organized as follows. Section 2 explores the properties of the Logistic-Exponential distribution and analyzes adaptive Type-I hybrid censoring techniques for estimating failure times in SS-PALT. Section 3 introduces the likelihood function that pertinent to SS-PALT. Section 4 present an application using real-world data and a simulation study, respectively. Finally, Section 5 offers concluding remarks and discusses implications for future research.

2 Model and Methods

2.1 Model

A continuous random variable X is said to follow a Logistic-Exponential distribution, abbreviated as $X \sim \text{LE}(\alpha, \beta)$, if its probability density function (PDF) is given as

$$f(x; \alpha, \beta) = \frac{\alpha\beta(e^{\alpha x} - 1)^{\beta-1}e^{\alpha x}}{(1 + (e^{\alpha x} - 1)^\beta)^2}, \quad x > 0, \quad (1)$$

where $\alpha > 0$ is the scale parameter and the shape parameter $\beta > 0$. The corresponding distribution function of (1) is given by

$$F(x; \alpha, \beta) = \frac{(e^{\alpha x} - 1)^\beta}{1 + (e^{\alpha x} - 1)^\beta}, \quad x > 0. \quad (2)$$

The reliability function on the support of X is

$$\bar{F}(x; \alpha, \beta) = \frac{1}{1 + (e^{\alpha x} - 1)^\beta}, \quad x > 0.$$

To make things easier and clearer, Table 1 provides a notation table.

The PDF under SS-PALT is given by

$$f(x) = \begin{cases} f_1(x), & \text{if } 0 < x \leq \tau, \\ f_2(x), & \text{if } x > \tau, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Table 1 Notation Table

Symbol	Description
n	Number of units under evaluation
m	Predetermined number of failures to be observed
τ	Stress change time
T_0	Specified time for terminating the experiment
j	Total number of failures recorded up to termination time
$\{R_i\}$	Set of units removed from the experiment at each failure
R_j^*	Number of surviving units after j failures
R_j^{**}	Final censored count of surviving units under the adaptive Type-I progressive hybrid censoring scheme
n_u	Number of failures observed under normal conditions

We can apply the variable-transformation technique to Eq. (1) along with the model proposed by DeGroot and Goel [10]:

$$X = \begin{cases} T, & \text{if } T \leq \tau, \\ \tau + \lambda^{-1}(T - \tau), & \text{if } T > \tau, \end{cases}$$

where T is the lifetime of an item under usual operating conditions, τ is the stress change time, and $\lambda > 1$ is the acceleration factor. Therefore, Eq. (3) can be written taking into consideration that $f_1(x)$ is as given in (1) and $f_2(x)$ is given by

$$f_2(x) = \frac{\alpha\beta(e^{\alpha(\tau+\lambda^{-1}(x-\tau))} - 1)^{\beta-1}e^{\alpha(\tau+\lambda^{-1}(x-\tau))}}{(1 + (e^{\alpha(\tau+\lambda^{-1}(x-\tau))} - 1)^\beta)^2}. \quad (4)$$

The scale parameter α adjusts with the acceleration factor λ . As λ increases, the effective scale of the distribution increases, leading to a higher rate of failure. The shape parameter β may also be influenced by stress conditions but is generally treated as constant in many models. However, it could be modeled to change, reflecting different failure mechanisms under varying stress. The link function is applied on the time scale through the transformation that modifies the time variable T . For $T \leq \tau$, X remains T , while for $T > \tau$, the transformation adjusts the time to reflect stress effects, explicitly extending the time to failure.

The chosen transformation reflects observed behaviors in materials under stress. When an item is subjected to stress beyond a certain threshold τ , the lifetime is effectively extended or adjusted according to the acceleration factor λ .

Under the progressive Type-I hybrid censoring scheme, the life test is concluded at a random time $\min(X_{m:m:n}, T_0)$, where $0 < T_0 < \infty$ and $1 \leq m \leq n$ are predetermined constants. The ordered failure times resulting from the experiment are denoted as $x_{1:m:n} \leq x_{2:m:n} \leq \dots \leq x_{j:m:n}$. Specifically, the experiment terminates at time $t_0 = x_{m:m:n}$ if the m th progressively censored observed failure occurs before time T_0 (i.e., $x_{m:m:n} < T_0$). Conversely, if the m th failure occurs after T_0 , the experiment concludes at time T_0 , with the condition $x_{j:m:n} < T_0 < x_{j+1:m:n}$, and all remaining surviving units are censored at this time. Here, j represents the number of failures that occur prior to $\min(x_{m:m:n}, T_0)$.

Under the SS-PALT scheme, units are initially subjected to a normal stress level S_0 . At time τ , the stress is increased to S_1 . Let m denote the predetermined number of failures to be monitored under both stress levels. The termination time T_0 and the removal sequences $(R_1, R_2, \dots, R_\tau, \dots, R_j)$ are established in advance. At the time of the i th failure $x_{i:m:n}$, R_i units are removed from the experiment. Additionally, at the stress change time τ , R_τ units are withdrawn from the remaining survivors, and this process continues accordingly. The observed data in the SS-PALT under the Type-I progressive hybrid censoring scheme is given by:

Case I: $x_{m:m:n} < T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{m-1:m:n}, R_{m-1}), (x_{m:m:n}, R_j^*).$$

Case II: $x_{m:m:n} = T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{m:m:n} = T_0, R_m = R_j^*).$$

Case III: $x_{m:m:n} > T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{j:m:n}, R_j), (T_0, R_j^*),$$

where n_u represents the number of failures observed under normal conditions. The total number of failures j and the final censored count R_j^* in the SS-PALT scheme for each case are defined as follows:

$$\begin{cases} j > m, R_j^* = n - m - \sum_{i=1}^m R_i & \text{if } \tau < x_{m:m:n} < T_0, \\ j = m, R_j^* = n - m - \sum_{i=1}^m R_i & \text{if } x_{m:m:n} = T_0, \\ j < m, R_j^* = n - j - \sum_{i=1}^j R_i & \text{if } x_{m:m:n} > T_0. \end{cases}$$

For the adaptive Type-I progressive hybrid censoring scheme, suppose n items are subjected to testing, and let the ordered lifetimes of these units be represented as $y_{1:n} \leq y_{2:n} \leq \dots \leq y_{n:n}$. If the m th failure occurs prior to the time T_0 (i.e., $y_{m:m:n} < T_0$), the test will not be halted; instead, it will continue to observe failures until T_0 is reached, without any further removals. Once time T_0 is attained, all remaining units, denoted as $R^{**} = n - j - \sum_j R_j$, are removed from the experiment, and the test concludes automatically. The observed data in the SS-PALT under the adaptive Type-I progressive hybrid censoring scheme is given by:

Case I: $x_{m:m:n} < T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{m-1:m:n}, R_{m-1}),$$

$$(x_{m:m:n}, R_m), (x_{m+1:m:n}, 0) \dots (x_{j:m:n}, 0), (T_0, R_j^{**}).$$

Case II: $x_{m:m:n} = T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{m:m:n} = T_0, R_m = R_j^{**}).$$

Case III: $x_{m:m:n} > T_0$

$$S_0 = (x_{1:m:n}, R_1), (x_{2:m:n}, R_2), \dots, (x_{n_u:m:n}, R_{n_u}), (\tau, R_\tau),$$

$$S_1 = (x_{n_u+1:m:n}, R_{n_u+1}), (x_{n_u+2:m:n}, R_{n_u+2}), \dots, (x_{j:m:n}, R_j), (T_0, R_j^{**}),$$

where n_u denotes the number of failures observed under normal conditions. The total number of failures j and the final censored count R_j^{**} in the SS-PALT scheme for each case are expressed as follows:

$$\begin{cases} j > m, R_j^{**} = n - j - \sum_{i=1}^m R_i & \text{if } \tau < x_{m:m:n} < T_0, \\ j = m, R_j^{**} = n - m - \sum_{i=1}^m R_i & \text{if } x_{m:m:n} = T_0, \\ j < m, R_j^{**} = n - j - \sum_{i=1}^j R_i & \text{if } x_{m:m:n} > T_0. \end{cases}$$

Under the adaptive progressive Type-I hybrid censoring scheme, the experimenter has the flexibility to adjust the value of T_0 to optimize the experimental duration and enhance the likelihood of observing additional failures. This scheme ensures that the life testing experiment concludes at a fixed time T_0 , thereby improving the efficiency of statistical inference. Moreover, it facilitates better control over the total number of failures, allowing the observed failures to remain close to the target number m . This control contributes to producing more accurate and reliable results. For further details, see Lone et al. [11].

2.2 Maximum Likelihood Estimation

Using the adaptive progressive hybrid censoring scheme, the likelihood function under the SS-PALT for the j ordered lifetime data set is given by:

$$L(\alpha, \beta, \lambda) = \prod_{i=1}^{n_u} f_1(x_{i:m:n}) [S_1(x_{i:m:n})]^{R_i} [S_1(\tau)]^{R_i}$$

$$\times \prod_{i=n_u+1}^m f_2(x_{i:m:n}) [S_2(x_{i:m:n})]^{R_i} [S_2(T_0)]^{R_i^{**}},$$

where $R_j^{**} = n - \sum_i R_i - j$, with $j = n_u + n_a$. Here, n_u represents the number of failed units under normal use conditions, and n_a denotes the number of failed units under accelerated conditions. Additionally,

$$S_1(x) = \frac{1}{1 + (e^{\alpha x} - 1)^\beta}$$

$$S_2(x) = \frac{1}{1 + (e^{\alpha(\tau + \lambda^{-1}(x - \tau))} - 1)^\beta}$$

Denote $g_1(x; \alpha) = e^{\alpha x} - 1$ and $g_2(x; \alpha, \lambda) = e^{\alpha(\tau + \lambda^{-1}(x - \tau))} - 1$, the likelihood function under SS-PALT is becoming as

$$L(\alpha, \beta, \lambda) = (\alpha\beta)^{n_u} e^{\sum_{i=1}^{n_u} x_i} \prod_{i=1}^{n_u} g_1(x_i; \alpha)^{\beta-1} (1 + (g_1(x_i; \alpha))^\beta)^{-(R_i+2)} (1 + (g_1(\tau; \alpha))^\beta)^{-R_i}$$

$$\times (\alpha\beta)^m e^{i=n_u+1} \sum_{g_2(x; \alpha, \lambda)} \prod_{i=n_u+1}^m (g_2(x; \alpha, \lambda))^{\beta-1} (1 + (g_2(x; \alpha, \lambda))^\beta)^{-(R_i+2)}$$

$$\times (1 + (g_2(T_0; \alpha, \lambda))^\beta)^{-R_i^*}$$

In order to simplify the calculations significantly, we utilize the log-likelihood function, as follows:

$$\ln L(\alpha, \beta, \lambda) = \sum_{i=1}^{n_u} x_i + \sum_{i=n_u+1}^j g_2(x; \alpha, \lambda) + (n_u + m) \ln(\alpha) + (n_u + m) \ln(\beta)$$

$$+ (\beta - 1) \left(\sum_{i=1}^{n_u} \ln(g_1(x_i; \alpha)) + \sum_{i=n_u+1}^j \ln(g_2(x_i; \alpha, \lambda)) \right)$$

$$- \sum_{i=1}^{n_u} (R_i + 2) \ln(1 + (g_1(x_i; \alpha))^\beta) - \sum_{i=n_u+1}^j (R_i + 2) \ln(1 + (g_2(x_i; \alpha, \lambda))^\beta)$$

$$- \sum_{i=1}^{n_u} R_i \ln(1 + (g_1(\tau; \alpha))^\beta) - \sum_{i=n_u+1}^j R_i^{**} \ln(1 + (g_2(T_0; \alpha, \lambda))^\beta)$$

Now, differentiating with respect to α, β and λ , and equating to zero, the likelihood equations are respectively obtained as:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n_u + m}{\alpha} + \sum_{i=n_u+1}^j \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \alpha} + (\beta - 1) \sum_{i=1}^{n_u} \frac{g_1'(x_i; \alpha)}{g_1(x_i; \alpha)}$$

$$+ (\beta - 1) \sum_{i=n_u+1}^j \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \alpha} \frac{1}{g_2(x_i; \alpha, \lambda)} - \sum_{i=1}^{n_u} (R_i + 2) \frac{\beta(g_1(x_i; \alpha))^{\beta-1} g_1'(x_i; \alpha)}{1 + (g_1(x_i; \alpha))^\beta}$$

$$- \sum_{i=n_u+1}^j (R_i + 2) \frac{\beta(g_2(x_i; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \alpha}}{1 + (g_2(x_i; \alpha, \lambda))^\beta} - \sum_{i=1}^{n_u} R_i \frac{g_1'(\tau; \alpha)}{1 + g_1(\tau; \alpha)}$$

$$- \sum_{i=n_u+1}^j R_i^{**} \frac{\beta(g_2(T_0; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \alpha}}{1 + (g_2(T_0; \alpha, \lambda))^\beta} = 0,$$

where $g_1(x; \alpha)$ and $g_2(x; \alpha, \lambda)$ are defined as before and

$$g_1'(x; \alpha) = xe^{\alpha x}.$$

$$\frac{\partial g_2(x; \alpha, \lambda)}{\partial \alpha} = e^{\alpha(\tau + \lambda^{-1}(x - \tau))} (\tau + \lambda^{-1}(x - \tau)).$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} &= \frac{n_u + m}{\beta} + \sum_{i=1}^{n_u} \ln(g_1(x_i; \alpha)) + \sum_{i=n_u+1}^j \ln(g_2(x_i; \alpha, \lambda)) \\ &\quad - \sum_{i=1}^{n_u} (R_i + 2) \frac{(g_1(x_i; \alpha))^\beta \ln(g_1(x_i; \alpha))}{1 + (g_1(x_i; \alpha))^\beta} \\ &\quad - \sum_{i=n_u+1}^j (R_i + 2) \frac{(g_2(x_i; \alpha, \lambda))^\beta \ln(g_2(x_i; \alpha, \lambda))}{1 + (g_2(x_i; \alpha, \lambda))^\beta} \\ &\quad - \sum_{i=1}^{n_u} R_i \frac{(g_1(\tau; \alpha))^\beta \ln(g_1(\tau; \alpha))}{1 + (g_1(\tau; \alpha))^\beta} - \sum_{i=n_u+1}^j R_i^{**} \frac{(g_2(T_0; \alpha, \lambda))^\beta \ln(g_2(T_0; \alpha, \lambda))}{1 + (g_2(T_0; \alpha, \lambda))^\beta} = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \lambda} &= \sum_{i=n_u+1}^j \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda} + (\beta - 1) \sum_{i=n_u+1}^j \frac{1}{g_2(x_i; \alpha, \lambda)} \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda} \\ &\quad - \sum_{i=n_u+1}^j (R_i + 2) \frac{\beta (g_2(x_i; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda}}{1 + (g_2(x_i; \alpha, \lambda))^\beta} \\ &\quad - R_i^{**} \frac{\beta (g_2(T_0; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \lambda}}{1 + (g_2(T_0; \alpha, \lambda))^\beta} = 0, \end{aligned}$$

where

$$\frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda} = -\alpha \lambda^{-2} (x - \tau) e^{\alpha(\tau + \lambda^{-1}(x - \tau))} (\tau + \lambda^{-1}(x - \tau)).$$

Obtaining closed-form solutions for the nonlinear equations mentioned above is nearly impossible. Consequently, we employ an iterative technique to find the maximum likelihood estimates (MLEs) of the parameters α , β , and λ . This approach allows us to derive the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$.

The asymptotic variances of the maximum likelihood estimators are represented by the elements of the inverse of the Fisher information matrix. The Fisher information matrix for $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$, denoted as $J(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = E(I; \hat{\alpha}, \hat{\beta}, \hat{\lambda})$, consists of elements I_{ij} for $i, j = 1, 2, 3$. Under the adaptive progressive hybrid censoring scheme, these elements are the negative of the second partial derivatives of the log-likelihood function, as shown below:

$$I = - \begin{bmatrix} I_{11} = \frac{\partial \ln L}{\partial \alpha^2} & I_{12} = \frac{\partial \ln L}{\partial \alpha \partial \beta} & I_{13} = \frac{\partial \ln L}{\partial \alpha \partial \lambda} \\ I_{21} = \frac{\partial \ln L}{\partial \beta \partial \alpha} & I_{22} = \frac{\partial \ln L}{\partial \beta^2} & I_{23} = \frac{\partial \ln L}{\partial \beta \partial \lambda} \\ I_{31} = \frac{\partial \ln L}{\partial \lambda \partial \alpha} & I_{32} = \frac{\partial \ln L}{\partial \lambda \partial \beta} & I_{33} = \frac{\partial \ln L}{\partial \lambda^2} \end{bmatrix}$$

Here, we focus on the diagonal elements, i.e., I_{11} , I_{22} and, I_{33} , as they are essential for calculating the variances and determining the confidence intervals.

$$\begin{aligned}
 I_{11} = & \frac{\partial \ln L}{\partial \alpha^2} = -\frac{n_u + m}{\alpha^2} + \sum_{i=n_u+1}^j \frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \alpha^2} \\
 & + (\beta - 1) \sum_{i=1}^{n_u} \frac{g_1''(x_i; \alpha) g_1(x_i; \alpha) - (g_1'(x_i; \alpha))^2}{(g_1(x_i; \alpha))^2} \\
 & + (\beta - 1) \sum_{i=n_u+1}^j \left(\frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \alpha^2} \cdot \frac{1}{g_2(x_i; \alpha, \lambda)} - \frac{(\frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \alpha})^2}{(g_2(x_i; \alpha, \lambda))^2} \right) \\
 & - \sum_{i=1}^{n_u} (R_i + 2) \frac{\beta(\beta - 1)(g_1(x_i; \alpha))^{\beta-2} g_1''(x_i; \alpha) + \beta(g_1(x_i; \alpha))^{\beta-1} g_1'(x_i; \alpha)}{1 + (g_1(x_i; \alpha))^\beta} \\
 & + \sum_{i=1}^{n_u} (R_i + 2) \left(\frac{\beta(g_1(x_i; \alpha))^{\beta-1} g_1'(x_i; \alpha)}{1 + (g_1(x_i; \alpha))^\beta} \right)^2 \\
 & - \sum_{i=n_u+1}^j (R_i + 2) \frac{\beta(\beta - 1)(g_2(x_i; \alpha, \lambda))^{\beta-2} \frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \alpha^2} + \beta(g_2(x_i; \alpha, \lambda))^{\beta-1} \frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \alpha^2}}{1 + (g_2(x_i; \alpha, \lambda))^\beta} \\
 & + \sum_{i=n_u+1}^j (R_i + 2) \left(\frac{\beta(g_2(x_i; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \alpha}}{1 + (g_2(x_i; \alpha, \lambda))^\beta} \right)^2 \\
 & - \sum_{i=1}^{n_u} R_i \frac{g_1''(\tau; \alpha)(1 + g_1(\tau; \alpha)) - (g_1'(\tau; \alpha))^2}{(1 + g_1(\tau; \alpha))^2} \\
 & - \sum_{i=n_u+1}^j R_i^{**} \frac{\beta(\beta - 1)(g_2(T_0; \alpha, \lambda))^{\beta-2} \frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \alpha} + \beta(g_2(T_0; \alpha, \lambda))^{\beta-1} \frac{\partial^2 g_2(T_0; \alpha, \lambda)}{\partial \alpha^2}}{1 + (g_2(T_0; \alpha, \lambda))^\beta} \\
 & + \sum_{i=n_u+1}^j R_i^{**} \left(\frac{\beta(g_2(T_0; \alpha, \lambda))^{\beta-1} \frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \alpha}}{1 + (g_2(T_0; \alpha, \lambda))^\beta} \right)^2,
 \end{aligned}$$

where $g_1'(x; \alpha)$ and $\partial g_2(x; \alpha, \lambda)/\partial \alpha$ are defined as before and

$$\begin{aligned}
 g_1''(x; \alpha) &= x^2 e^{\alpha x}. \\
 \frac{\partial^2 g_2(x; \alpha, \lambda)}{\partial \alpha^2} &= e^{\alpha(\tau + \lambda^{-1}(x - \tau))} (\tau + \lambda^{-1}(x - \tau))^2.
 \end{aligned}$$

By substituting these functions into the equation above, one can obtain I_{11} , resulting in a lengthy but well-structured expression. Similarly, the same process can be applied to I_{22} and I_{33} . This is illustrated below:

$$\begin{aligned}
I_{22} &= \frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_u + m}{\beta^2} - \sum_{i=1}^{n_u} (R_i + 2) \frac{g_1(x_i; \alpha)^\beta [\ln(g_1(x_i; \alpha))]^2}{1 + g_1(x_i; \alpha)^\beta} \\
&\quad + \sum_{i=1}^{n_u} (R_i + 2) \left(\frac{g_1(x_i; \alpha)^\beta \ln(g_1(x_i; \alpha))}{1 + g_1(x_i; \alpha)^\beta} \right)^2 \\
&\quad - \sum_{i=n_u+1}^j (R_i + 2) \frac{g_2(x_i; \alpha, \lambda)^\beta [\ln(g_2(x_i; \alpha, \lambda))]^2}{1 + g_2(x_i; \alpha, \lambda)^\beta} \\
&\quad - \sum_{i=1}^{n_u} R_i \frac{g_1(\tau; \alpha)^\beta [\ln(g_1(\tau; \alpha))]^2}{1 + g_1(\tau; \alpha)^\beta} \\
&\quad + \sum_{i=n_u+1}^j (R_i + 2) \left(\frac{g_2(x_i; \alpha, \lambda)^\beta \ln(g_2(x_i; \alpha, \lambda))}{1 + g_2(x_i; \alpha, \lambda)^\beta} \right)^2 \\
&\quad + \sum_{i=1}^{n_u} R_i \left(\frac{g_1(\tau; \alpha)^\beta \ln(g_1(\tau; \alpha))}{1 + g_1(\tau; \alpha)^\beta} \right)^2 \\
&\quad - \sum_{i=n_u+1}^j R_i^{**} \frac{g_2(T_0; \alpha, \lambda)^\beta [\ln(g_2(T_0; \alpha, \lambda))]^2}{1 + g_2(T_0; \alpha, \lambda)^\beta} \\
&\quad + \sum_{i=n_u+1}^j R_i^{**} \left(\frac{g_2(T_0; \alpha, \lambda)^\beta \ln(g_2(T_0; \alpha, \lambda))}{1 + g_2(T_0; \alpha, \lambda)^\beta} \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
I_{33} &= \frac{\partial^2 \ln L}{\partial \lambda^2} = \sum_{i=n_u+1}^j \frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \lambda^2} + (\beta - 1) \sum_{i=n_u+1}^j \left[\frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \lambda^2} - \left(\frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda} \right)^2 \right] \\
&\quad - \sum_{i=n_u+1}^j (R_i + 2) \frac{\beta [g_2(x_i; \alpha, \lambda)]^{\beta-1} \frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \lambda^2} + \beta(\beta - 1) [g_2(x_i; \alpha, \lambda)]^{\beta-2} \left(\frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda} \right)^2}{1 + [g_2(x_i; \alpha, \lambda)]^\beta} \\
&\quad + \sum_{i=n_u+1}^j (R_i + 2) \left(\frac{\beta [g_2(x_i; \alpha, \lambda)]^{\beta-1} \frac{\partial g_2(x_i; \alpha, \lambda)}{\partial \lambda}}{1 + [g_2(x_i; \alpha, \lambda)]^\beta} \right)^2 \\
&\quad - \sum_{i=n_u+1}^j R_i^{**} \frac{\beta [g_2(T_0; \alpha, \lambda)]^{\beta-1} \frac{\partial^2 g_2(T_0; \alpha, \lambda)}{\partial \lambda^2} + \beta(\beta - 1) [g_2(T_0; \alpha, \lambda)]^{\beta-2} \left(\frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \lambda} \right)^2}{1 + [g_2(T_0; \alpha, \lambda)]^\beta} \\
&\quad + \sum_{i=n_u+1}^j R_i^{**} \left(\frac{\beta [g_2(T_0; \alpha, \lambda)]^{\beta-1} \frac{\partial g_2(T_0; \alpha, \lambda)}{\partial \lambda}}{1 + [g_2(T_0; \alpha, \lambda)]^\beta} \right)^2,
\end{aligned}$$

where

$$\frac{\partial^2 g_2(x_i; \alpha, \lambda)}{\partial \lambda^2} = 2\alpha\lambda^{-3}(x - \tau)e^{\alpha(\tau + \lambda^{-1}(x - \tau))} + \alpha^2\lambda^{-4}(x - \tau)^2e^{\alpha(\tau + \lambda^{-1}(x - \tau))}.$$

The exact mathematical expressions for $J(\alpha, \beta, \lambda) = E(I, \alpha, \beta, \lambda)$ are complex to derive. Consequently, we can use the observed Fisher information matrix in place of the theoretical Fisher information matrix. The variance-covariance matrix can be approximated as $V_{ij} = I_{ij}^{-1}$. The asymptotic distribution of the maximum likelihood estimator can be expressed as follows (see Miller [12]).

$$\left[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{\lambda} - \lambda) \right] \sim N_3(0, V). \tag{5}$$

Given that V is a function of the parameters $\alpha, \beta,$ and λ , we substitute the corresponding maximum likelihood estimates (MLEs) to derive an estimate of V , which we denote as \hat{V} . Utilizing this estimate, we can construct approximate $100(1 - \gamma)\%$ confidence intervals for the parameters $\alpha, \beta,$ and λ as follows:

$$\hat{\alpha} \pm Z_{\gamma/2} \sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm Z_{\gamma/2} \sqrt{\hat{V}_{22}}, \quad \hat{\lambda} \pm Z_{\gamma/2} \sqrt{\hat{V}_{33}},$$

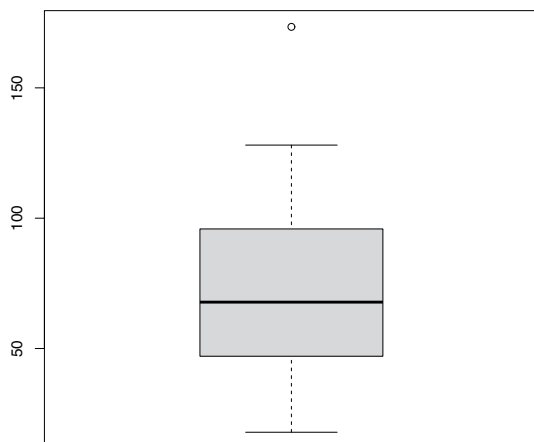
where Z_γ is the upper 100γ -th percentile of the standard normal distribution.

3 Real Data Analysis

In this section, we analyze a real data set originally presented by Lieblein and Zelen [13]. This data set, frequently cited in subsequent research, provides the number of million revolutions before failure for 23 ball bearings. The data is as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

The dataset consists of $n = 23$ observations, with a mean of 72.22, a median of 67.80, and a standard deviation of 36.66. The minimum value in the dataset is 17.88, while the maximum is 173.40. The first quartile (Q_1) and third quartile (Q_3) values

Fig. 1 The box-plot for 23 ball bearings



are 47.04 and 95.88, respectively, indicating a moderate spread around the central value. The box-plot is given in Fig. 1:

The P-P plots for the Logistic, Exponential, Logistic-Exponential, and Normal distributions are given in Fig. 2. The P-P plot visually evaluates how closely the empirical data follows the theoretical distribution. If the points closely follow the reference line, it suggests a good fit. It is evident that the Logistic-Exponential distribution fits the given data well.

Figure 3 shows histograms with the fitted PDFs (left panel) and empirical and theoretical CDFs (right panel) for the Logistic, Exponential, Logistic-Exponential, and Normal distributions. All parameters used here are provided in Table 2.

The maximum likelihood estimates and standard errors for involved parameters of the Logistic-Exponential, Exponential, Logistic, and Normal distributions are given in Table 2. The result shown in this table indicates a variety of distributions that could model different data types. The Logistic-Exponential distribution suggests increasing event rates, while the normal distribution indicates significant variability. The pro-

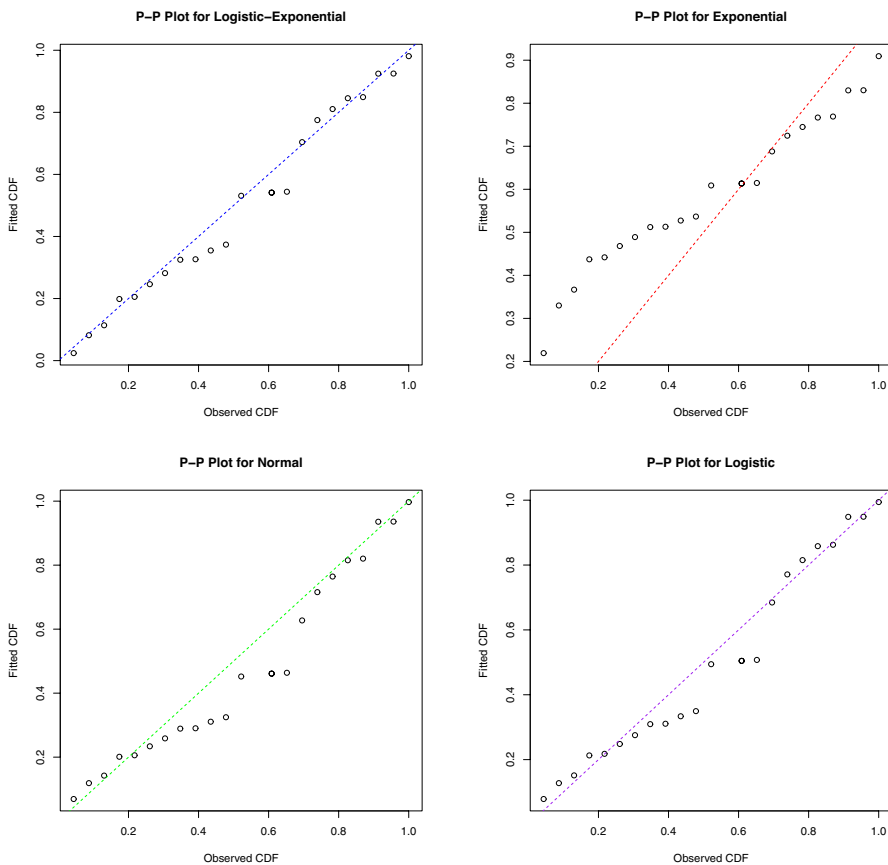


Fig. 2 P-P plots for the Logistic, Exponential, Logistic-Exponential, and Normal distributions, using the parameters provided in Table 2

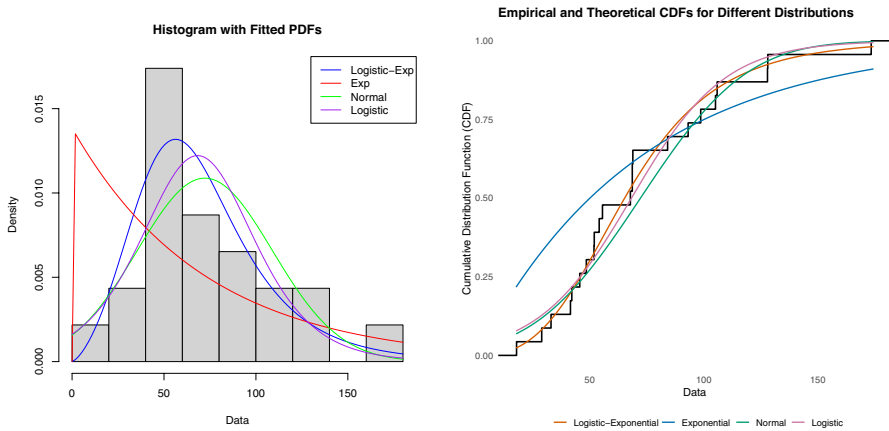


Fig. 3 The histogram with the PDFs (left) and the empirical and theoretical CDFs (right), using the parameters provided in Table 2

Table 2 Maximum likelihood estimates and standard errors for parameters

Distribution	Parameter 1	SE	Parameter 2	SE
Logistic-exponential	$\alpha = 0.0106$	0.0011	$\beta = 2.3630$	0.4135
Exponential	$\lambda = 0.0138$	0.0029	NA	NA
Normal	$\mu = 72.2243$	7.6451	$\sigma = 36.6637$	7.7059
Logistic	Location = 68.2713	7.4521	Scale = 20.4619	3.5552

vided standard errors give insight into the reliability of these estimates, with smaller values indicating more precise estimates.

To compare the two distribution models, we utilize criteria such as the Kolmogorov-Smirnov (KS) test, Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) for the data set. A model is considered better if it yields lower values for KS, AIC, and CAIC.

$$AIC = -2 \ln(L) + 2k.$$

$$CAIC = AIC + \frac{2k(k + 1)}{n - k - 1}.$$

$$BIC = -2 \ln(L) + k \ln(n).$$

In the Kolmogorov-Smirnov test, a small p-value (typically $p < 0.05$) indicates that there is significant evidence against the null hypothesis, suggesting that the sample does not come from the specified distribution. A large p-value indicates that the data is consistent with the specified theoretical distribution. The results are given in Table 3.

The values in Table 3 suggest that the Logistic-Exponential distribution provides a superior fit compared to all other models.

Table 3 The values of $-\log L$, AIC, CAIC, BIC, KS test and p -value

Distribution	$-\log L$	AIC	CAIC	BIC	KS-Statistic	KS p -value
Logistic-Exponential	113.244	230.489	231.089	232.760	0.10802	0.92499
Exponential	121.435	244.870	245.060	246.005	0.30679	0.02029
Normal	115.489	234.977	235.577	237.248	0.18851	0.34319
Logistic	195.243	394.485	395.085	396.756	0.14474	0.66837

4 Simulation

In this section, Monte Carlo simulation techniques were employed to estimate the unknown parameters of the distribution and the acceleration factor (AF). The maximum likelihood estimates (MLEs) for Type-I Progressive Hybrid Censoring Scheme (PHCS) are compared based on Root Mean Square Errors (RMSEs) and Relative Absolute Biases (RABs). The simulation was conducted with predetermined values for n, m, τ and T_0 .

A Markov Chain Monte Carlo (MCMC) algorithm is employed to obtain samples from the Logistic-Exponential distribution. Rather than sampling directly from this distribution, we can utilize the Metropolis-Hastings algorithm, which serves as an extension of the traditional rejection-acceptance sampling method. For more details on MCMC and the Metropolis-Hastings algorithms, one may refer to the book by Robert and Casella [14]. The proposed algorithm is outlined as follows:

- Step 1. Choose the values of n and $\theta = (\alpha, \beta, \lambda)$.
- Step 2. Choose a starting point $x^{(t)}$ such that $\psi(x^{(t)}) > 0$.
- Step 3. Generate Y_t from a proposal distribution $q(y|x^{(t)})$, where q is symmetric.
- Step 4. Set $X^{(t+1)} = Y_t$ with probability $\rho(x^{(t)}, Y_t)$ or otherwise set $X^{(t+1)} = x^{(t)}$ with probability $1 - \rho(x^{(t)}, Y_t)$, where $\rho(x, y)$ is defined as:

$$\rho(x, y) = \min \left(1, \frac{\psi(y)q(x|y)}{\psi(x)q(y|x)} \right).$$

Here, $\psi(\cdot)$ represents the target density.

- Step 5. Evaluate the MLEs.
- Step 6. Continue the process until the chain has stabilized and the samples represent the target distribution accurately.

From Tables 4 and 5, the results show that larger sample sizes ($n = 150, 200$) generally yield more precise estimates of the parameters $\alpha = 0.2$, $\beta = 0.1$, and $\lambda = 1.5$, as indicated by lower RMSE and RAB values. This improvement in accuracy with increasing n is consistent with statistical theory, as larger samples enhance estimation reliability. These findings suggest that, under an adaptive Type-I progressively hybrid censoring scheme, larger sample sizes and moderate censoring levels are preferable for reliable parameter estimation.

Figure 4 presents the results from the MCMC algorithm applied to the SS-PALT model under adaptive Type-I progressively hybrid censored data. The figure illus-

Table 4 MLE, RMSE and RAB values under APHC for the true parameters $\alpha = 0.2, \beta = 0.1$, and $\lambda = 1.5$, and for different values of n, m, τ , and T_0

Scheme	(n, m)	τ	T_0	MLE		RMSE		RAB		λ		
				α	β	α	β	α	β			
1	(80, 50)	0.6	0.8	0.6769	0.1110	1.8489	0.4831	0.0110	0.3489	2.4157	0.1098	0.2326
2	(100, 60)	0.6	0.8	0.5721	0.1089	1.7730	0.3767	0.0089	0.2730	1.8834	0.0885	0.1820
3	(120, 70)	0.6	0.8	0.4468	0.1059	1.6820	0.2525	0.0059	0.1820	1.2623	0.0590	0.1213
4	(150, 100)	0.6	0.8	0.2469	0.1012	1.5350	0.0491	0.0012	0.0350	0.2457	0.0116	0.0233
5	(200, 100)	0.6	0.8	0.2189	0.1004	1.5140	0.0195	0.0004	0.0140	0.0974	0.0044	0.0093
1	(80, 50)	0.7	0.8	0.7545	0.1118	1.9058	0.5591	0.0118	0.4058	2.7954	0.1179	0.2705
2	(100, 60)	0.7	0.8	0.5032	0.1065	1.7230	0.3112	0.0065	0.2230	1.5560	0.0646	0.1486
3	(120, 70)	0.7	0.8	0.3618	0.1033	1.6190	0.1646	0.0033	0.1190	0.8231	0.0330	0.0793
4	(150, 100)	0.7	0.8	0.2469	0.1011	1.5350	0.0491	0.0011	0.0350	0.2457	0.0113	0.0233
5	(200, 100)	0.7	0.8	0.2093	0.1002	1.5070	0.0099	0.0002	0.0070	0.0494	0.0020	0.0047
1	(80, 50)	0.6	1	0.7254	0.1120	1.8850	0.5306	0.0120	0.3850	2.6528	0.1205	0.2567
2	(100, 60)	0.6	1	0.4470	0.1057	1.6820	0.2522	0.0057	0.1820	1.2608	0.0569	0.1213
3	(120, 70)	0.6	1	0.4386	0.1056	1.6750	0.2414	0.0056	0.1750	1.2071	0.0559	0.1167
4	(150, 100)	0.6	1	0.3124	0.1025	1.5839	0.1181	0.0025	0.0839	0.5903	0.0246	0.0559
5	(200, 100)	0.6	1	0.2469	0.1011	1.5350	0.0491	0.0011	0.0350	0.2457	0.0110	0.0233
1	(80, 50)	0.7	1	0.6574	0.1101	1.8315	0.4642	0.0101	0.3315	2.3211	0.1012	0.2210
2	(100, 60)	0.7	1	0.4773	0.1061	1.7030	0.2795	0.0061	0.2030	1.3977	0.0615	0.1353
3	(120, 70)	0.7	1	0.3505	0.1036	1.6120	0.1567	0.0036	0.1120	0.7837	0.0362	0.0747
4	(150, 100)	0.7	1	0.2565	0.1013	1.5420	0.0587	0.0013	0.0420	0.2937	0.0132	0.0280
5	(200, 100)	0.7	1	0.2480	0.1011	1.5350	0.0480	0.0011	0.0350	0.2400	0.0107	0.0233
1	(80, 50)	0.8	1	0.6263	0.1090	1.8150	0.4377	0.0090	0.3150	2.1885	0.0905	0.2100
2	(100, 60)	0.8	1	0.5233	0.1069	1.7367	0.3295	0.0069	0.2367	1.6477	0.0693	0.1578
3	(120, 70)	0.8	1	0.4095	0.1042	1.6535	0.2129	0.0042	0.1535	1.0646	0.0420	0.1024
4	(150, 100)	0.8	1	0.2745	0.1015	1.5560	0.0791	0.0015	0.0560	0.3954	0.0147	0.0373
5	(200, 100)	0.8	1	0.2096	0.1002	1.5070	0.0096	0.0002	0.0070	0.0480	0.0016	0.0047

Table 5 MLE, RMSE and RAB values under APHC for the true parameters $\alpha = 0.2, \beta = 0.1$, and $\lambda = 1.5$, and for different values of n, m, τ , and T_0

Scheme	(n, m)	τ	T_0	MLE		RMSE		RAB		λ		
				α	β	α	β	α	β			
1	(80, 50)	0.9	1	0.7336	0.1103	1.8884	0.5416	0.0103	0.3884	2.7080	0.1034	0.2589
2	(100, 60)	0.9	1	0.4360	0.1047	1.6750	0.2440	0.0047	0.1750	1.2200	0.0469	0.1167
3	(120, 70)	0.9	1	0.3533	0.1028	1.6120	0.1539	0.0028	0.1120	0.7694	0.0282	0.0747
4	(150, 100)	0.9	1	0.2762	0.1014	1.5560	0.0774	0.0014	0.0560	0.3869	0.0136	0.0373
5	(200, 100)	0.9	1	0.2192	0.1004	1.5140	0.0192	0.0004	0.0140	0.0960	0.0038	0.0093
1	(80, 50)	0.6	1.2	0.7345	0.1132	1.8920	0.5407	0.0132	0.3920	2.7037	0.1318	0.2613
2	(100, 60)	0.6	1.2	0.5044	0.1071	1.7240	0.3101	0.0071	0.2240	1.5503	0.0714	0.1493
3	(120, 70)	0.6	1.2	0.3618	0.1038	1.6190	0.1646	0.0038	0.1190	0.8231	0.0380	0.0793
4	(150, 100)	0.6	1.2	0.2858	0.1020	1.5630	0.0870	0.0020	0.0630	0.4349	0.0196	0.0420
5	(200, 100)	0.6	1.2	0.2189	0.1005	1.5140	0.0195	0.0005	0.0140	0.0974	0.0047	0.0093
1	(80, 50)	0.7	1.2	0.5777	0.1083	1.7800	0.3903	0.0083	0.2800	1.9514	0.0828	0.1867
2	(100, 60)	0.7	1.2	0.5052	0.1063	1.7234	0.3092	0.0063	0.2234	1.5460	0.0627	0.1490
3	(120, 70)	0.7	1.2	0.2949	0.1023	1.5700	0.0971	0.0023	0.0700	0.4857	0.0229	0.0467
4	(150, 100)	0.7	1.2	0.2748	0.1017	1.5560	0.0788	0.0017	0.0560	0.3940	0.0171	0.0373
5	(200, 100)	0.7	1.2	0.2093	0.1002	1.5070	0.0099	0.0002	0.0070	0.0494	0.0021	0.0047
1	(80, 50)	0.8	1.2	0.7624	0.1121	1.9130	0.5704	0.0121	0.4130	2.8520	0.1213	0.2753
2	(100, 60)	0.8	1.2	0.5052	0.1063	1.7240	0.3092	0.0063	0.2240	1.5460	0.0628	0.1493
3	(120, 70)	0.8	1.2	0.3810	0.1037	1.6330	0.1838	0.0037	0.1330	0.9191	0.0375	0.0887
4	(150, 100)	0.8	1.2	0.3239	0.1027	1.5904	0.1257	0.0027	0.0904	0.6283	0.0272	0.0603
5	(200, 100)	0.8	1.2	0.2186	0.1005	1.5140	0.0198	0.0005	0.0140	0.0989	0.0048	0.0093
1	(80, 50)	0.9	1.2	0.6368	0.1092	1.8220	0.4464	0.0092	0.3220	2.2322	0.0918	0.2147
2	(100, 60)	0.9	1.2	0.5233	0.1066	1.7380	0.3295	0.0066	0.2380	1.6477	0.0656	0.1587
3	(120, 70)	0.9	1.2	0.3717	0.1032	1.6260	0.1739	0.0032	0.1260	0.8697	0.0323	0.0840
4	(150, 100)	0.9	1.2	0.2762	0.1015	1.5560	0.0774	0.0015	0.0560	0.3869	0.0153	0.0373
5	(200, 100)	0.9	1.2	0.2093	0.1002	1.5070	0.0099	0.0002	0.0070	0.0494	0.0020	0.0047

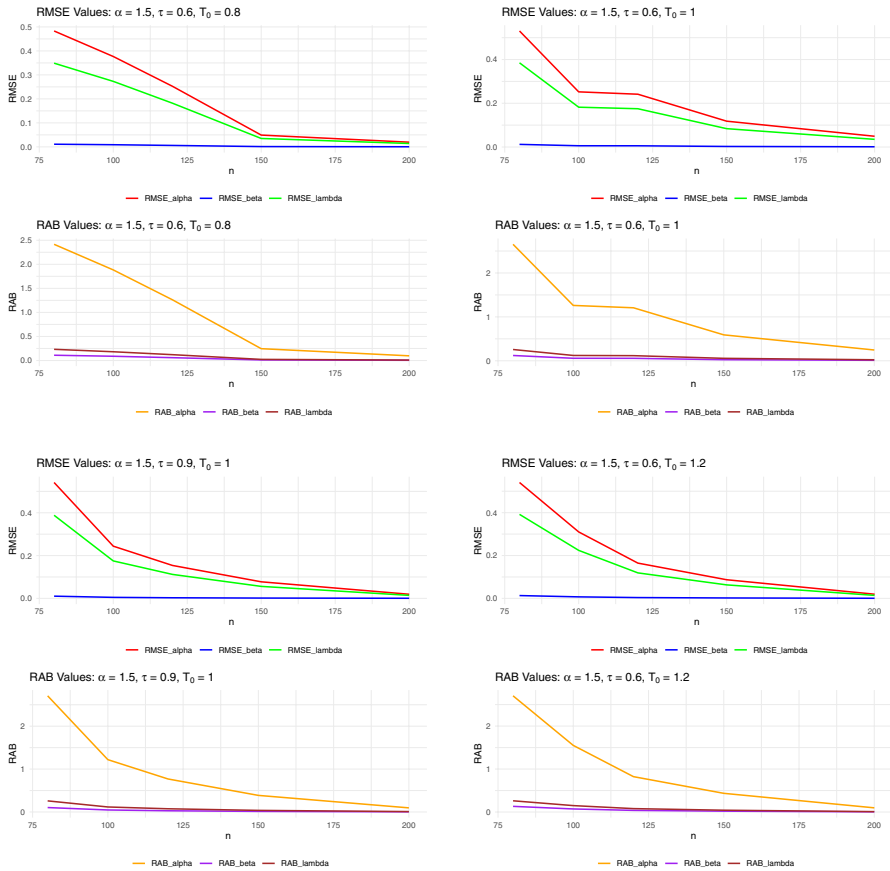


Fig. 4 The RMSEs and RABs plots for different parameters ($\alpha = 1.5, \beta = 0.1$ and $\lambda = 1.5$) and some values of τ and T_0 as shown in each figure

trates the Root Mean Square Errors (RMSEs) and Relative Absolute Biases (RABs) for the parameters $\alpha = 1.5, \beta = 0.1$, and $\lambda = 1.5$, across various values of τ and T_0 . Although results for other parameter values are available in separate figures, they are not included here for simplicity.

5 Conclusion

The estimation and analysis of failure time data were performed within the SSPALT framework, employing an adaptive Type-I progressive hybrid censoring method based on a maximum likelihood approach. It is assumed that the lifetimes of the test units follow a Logistic-Exponential distribution. The maximum likelihood estimates (MLEs) of the model parameters were computed, and their performance was assessed using the root mean square error (RMSE) and relative absolute bias (RAB). The results demonstrate that the adaptive progressive hybrid censoring scheme yields

more efficient estimators for the parameters. Although the real dataset sample size used in this study was relatively small, extending the analysis to larger samples and examining the asymptotic behavior of the estimators would provide valuable additional insights. Based on the current findings, we have the intuition that the results would not change substantially, as the efficiency advantages of the adaptive progressive hybrid censoring method appear robust; however, this remains an interesting challenge for future investigation. Overall, if time is not a critical factor for the experimenter, the adaptive progressive hybrid censoring method is recommended for achieving greater efficiency in parameter estimation.

Acknowledgements The author expresses their sincere gratitude to the editor and anonymous reviewers for their invaluable comments and suggestions. The reviewers' critical and insightful feedback was particularly helpful, playing a pivotal role in shaping and refining the paper into its current form.

Author Contributions Not applicable.

Data Availability We have used secondary data. All data sets are available in the citations given.

Declarations

Conflict of Interest The author's declare that they have no conflict of interest.

Ethical Approval Not applicable.

Consent for Publication Not applicable.

Consent to Participate Author read and proved the final version of this article.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

References

1. Balakrishnan, N.: Progressive censoring methodology: an appraisal. *TEST* **16**, 211–296 (2007)
2. Balakrishnan, N., Aggarwala, R.: *Progressive Censoring: Theory, Methods and Applications*. Birkhäuser, Boston (2000)
3. Balakrishnan, N., Iliopoulos, G.: Stochastic monotonicity of the mle of exponential mean under different censoring schemes. *Ann. Inst. Stat. Math.* **61**(3), 753–772 (2009)
4. Liang, T.C., Yang, M.-C.: Optimal bayes sampling plans for exponential distributions based on hybrid censored samples. *J. Stat. Comput. Simul.* **83**(5), 922–940 (2013)
5. Lin, C.T., Ng, H.K.T., Chan, P.S.: Statistical inference of type-II progressively hybrid censored data with Weibull lifetimes. *Commun. Stat.-Theor. M* **38**(10), 1710–1729 (2009)

6. Shakoor, A., Khan, S.: Inference for step-stress partially accelerated life test model with an adaptive type-I progressively hybrid censored data. *J. Mod. Appl. Stat. Methods* **20**(1), 1–15 (2021)
7. Khawar Kiani, S., Aslam, M., Bhatti, M.I.: Investigation of half-normal model using informative priors under Bayesian structure. *Stat. Transition New Ser.* **24**(4), 19–36 (2023)
8. Basit, Z., Bhatti, M.I.: Efficient classes of estimators of population variance in two-phase successive sampling under random non-response. *Statistica (Bologna)* **82**(2), 177–198 (2022)
9. Basit, Z., Masood, S., Bhatti, I.: A class of estimators for estimation of population mean under random non-response in two phase successive sampling. *J. Stat. Theory Appl.* **22**(4), 309–338 (2023)
10. DeGroot, M.H., Goel, P.K.: Bayesian estimation and optimal design in partially accelerated life testing. *Nav. Res. Logist. Q.* **26**(2), 223–235 (1979)
11. Lone, S.A., Rahman, A., Tarray, T.A.: Inference for step-stress partially accelerated life test model with an adaptive type-I progressively hybrid censored data. *J. Mod. Appl. Stat. Methods* **19**(1), 13 (2021)
12. Miller, J.R.: *Survival Analysis*. John Wiley & Sons, New York (1981)
13. Lieblein, J., Zelen, M.: Statistical investigation of the fatigue life of deep-groove ball bearings. *J. Res. Natl. Bur. Stand.* **57**, 273–315 (1956)
14. Robert, C.P., Casella, G.: *Monte Carlo Statistical Methods*, 2nd edn. Springer (2013)