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On the behavior of growth of polygons in semi-regular hyperbolic tessellations

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Abstract: In this work we consider tessellations (or tilings) of the hyperbolic plane by copies of a semi-regular polygon with alternating angles and we study the behavior of the growth of the polygons, edges, and vertices when the distance increase from a fixed initial polygon.

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1. Introduction

It is a basic fact of hyperbolic geometry that there exists a tessellation (tiling) of the hyperbolic plane by a regular polygon with p sides and with q other p -gons meeting in each vertex if, and only if, $1/p + 1/q < 1/2$. Beside the regular case, the study of non regular tilings, which includes archimedean e quasiregular tilings, is an active research topic, see Dolbilin and Schattschneider (1997), Durand (1999), Lucic and Molnár (1990), Martinez (1986), and Sadoc (1990) (with applications in crystallography).

Fixed a p -gon P_0 , with alternating inner angles π/m and π/n , let F_1 be the family of copies of P_0 obtained either by reflections (if $n = m$) or translations of it on its edges. Let F_2 obtained from F_1 in the same way, and so on. Giving the family F_k , subset of a tessellation $\{p, m, n\}$, how is the behavior of vertices, edges, and polygons when F_k grows? If k is really great, from a fixed external vertex v of

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This work may have applications in communication theory, in order to produce signal constellations in the hyperbolic plane, a situation in which a finite number of tiles of a tessellation with suitable properties needs to be chosen. In such a situation, the constructive approach adopted in this work may be useful.

PUBLIC INTEREST STATEMENT

From simple geometric arguments, we may prove that the Euclidean plane may be paved with copies of a same regular polygon using only squares, hexagons, and equilateral triangles. Is the same property true in the Hyperbolic plane? Roughly speaking, hyperbolic geometry is a geometry which does not satisfy Euclide's fifth Axiom: in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the given point. It may proved that there are infinite options of regular polygons to pave the Hyperbolic plane.

On the other hand, when we are interested to pave a plane, either Euclidean or Hyperbolic using non regular polygons, we are faced with a much more complicated problem. In this work, we give conditions for the existence of tilings, using semi-regular polygons which have alternating inner angles.

F_k , is it possible that the tiles which origin from the edges of v will intercept, creating a closed “hole” in some next set $F_j, j \gg k$?

This question is meaningful. For instance, if \mathcal{T} is a torus and $\{P_j\}$ is a tiling of \mathcal{T} by squares, fixing a square P_0 and considering the sets F_k as described above, for k great enough, the external tiles of F_k will meet, giving origin to a “hole”. This may happen even for a noncompact surface. If C is a infinite circular cylinder, considering a square tiling of C we will also get a “hole” for F_k with k great enough.

In a more technical setting, in this work we are interested in tessellations (or tilings) of the hyperbolic plane by copies of a semi-regular polygon, which has edges with same length and alternating angles, and the distance in consideration is the natural distance within the usual setting of group theory. Given two of such tiles, we first take an interior point in each of them and consider continuous paths connecting those points avoiding the vertices of the polygons in the tessellation. Then, the distance between the tiles will be the minimal number of edges crossed by such a path. Fixed a polygon in a such tessellation called initial polygon, we study the behavior of the growth of the tessellation from this initial point and we prove several geometric results, including there is not holes when the tessellation grows.

We also call the attention that hyperbolic geometry has become an important mathematical tool in communication theory (Silva, Costa, Firer, & Palazzo, 2006; Silva, Firer & Palazzo, 2003) and quantum error correction codes (Albuquerque, Palazzo, & Silva, 2009), where the tilings play a fundamental role.

2. The semi-regular hyperbolic tessellations $\{p, m, n\}$

Let us consider in the hyperbolic plane \mathbf{H}^2 a polygon $\mathcal{P}_{p,m,n}$ such that it has p equal-length edges and p alternating internal angles $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$, where m and n are positive integers. If $m = n = q$, such polygons exist for every positive integers p and q , whenever $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, and in this case we have a regular polygon denoted by $\mathcal{P}_{p,q}$. If $m \neq n$, then p must be even, and the following theorem (see Beardon, 1983, Th. 7.16.2) guarantees its existence.

THEOREM 1 *Let m, n and r be integers such that $m, n > 2$ and $r \geq 2$. Then, there exists a polygon P with $p = 2r$ equal-length edges and internal angles alternating between $2\pi/m$ and $2\pi/n$, if and only if*

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{r} < 1.$$

Besides, P has incenter.

It is possible to tile \mathbf{H}^2 with $\mathcal{P}_{p,q}$, for all $1/p + 1/q < 1/2$, in the sense there is a family $\{\mathcal{P}_n | n \in \mathbf{N}\}$ of isometric copies of $\mathcal{P}_{p,q}$ such that $\mathbf{H}^2 = \cup_{n \in \mathbf{N}} \mathcal{P}_n$ and in the case when $\mathcal{P}_n \cap \mathcal{P}_m \neq \emptyset$, this intersection is either a common edge or a common vertex. We say that the family $\mathcal{T}_{p,q} = \{\mathcal{P}_n | n \in \mathbf{N}\}$ defines a *tiling (or tessellation)* and each \mathcal{P}_n is called a *tile*. An *edge (vertex) of the tiling* is an edge (vertex) of a tile. We denote, respectively, by $\mathcal{E}_{p,q}$ and $\mathcal{V}_{p,q}$ the set of edges and vertices of $\mathcal{T}_{p,q}$. For some polygons $\mathcal{P}_{p,m,n}$ it is also possible to tile \mathbf{H}^2 , which is given for the following result.

THEOREM 2 *Let m, n and l be integers such that $m, n, l \geq 3$ and satisfying $\frac{1}{m} + \frac{1}{n} = \frac{2}{l}$. Then, there exists a family $\mathcal{T}_{p,m,n} = \{P_j | j \in \mathbf{N}\}$ of polygons $P_j = \mathcal{P}_{p,m,n}^j$ with $p = 2k, k \geq 2$, which tile \mathbf{H}^2 .*

Proof Initially, we consider a more general construction. Let k, l be integers, $k \geq 2$ and $l \geq 3$. Construct a hyperbolic triangle with angles of $\pi/k, \alpha$, and β such that $\alpha + \beta = \pi/l$. Apply reflections in opposite sides to α and β (also for images of the triangle). As a result, we obtain an equilateral semi-regular $2k$ -gon with alternating angles equal to 2α and 2β . The symmetry group of the $2k$ -gon is the dihedral group D_k , which has the rotation of order k and k reflections in straight lines.

Take the mid-point of each side of the $2k$ -gon as the center of half-rotation and rotate the polygon around each center by angle of π . In this way, we obtain neighbors along all $2k$ sides of the polygon. Apply the rotation around each vertex of the $2k$ -gon by angle of π/l . Repeating the rotations at each vertex of new polygons yields a tiling of the hyperbolic plane. The fundamental domain of this tiling is the first constructed triangle. To obtain desired tilings of theorem, we take the angles α and β be equal to π/m and π/n , respectively. We find such natural numbers m, n , and l by solving the Diophantine equation $1/m + 1/n = 2/l$, where $m \geq 3, n \geq 3, l \geq 3, m, n$, and l are integers. \square

Remark 3 We list some sets of solutions of the equation $1/m + 1/n = 2/l$: $(m, n, l) = (3, 6, 4), (3, 15, 5), (4, 12, 6), (4, 28, 7), (5, 20, 8), (5, 45, 9), (6, 12, 8), (6, 18, 9), (6, 30, 10), (6, 66, 11), (7, 42, 12), (7, 91, 13), (8, 24, 12), (8, 56, 14), (8, 120, 15), (9, 18, 10), (9, 45, 15), (9, 72, 16), (9, 153, 17), (10, 15, 12), (10, 30, 15), (10, 40, 16), (10, 90, 18), (10, 190, 19)$, and so on.

Of course, if $m = n$ then $q = m$ and we recover the regular case, for any p, q such that $1/p + 1/q < 1/2$.

Let us denote by Δ_0 the initial tile of our tessellation and let p_0 be its incenter, and $\Gamma_{p,m,n}$ denotes the group of isometries which generate the tiling such that every tile $\Delta_n \in \mathcal{T}_{p,m,n}$ is the image $f(\Delta_0)$ for a unique $f \in \Gamma_{p,m,n}$. If we consider on $\Gamma_{p,m,n}$ the Cayley metric $d_c(\cdot, \cdot)$, we find that

$$d(f(\Delta_0), h(\Delta_0)) = d_c(f, h), \quad \forall f, h \in \Gamma_{p,m,n}. \tag{1}$$

Since $d(g(\Delta_0), h(\Delta_0)) = d_c(g, h), \quad \forall g, h \in \Gamma_{p,m,n}$ we may consider metric constructions in $\mathcal{T}_{p,m,n}$ or in $\Gamma_{p,m,n}$ without any distinction.

Let B_k denotes the closed ball in $\Gamma_{p,m,n}$ with center in the identity and with radius k , and let $C_k := B_k \setminus B_{k-1}$ be the circumference of radius k , which we call the k -th stage (level) of the tessellation:

$$B_k := \{g \in \Gamma_{p,m,n} \mid d_c(g, e) \leq k\},$$

$$C_k := \{g \in \Gamma_{p,m,n} \mid d_c(g, e) = k\}.$$

From the correspondence between the elements of the group and the polygons of the tessellation, we denote by P_k the set of polygons of tessellation corresponding to the ball B_k in the group, and by $NP_k := P_k \setminus P_{k-1}$ the set corresponding to the circumference C_k , where the prefix N stands for new:

$$P_k = \{g(\Delta_0) \mid g \in B_k\},$$

$$NP_k = \{g(\Delta_0) \mid g \in C_k\}.$$

Remark 4 Naturally, we have that

$$|P_k| = |B_k|, \quad |NP_k| = |C_k|$$

where $|P_k|$ the cardinality of P_k , is the k -th coefficient of the growth series of the group $\Gamma_{p,m,n}$. This suggests, what is actually true, that the techniques used in this work may be used to determine the growth series of the group $\Gamma_{p,m,n}$ studied in Bartholdi and Ceccherini-Silberstein (2002) and Cannon (1984), for $m = n$ and for (Floyd & Plotnick, 1994) for some particular cases of $m \neq n$.

In this work, we study the growth of the tessellations $\mathcal{T}_{p,m,n}$ and hence the growth of the groups $\Gamma_{p,m,n}$ considering the topological shape of this growth, in the senses that, for every $k \geq 0$, the set P_k has a trivial topology.

3. The growth of the tessellation $\mathcal{T}_{p,m,n}$

Considering the polygons in P_{k-1} , let us observe that $\varepsilon \in E_{k-1}$ is an edge of either one or two polygons in P_{k-1} : if ε is an edge of only one polygon $\Delta \in P_{k-1}$, we have $\rho_\varepsilon(\Delta)$ (the isometry giving the neighbor polygon in the next level) is a new polygon in NP_k ; if ε is an edge of two polygons in P_{k-1} , then the

polygon gives rise to none new element in P_k . If an edge $\varepsilon \in E_{k-1}$ is edge of only one polygon, then $\varepsilon \in NE_{k-1}$.

We define the function $t_k: V_k \rightarrow \{2, \dots, q\}$ which for each vertex $v \in V_k$ gives the number of edges of E_k which have v as vertex. We say $t_k(v)$ is the k -type of the vertex v , and the notation v_k^w is used to say that $w = t_k(v)$. We also need the following notations:

$$V_k = \{v_{k,1}^{w_1}, v_{k,2}^{w_2}, \dots, v_{k,|V_k|}^{w_{|V_k|}}\} \text{ and } NV_k = \{v_{k,1}^{w_1}, v_{k,2}^{w_2}, \dots, v_{k,|NV_k|}^{w_{|NV_k|}}\}.$$

where k is the stage of the tessellation and $w_i = t_k(v_{k,i})$ with $i = 1, \dots, |V_k|$ for V_k and $i = 1, \dots, |NV_k|$ for NV_k .

Considering an edge $\varepsilon \in E_k$, we characterize it using the k -type of both its vertices: we say that ε has k -**type** $(t_k(i(\varepsilon)), t_k(\tau(\varepsilon)))$, where $i(\varepsilon)$ and $\tau(\varepsilon)$ are the initial and final vertices, respectively, which determine ε . It is assumed that $t_k(i(\varepsilon)) \leq t_k(\tau(\varepsilon))$ and, without ambiguity, we may assume the notation $\varepsilon_k^{i_j, w_j}$ where $i_j = t_k(i(\varepsilon))$ and $w_j = t_k(\tau(\varepsilon))$ with $j = 1, \dots, |E_k|$ for E_k and $j = 1, \dots, |NE_k|$ for NE_k . In a similar way as done for the vertices, we use the following notation for the set of edges in a given level k :

$$E_k = \{\varepsilon_{k,1}^{i_1, w_1}, \varepsilon_{k,2}^{i_2, w_2}, \dots, \varepsilon_{k,|E_k|}^{i_{|E_k|}, w_{|E_k|}}\} \text{ and } NE_k = \{\varepsilon_{k,1}^{i_1, w_1}, \varepsilon_{k,2}^{i_2, w_2}, \dots, \varepsilon_{k,|NE_k|}^{i_{|NE_k|}, w_{|NE_k|}}\}.$$

Let us note that the vertices and edges of the tessellation were defined in accordance with the tessellation level and this information will be used to distinguish vertices NV_k from $V_k \setminus NV_k$.

We want to determine the types of vertices and edges which generate new polygons in the next stage and the type of these new vertices and edges.

An edge is k -**outer (external)** if it is an edge of a unique polygon in the stage k , and we say it is **inner (internal)** on the contrary case. When an edge $\varepsilon := \varepsilon_{k,j}^{i_j, \tau_j}$ has $\tau_j = q$ edges adjacent to its terminal vertex, we say that the edges having $\tau(\varepsilon)$ as a vertex is a **closed cycle of edges** if there are also q polygons in P_k having $\tau(\varepsilon)$ as a vertex. We note that $\tau_j = q$ does not imply that we have a closed cycle of edges at τ_j , since there may be an outer edge among those q edges. If $\varepsilon_{k,j}^{i_j, \tau_j}$ is such an outer edge, then the reflection $\rho_{\varepsilon_{k,j}^{i_j, \tau_j}}$ **closes the cycle of the edges**, since q is the maximum number of polygons in the tilling having a common vertex.

Let $v_{k,t}^j \in V_k$ and $\mathcal{A} := \mathcal{A}(v_{k,t}^j) = \{\varepsilon_{k,1}^{i_1, \tau_1}, \dots, \varepsilon_{k,j}^{i_j, \tau_j}\}$ be the set of edges of E_k which have $v_{k,t}^j$ as a vertex, considering the edges numbered anticlockwise. Since we are interested only in the case of vertices which in the stage in question have some external edge (because otherwise all polygons having it as a vertex had already been counted), we may assume \mathcal{A} has one, and consequently, at least, two exterior edges. Thus, we assume the ordinance of the edges of \mathcal{A} is done in such a way that $\varepsilon_{k,1}^{i_1, \tau_1}$ and $\varepsilon_{k,j}^{i_j, \tau_j}$, the first and the last edge, are k -exterior edges. Such an ordering of \mathcal{A} depends only on the choice of the first edge and, as we will see later (following Proposition 6), there is a canonical way to do it.

We begin with some definitions. Two edges with a common vertex are **consecutive** if the angle between them (at the common vertex) is either $\frac{2\pi}{m}$ or $\frac{2\pi}{n}$. If $\varepsilon_{k,i}^{i_i, \tau_i}$ and $\varepsilon_{k,i+1}^{i_{i+1}, \tau_{i+1}}$ are consecutive, for all $i = 1, 2, \dots, j-1$, then the set of all edges of $v_{k,t}^j$ is called a **consecutive set of edges**. Otherwise, **there is a discontinuity** or a hole at the edges containing $v_{k,t}^j$.

Given a set of consecutive edges $\mathcal{A} = \{\varepsilon_{k,1}^{i_1, \tau_1}, \varepsilon_{k,2}^{i_2, \tau_2}, \dots, \varepsilon_{k,j}^{i_j, \tau_j}\}$ of the vertex $v_{k,t}^j$, the only k -exterior edges of \mathcal{A} are $\varepsilon_{k,1}^{i_1, \tau_1}$ and $\varepsilon_{k,j}^{i_j, \tau_j}$, since the edges $\varepsilon_{k,2}^{i_2, \tau_2}, \dots, \varepsilon_{k,j-1}^{i_{j-1}, \tau_{j-1}}$ are edges of two tiles.

In Proposition 1 will be proved the edges of any given level, having a common vertex is always a set of consecutive edges and this will remove any ambiguity in the numbering of $\mathcal{A} = \{\varepsilon_{k,1}^{i_1, \tau_1}, \dots, \varepsilon_{k,j}^{i_j, \tau_j}\}$. To prove this proposition, we will need some results from the theory of Coxeter groups. To avoid the introduction of new terms, we enunciate those results in the context in question (groups $\Gamma_{p,q}$), replacing the original terms “chambers” and “walls” by “polygons” and “edges,” respectively. All of these concepts and results may be found in Humphreys (1990).

Given a set $C \subset \mathbf{H}^2$, its (topological) interior is denoted by $(C)^\circ$ and its closure by \overline{C} . Given a tessellation \mathcal{T} with a set of vertices \mathcal{V} , we say that $c: I \rightarrow \mathbf{H}^2$ is a **regular path** if $c(t)$ is a continuous function and $c(t) \notin \mathcal{V}_{p,m,n}, \forall t \in I = [0, 1]$.

Definition 5 Let $c: I \rightarrow \mathbf{H}^2$ be a regular path with $c(0) \in (g(\Delta_0))^\circ$ and $c(1) \in (h(\Delta_1))^\circ$, with $g, h \in \Gamma_{p,q}$. The simplicial length of c is given by

$$|c|_{g,h} = |\{\text{crossed edges by } c(t), \text{ counted with multiplicities}\}|.$$

A geodesic δ determines two disjoint connected open half-spaces, \mathbf{H}_δ^+ and \mathbf{H}_δ^- such that $\delta = \overline{\mathbf{H}_\delta^+} \cap \overline{\mathbf{H}_\delta^-}$. Hence, a support geodesic may be characterized by the existence of two polygons Δ_1 and Δ_2 contained in each one of these two disjoint half-spaces determined by δ and satisfying that $\overline{\Delta_1} \cap \overline{\Delta_2}$ is an edge contained in δ and, in this case, we say that δ **supports the polygons** Δ_1, Δ_2 , as well as the edge $\overline{\Delta_1} \cap \overline{\Delta_2}$.

Let δ be a support geodesic of the tessellation ($\rho_\delta \in \Gamma_{p,m,n}$). We say δ **separates** the tiles $g(\Delta_0)$ and $h(\Delta_0)$ if those tiles belong to different connected components of $\mathbf{H}^2 \setminus \delta$, that is, every continuous path connecting these polygons intercepts δ .

Let ε be an edge of the tessellation with initial vertex v and δ its support geodesic. We denote by δ_i^+ the geodesic ray with initial point in v and containing ε and δ_i^- your opposite ray.

Given a support geodesic of the tessellation $\delta \in S$ and $\Delta_i, \Delta_j \in P_k$, we are considering the following notation:

$$\begin{aligned} \Delta_i |_\delta \Delta_j &: \delta \text{ separates the polygons } \Delta_i \text{ e } \Delta_j, \\ \Delta_i \nmid_\delta \Delta_j &: \delta \text{ does not separate the polygons } \Delta_i \text{ and } \Delta_j. \end{aligned}$$

Finally, it may be proved that the tessellation $\{p, q\}$ does not generate, in any stage, a set of discontinuous edges with a common vertex.

Proposition 6 Given a vertex $v_k^l \in V_k$, the set of edges with terminal vertex v_k^l is a set of consecutive edges.

Proof The proof will be made by absurd, assuming that some vertex does not possess a consecutive set of edges and thus contradicting the minimality of the distance in some level of the tessellation. We keep the notation introduced above for the group $\Gamma_{p,m,n}$ and the base tile Δ_0 . Whenever there is no ambiguity about the level of the tessellation in consideration, it will be omitted unnecessary indices of the notation.

Let us consider a vertex $v = v_k^l$ with k -type l and $l < q$ (in the case $l = q$ the edges are consecutive because the cycle of edges is closed). The number of edges is counted module q , since the angle between two consecutive edges is either $\frac{2\pi}{m}$ or $\frac{2\pi}{m}$, and we may have at most q edges with v as a vertex.

Let us suppose by absurd that the edges of v may be separated into two disjoint sets of consecutive edges. We order these edges in the counterclockwise direction,

$$\mathcal{A}_1 = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s\} \text{ and } \mathcal{A}_2 = \{\varepsilon_{s+r}, \dots, \varepsilon_l\}$$

with $r > 1$. Let δ_i^+ be the geodesic rays starting at v , which support the edges of ε_i with $i \in \{1, 2, \dots, s, s+r, \dots, l\}$ and δ_i^- be the their opposite ray (both δ_i^+ and δ_i^- are contained in the same geodesic δ_i , with initial point v but opposite direction). Let us consider the tiles that have edges in \mathcal{A}_1 and in \mathcal{A}_2 ,

$$\mathcal{P}_1 = \{\Delta_1, \dots, \Delta_{s-1}\} \text{ and } \mathcal{P}_2 = \{\Delta_{s+r}, \dots, \Delta_{l-s-r}\}$$

ordered in the counterclockwise direction, such that ε_i and ε_{i+1} are the two consecutive edges of Δ_i with common vertex v . By the hypothesis, this is a set with discontinuity of the edges and it follows that ε_1 is not a consecutive edge with ε_l and ε_s is not a consecutive edge with ε_{s+r} , i.e. the angle θ_1 between the geodesic rays δ_1^+ and δ_l^+ which contains ε_1 and ε_l , respectively, has measure $|\theta_1| > \max\left\{\frac{2\pi}{m}, \frac{2\pi}{n}\right\}$. In the same way, the angle θ_2 between the geodesic rays δ_s^+ and δ_{s+r}^+ which contains ε_s and ε_{s+r} , respectively, has measure $|\theta_2| > \max\left\{\frac{2\pi}{m}, \frac{2\pi}{n}\right\}$.

The geodesics $\delta_1, \delta_s, \delta_{s+r}$ and δ_l are not necessarily distinct, but they may not be all equal, since both θ_1 and θ_2 are greater than $\max\left\{\frac{2\pi}{m}, \frac{2\pi}{n}\right\}$. Hence, they determine at least two pairs of distinct hyperplanes and Δ_0 is contained in at least one of them.

This holds since \mathbf{H}^2 is simply connected, that is, the fundamental group is trivial ($\Pi_1(\mathbf{H}^2) = \{0\}$).

We claim that at least one of the exterior tiles $\Delta_1, \Delta_{s-1}, \Delta_{s+r}$ or Δ_l is separated from Δ_0 by one of these geodesics. This fact is obvious if it happens that Δ_0 is contained in more than one of these hyperplanes. Also, if Δ_0 is contained in only one of those hyperplanes, we must have $\delta_1 = \delta_{s+r}$ and $\delta_s = \delta_l$ and also in this case the claim follows easily. Let us suppose, losing no generality, that δ_1 separates Δ_0 and Δ_1 . Then, every path connecting Δ_0 to Δ_1 must cross the geodesic δ_1 . Moreover, considering the isometry ρ_{δ_1} which copies Δ_1 to the next level, since we are assuming that Δ_1 is a k -external tile, we have that $d(\Delta_0, \rho_{\delta_1}(\Delta_1)) = d(\Delta_0, \Delta_1) + 1$. Let c_0 be a minimal path connecting Δ_0 to Δ_1 , c_1 a minimal path connecting Δ_1 to $\rho_{\delta_1}(\Delta_1)$, with the initial point of c_1 coinciding with the final point of c_0 . Thus, the path c obtained by the concatenation of c_0 and c_1 is a path connecting Δ_0 to $\rho_{\delta_1}(\Delta_1)$. By the construction, one has $|c| = |c_0| + |c_1| = k + 1$ and since $\rho_{\delta_1}(\Delta_1) \in NP_{k+1}$, c is a minimal path. However, both c_0 and c_1 cross the geodesic δ_1 . But a minimal path may not cross a reflection geodesic twice, as assured by well-known result concerning to Coxeter groups (as stated in Ronan, 2009, Proposition 2.6). \square

The following result, tell us about the type of the vertices which appear when the tessellation grows.

Proposition 7 Let $\mathcal{T}_{p,m,n}$ be a tessellation of \mathbf{H}^2 . If $q = 2mn/(m+n)$ is even, then the k -type of a vertex is always even.

Proof The proof will be done by induction over the stage k of the tessellation, and we remember the maximum number of edges with a common vertex is $q = 2s$.

For the initial case, the base tile Δ_0 has vertices $v_{0,1}^2, \dots, v_{0,p}^2$, which clearly have 0-type equal to 2.

Now, let us assume that in the stage k all vertices have even k -type. We prove here the vertex in the stage $k+1$ has even $(k+1)$ -type.

To obtain the tiles in NP_{k+1} , we need to reflect tiles of NP_k in the k -outer edges of the tessellation. By Proposition 6, each vertex with a cycle that is not completed has exactly two k -outer edges. Each one of these edges will contribute in the next stage with a new adjacent edge to this vertex (we may have none if the cycle of edges is completed in the stage k). Since q is even, the two edges generated by the outer edges of the previous stage cannot be the same.

Thus, by the induction hypothesis, the $(k + 1)$ -type of the vertices of NV_{k+1} is even. Since k -type of vertices is even, the same holds for the edges of the tessellation. \square

In the next proposition, it is given more precise information about the types of the new vertices in the tessellation.

Proposition 8 Consider a $\mathcal{T}_{p,m,n}$ tessellation and let $v_k^i \in V_k$. If $q = 2mn/(m + n)$ is even, then $v_k^i \in NV_k$ if and only if $i = 2$. If q is odd, then $v_k^i \in NV_k$ if and only if either $i = 2$ or $i = 3$ and if $i = 3$ there is an edge ε with $v = i(\varepsilon)$ and $t_k(\tau(\varepsilon)) = q$.

To prove this proposition, we will need some results and new definitions.

LEMMA 9 (*duality of consecutive vertices*). Let $\Delta \in NP_k$ where its vertices are numbered in anticlockwise $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p$. The indices of the vertices may be decomposed into two disjoint sets, $I = \{i, i + 1, \dots, i + r\}$ and $J = \{i + r + 1, i + r + 2, \dots, i + p - 1\}$, (where the indices are taken modulo p) in such a way that $\varepsilon_l \in NE_k$ if and only if $l \in I$.

Proof Let $q = 2mn/(m + n)$ integer. The dual tessellation of $\mathcal{T}_{p,m,n}$ is a tessellation \mathcal{T}_q^* constructed as follows:

- (1) The vertices of \mathcal{T}_q^* are the incenters of the polygons in $\mathcal{T}_{p,m,n}$.
- (2) Two vertices v, v' in \mathcal{T}_q^* determined by polygons Δ and Δ' in $\mathcal{T}_{p,m,n}$ are connected by an edge iff Δ and Δ' have a common edge.
- (3) The polygons of \mathcal{T}_q^* are determined by the edges of \mathcal{T}_q^* that intersect any edge of $\mathcal{T}_{p,m,n}$ which has a given vertex of $\mathcal{T}_{p,m,n}$ as initial or final point.

Given an edge $\varepsilon \in \mathcal{T}_{p,m,n}$, let us denote by $\tilde{\varepsilon}$ the unique edge of the dual tessellation \mathcal{T}_q^* that intersects ε .

Since the vertices of Δ are labeled modulo p , it may assume that $\varepsilon_1 \in NE_k$ and $\varepsilon_1 \notin NE_{k-1}$. To suppose that the indices of vertices of Δ may not be decomposed as stated, it is equivalent to assume the existence of $1 \leq r < s \leq t < p$ such that

$$\begin{aligned} \varepsilon_r \in NE_k \quad \text{and} \quad \varepsilon_{r+1} \in E_k \setminus NE_k, \\ \varepsilon_s \in NE_k \quad \text{and} \quad \varepsilon_{s-1} \in E_k \setminus NE_k, \\ \varepsilon_t \in NE_k \quad \text{and} \quad \varepsilon_{t+1} \in E_k \setminus NE_k, \end{aligned}$$

The dual edges $\tilde{\varepsilon}_1, \tilde{\varepsilon}_r, \tilde{\varepsilon}_s, \tilde{\varepsilon}_t, \tilde{\varepsilon}_p$ are edges belonging to the k -stage of the dual tessellation which have the following property:

$$\begin{aligned} \tilde{\varepsilon}_1, \tilde{\varepsilon}_r \in NE_k(\mathcal{T}_q^*), \\ \tilde{\varepsilon}_{r+1}, \tilde{\varepsilon}_{s-1} \in E_k(\mathcal{T}_q^*) \setminus NE_k(\mathcal{T}_q^*), \\ \tilde{\varepsilon}_s, \tilde{\varepsilon}_t \in NE_k(\mathcal{T}_q^*), \\ \tilde{\varepsilon}_{t+1}, \tilde{\varepsilon}_p \in E_k(\mathcal{T}_q^*) \setminus NE_k(\mathcal{T}_q^*). \end{aligned}$$

But all those edges in \mathcal{T}_q^* share a common vertex, and hence it constitutes a set of non-consecutive vertices, thus contradicting Proposition 6. Thus, the proof is complete. \square

Definition 10 A k -hole in a tessellation is an open set $D \subset P_k \subset \mathbb{H}^2$ with a non trivial fundamental group ($\pi_1(D) \neq \{0\}$) such that its boundary ∂D may be described as a union of edges in NE_k .

We remark that if a tessellation has a k -hole, then this is equivalent to state that the fundamental group of P_k is nontrivial, i.e. $\pi_1(P_k) \neq \{0\}$. The next proposition ensures it does not happen.

Proposition 11 A tessellation has no k -hole for any k , i.e. $\Pi_1(P_k) = \{0\}$, for all k .

Proof Suppose that D is a k -hole. Let us assume that k is the minimal stage with such property and let $\varepsilon_1, \dots, \varepsilon_i \in NP_k$ be the edges that determine the boundary ∂D , the edges being labeled anticlockwise.

We may have more than one such edge belonging to the same polygon D_s , but in this case the previous proposition ensures there are i_s and $j_s \geq 0$ such that $\varepsilon_{i_s}, \varepsilon_{i_s+1}, \dots, \varepsilon_{i_s+j_s} \subseteq D_s$ and these are all such vertices.

To simplify the notation, we denote

$$\hat{\varepsilon}_s = \bigcup_{i=0}^{j_s} \varepsilon_{i_s+i}$$

Thus, each $\hat{\varepsilon}_s$ is a polygonal line with $\hat{\varepsilon}_s \subset \partial D_s$, $D_s \in NP_k$ and $D_s \neq D_t$ for $s \neq t$.

Given $\hat{\varepsilon}_p$, let $\hat{\varepsilon}_i^\perp$ be the complement of $\hat{\varepsilon}_i$ in the boundary of D_p , i.e. $\hat{\varepsilon}_i \cup \hat{\varepsilon}_i^\perp = \partial D_i$ and $\hat{\varepsilon}_i$ and $\hat{\varepsilon}_i^\perp$ intersect only at its common initial and final vertices. By Lemma 9, if we write $\hat{\varepsilon}_i^\perp = \bigcup_{j=1}^{p-j_i} \hat{\varepsilon}_{i,j}^\perp$ with $\hat{\varepsilon}_{i,j}^\perp \in \partial D_p$, we find that those constitute a set of consecutive edges and $\hat{\varepsilon}_{i,j}^\perp \in E_{k-1}$, for all $j = 1, \dots, p-j_i$.

Consider the vertices of $\hat{\varepsilon}_i^\perp$ labeled in the following way: v_0, v_1 are the vertices of $\hat{\varepsilon}_{i,1}^\perp$, v_2, v_3 vertices of $\hat{\varepsilon}_{i,2}^\perp, \dots, v_{p-j_i-1}, v_{p-j_i}$ vertices of $\hat{\varepsilon}_{i,p-j_i}^\perp$ and $\hat{\varepsilon}_i \cap \hat{\varepsilon}_i^\perp = \{v_0, v_{p-j_i}\}$. We observe that for $v_l (l \neq 0, p-j_i)$ one has $v_l \in V_{k-1}$. If we consider the paths $\hat{\varepsilon}_i$ (obtained by concatenation of the external edges of Δ_i) and $\hat{\varepsilon}_i^\perp$ (obtained by concatenation of the inner edges of Δ_i), since their union represents the border of Δ_i , a polygon, then $\hat{\varepsilon}_i$ and $\hat{\varepsilon}_i^\perp$ are homotopic in P_k . Thus, the paths γ (obtained by concatenation of the polygonal segments $\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_m$) and γ^\perp (obtained by concatenation of the polygonal segments $\hat{\varepsilon}_1^\perp, \dots, \hat{\varepsilon}_m^\perp$) are homotopic in P_k .

We are assuming that D is a k -hole, so that its boundary ∂D is a curve not homotopic to a constant in P_k . But ∂D is just the polygonal line γ , so that γ is not homotopic to a constant in P_k . Since γ and γ^\perp are homotopic in P_k , we find that also γ^\perp is not homotopic to a constant in P_k , and also so in P_{k-1} since $P_{k-1} \subset P_k$. But $\gamma^\perp \subset P_{k-1}$, and this contradicts the minimality of k . \square

Proof of Proposition 8.

Proof We first prove the proposition for q even, since the proof in the odd case follows the same steps.

(\Rightarrow) Let $v_{k,l}^i \in V_k$ with $i = 2$. We suppose $v_{k,l}^i \notin NV_k$, i.e. $v_{k,l}^i \in NV_t$ for some $t < k$. By the Proposition 7, the k -type of a vertex is always even, then the t -type of $v_{k,l}^i$ in the stage t is at least $i = 2$, so in the stage k should have type $\min\{2 + 2(k-t), q\} \geq 2$, what is an absurd.

(\Leftarrow) Let us consider $v = v_{k-1,l}^i \in NV_k$. We suppose $i \neq 2$. Since by Proposition 7 i is even, then $i \geq 4$. Let us enumerate counterclockwise the edges of $v_{k-1,l}^i$. Since it is a set of consecutive edges, we may write $\varepsilon_{k-1,1}^{i,\tau}, \varepsilon_{k-1,2}^{i,\tau}, \dots, \varepsilon_{k-1,i}^{i,\tau}$ and the only $(k-1)$ -outer edges are $\varepsilon_{k-1,1}^{i,\tau}$ and $\varepsilon_{k-1,i}^{i,\tau}$. We label the polygons containing these edges as $\Delta_1, \Delta_2, \dots, \Delta_{i-1}$, with Δ_j containing the edges $\varepsilon_{k-1,j}^{i,\tau}$ and $\varepsilon_{k-1,j+1}^{i,\tau}$, for $j = 1, \dots, i$.

Thus, Δ_1 and $\Delta_{i-1} \in NP_{k-1}$. To conclude the proof it is enough to show that Δ_2 (and $\Delta_3, \dots, \Delta_{i-2}$) $\in P_{k-1}$. Indeed, if $\Delta_2 \in P_r$ then $\Delta_2 \in NP_r$, $r \leq k$. If $\varepsilon_2, \varepsilon_3 \in NP_r$, then $\iota(\varepsilon_2) = \iota(\varepsilon_3) = v \in NP_r$, contradicting the hypothesis of $v \in NP_{k+1}$, since $NP_r \cap NP_{r+1} = \emptyset$.

If we suppose that $\Delta_2 \notin P_{k-1}$, then $\Delta_2 \in NP_k$. Since both $\Delta_1, \Delta_2 \in NP_k$, there are $\Delta_1^-, \Delta_2^- \in NP_{k-1}$ such that $\rho_{\delta_1}(\Delta_1^-) = \Delta_1$ and $\rho_{\delta_2}(\Delta_2^-) = \Delta_2$, where $\delta_i = \Delta_i \cap \Delta_i^-$ is an edge and ρ_{δ_i} is the isometry acting in the level $k-1$ and containing ε_i .

We recall that $\varepsilon_2 = \Delta_1 \cap \Delta_2$ (since the edges are consecutive). The edge ε_1 is the first segment of a line connecting the common vertex $\{v\} = \cap_{j=1}^i \varepsilon_j$ to P_{k-1} along the boundary of Δ_1 . There is also a line connecting v to P_{k-1} such that it has ε_l as initial segment. This gives rise to a k -hole, unless $\varepsilon_1 = \varepsilon_l$. But, in this case, we changed the parity of vertices v when passing from stage $k - 1$ to stage k , contradicting Proposition 7, and we conclude the proof. \square

As a consequence, we have that an edge ε is in NE_k if it has a vertex with k -type 2 and the polygon containing the edge $\varepsilon = \varepsilon_{k+1}^{2,\tau}$ is a new polygon in the tessellation. Thus, each edge with k -type 2 will give rise to a new polygon in the stage $k + 1$.

We keep in mind that given an edge $\varepsilon_k^{i,\tau} \in E_k$ we denote by $i(\varepsilon_k^{i,\tau})$ and $\tau(\varepsilon_k^{i,\tau})$ the initial and final vertices of $\varepsilon_k^{i,\tau}$, respectively. Then we may define the following set

$$n_k(\varepsilon_k^{i,\tau}) = |\{\Delta \in P_k \mid i(\varepsilon_k^{i,\tau}) \in \Delta\}|, \tag{2}$$

i.e. $n_k(\varepsilon_k^{i,\tau})$ is the cardinality of polygons in the tessellation having in common the initial vertex of the edge $\varepsilon_k^{i,\tau}$.

Proposition 12 Let $\varepsilon = \varepsilon_k^{2,\tau} \in NE_k$. Then $n_k(\varepsilon) = 1$ and $n_{k+1}(\varepsilon) = 3$.

Proof Let $v = i(\varepsilon)$. Since v has k -type 2 there is another edge $\varepsilon' \in V_k$ such that $i(\varepsilon') = v$. It follows that ε and ε' are edges of the same polygon $\Delta \in P_k$ hence $n_k(\varepsilon) = 1$. Since both $i(\varepsilon)$ and $i(\varepsilon')$ have k -type 2, we have that $\varepsilon, \varepsilon' \in NE_k$ and both will give rise to new polygons in the next stage, so that $n_{k+1}(\varepsilon) = 3$, and concluding the proof. \square

If $\varepsilon_k^{2,\tau}$ is a k -outer edge, then it contributes with a new polygon in the level $k + 1$. If $\tau < q$ then each isometry acting on the polygons sharing a same vertex will generate different tiles. However, if $\tau = q$ this tile will be counted twice, since there is another k -outer edge that generates this same polygon. It follows that the vertex $v = i(\varepsilon_k^{2,\tau})$ will have $k + 1$ -type 4. In the special case that $p = 3$ and $m = n$, this will imply the existence of an edge $\varepsilon' \in NE_{k+1}$ of $k = 1$ -type (4, 4) and we have proved the following:

COROLLARY 13 If $\varepsilon = \varepsilon_{k,l}^{i,\tau} \in NV_k$ then $(i, \tau) = (2, \tau)$ except for the case $\tau = q, p = 3$ and $m = n$, when it may happen that $(i, \tau) = (4, 4)$.

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