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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# Stability conditions a class of linear delay difference systems

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**Abstract:** In this paper, we give new necessary and sufficient conditions for the asymptotic stability of a linear delay difference system:  $x_{n+1} - ax_n - Ax_{n-k} = 0$ , where  $A$  is a  $2 \times 2$  constant matrix,  $k$  is a nonnegative integer and  $a$  is a real number. To prove the theorems, root analysis is used for characteristic equation of system.

**Subjects:** Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Differential Equations

**Keywords:** difference equations; difference system; characteristic equation; asymptotic stability

**2010 Mathematics subject classifications:** 39A13; 39A30

### 1. Introduction

In this paper we study the asymptotic stability of the solutions of linear delay difference systems;

$$x_{n+1} - ax_n - Ax_{n-k} = 0 \quad (1)$$

where  $A$  is a  $2 \times 2$  constant matrix,  $k$  is a nonnegative integer and  $a$  is a real number. Difference equations and their stability are the appropriate mathematical representations for discrete processes, which have special importance in areas such as population models. Difference equations are mentioned below which usually happen as a result of linearization the population models. Recently, stability of the difference equations as the system (1) has been investigated by many researchers. For instance Levin and May (1976) obtained necessary and sufficient conditions for the asymptotic stability of the delay difference equation.

### ABOUT THE AUTHORS

My key research activities include: Delay Differential Equations, Delay Difference Equations, Neutral Differential Equations, Neutral Difference Equations and stability of these equations. More generally, we can say Differential Equations, Difference Equations and stability of these equations.

The research reported in this paper relates to the stability of the systems that cannot be examined by differential equations such as populations and economics. In order to investigate the stability of these systems, difference equations should be used. Our research represents a generalized method to define the stability of these systems which involve populations and economics.

### PUBLIC INTEREST STATEMENT

In this paper, stability analysis of a class of Linear Delay Difference Systems was investigated. This system is a generalized version of the difference equation of a population model using a matrix  $A$  instead of scalar  $a$ . In stability analysis of a class of Linear Delay Difference Systems, we researched the asymptotic stability of these systems. It is known that these systems are asymptotically stable if and only if all the roots of characteristic equation of the system are inside the unit disk. We applied a root analysis method using a qualitative approach in order to prove the asymptotic stability of the systems. We have created new necessary and sufficient conditions for the systems which are asymptotically stable; in case of matrix,  $A$  has complex coefficients.

$$x_{n+1} - x_n + bx_{n-k} = 0, \tag{2}$$

where  $b$  is a real number and  $k$  is a nonnegative integer. As a result, Levin and May (1976) obtained zero solution of (2) which is asymptotically stable iff

$$0 < b < 2 \cos \frac{k\pi}{2k+1}.$$

Later, Clark (1976) studied the delay difference equation

$$x_{n+1} - ax_n + bx_{n-k} = 0, \tag{3}$$

where  $a, b$  are arbitrary real numbers and  $k$  is a positive integer. Clark indicated if

$$|a| + |b| < 1,$$

then (3) is asymptotically stable, although his work brings significant innovations which only give sufficient condition for the asymptotic stability of (3). After that, Kuruklis (1994) demonstrated that the zero solution of (3) is asymptotically stable iff  $|a| < \frac{k+1}{k}$ , and

$$|a| - 1 < b < (a^2 - 2a \cos \theta + 1)^{\frac{1}{2}} \quad \text{if } k \text{ is odd,}$$

$$|b - a| < 1 \text{ and } |b| < (a^2 - 2a \cos \theta + 1)^{\frac{1}{2}} \quad \text{if } k \text{ is even,}$$

where  $\theta$  is the solution of  $\frac{\sin k\theta}{\sin(k+1)\theta} = \frac{1}{|a|}$  interval  $(0, \frac{\pi}{k+1})$  (Figure 1).

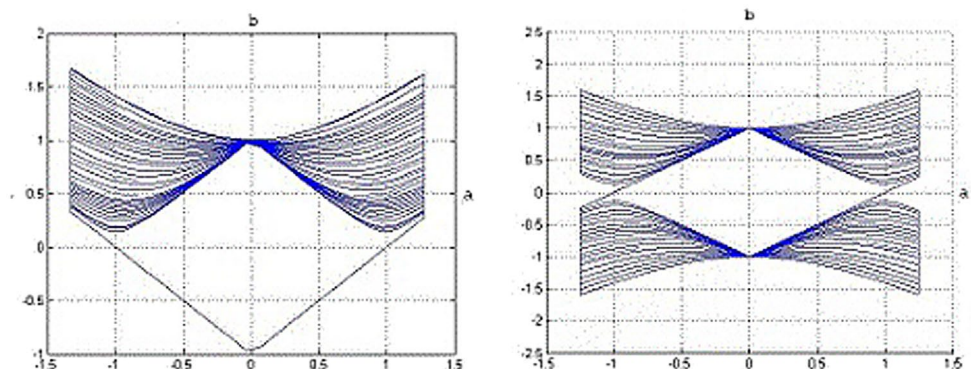
Matsunaga and Hara (1999) considered difference system

$$x_{n+1} - x_n + Ax_{n-k} = 0,$$

and they obtained necessary and sufficient conditions for the asymptotic stability of the zero solution of the system, where  $A$  is a  $2 \times 2$  constant matrix and  $k$  is a nonnegative integer. Matsunaga (2004) showed new stability conditions of generalized linear delay difference system

$$x_{n+1} - ax_n + Bx_{n-k} = 0,$$

**Figure 1. The stability regions for the result of Kuruklis with  $k = 3$  and  $k = 4$ .**



which was an extension of Matsunaga (1999). Kipnis and Malygina (2011) and Cermák and Jánský (2014) have also investigated a similar problem and the authors should clearly demonstrate the novelty and originality of their results like ours

where  $B$  is a  $2 \times 2$  constant matrix,  $k$  is a nonnegative integer and  $a$  is a real number. The purpose of this paper is to obtain new results for the asymptotic stability of zero solution of system (1) when  $A$  is a constant matrix. Thus, we need to show that zero solution of system (1) is asymptotically stable iff all the roots of its characteristic equation are inside the unit disk (Elaydi, 2005). Now we will give some basic information that we use the lemmas.

## 2. Preliminary

We consider system (1). Characteristic equation of system (1):

$$F(\lambda) := \det(\lambda^{k+1}I - a\lambda^kI - A) = 0, \tag{4}$$

where  $I$  is a  $2 \times 2$  identity matrix. If we write  $x_n = Py_n$  for a regular matrix  $P$ , then we get following system:

$$y_{n+1} - ay_n - P^{-1}APy_{n-k} = 0, \quad n \in \{0, 1, 2, \dots\}.$$

Thus,  $A$  can be given one of the following two matrices in Jordan form (Elaydi, 2005):

- (I)  $A = \begin{pmatrix} b_1 & d \\ 0 & b_2 \end{pmatrix}$ ,  $b_1, b_2$  and  $d$  are real constants,
- (II)  $A = ib \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ ,  $b, \theta$  are real constants and  $\frac{\pi}{2} < |\theta| < \pi$ .

Now, (I) and (II) will be investigated, respectively (Figure ref2);

(I) We consider (4) where  $A$  is given by (I),

$$\begin{aligned} F(\lambda) &= \det\left(\begin{pmatrix} \lambda^{k+1} & 0 \\ 0 & \lambda^{k+1} \end{pmatrix} - \begin{pmatrix} a\lambda^k & 0 \\ 0 & a\lambda^k \end{pmatrix} - \begin{pmatrix} b_1 & d \\ 0 & b_2 \end{pmatrix}\right) = 0, \\ &= (\lambda^{k+1} - a\lambda^k - b_1)(\lambda^{k+1} - a\lambda^k - b_2) = 0. \end{aligned} \tag{5}$$

If the results of Kuruklis are applied to (5), then Theorem 1 is obtained.

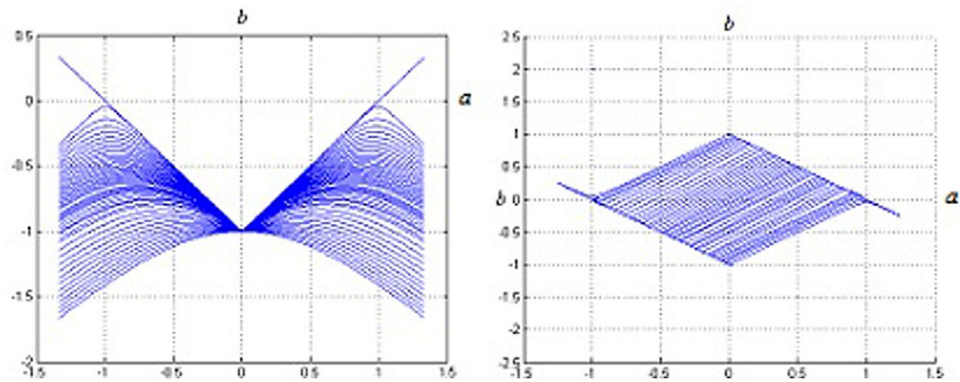
**THEOREM 1** Assume that  $a \neq 0$ . Then, the zero solution of system (1) is asymptotically stable if and only if  $|a| < \frac{k+1}{k}$ , and for  $j = 1, 2$

$$-(a^2 - 2a \cos \theta + 1)^{\frac{1}{2}} < b_j < |a| - 1 \quad \text{if } k \text{ is odd,}$$

$$|a + b_j| < 1 \quad \text{and} \quad |b_j| < (a^2 - 2a \cos \theta + 1)^{\frac{1}{2}} \quad \text{if } k \text{ is even,}$$

where  $\theta$  is the solution of  $\frac{\sin k\theta}{\sin (k+1)\theta} = \frac{1}{|a|}$  in the interval  $(0, \frac{\pi}{k+1})$ .

**Figure 2. The stability regions for the result of (I) with  $k = 3$  and  $k = 4$ .**



(II) We consider (4) where  $A$  is given by (II),

$$F(\lambda) = \det \left( \begin{pmatrix} \lambda^{k+1} & 0 \\ 0 & \lambda^{k+1} \end{pmatrix} - \begin{pmatrix} a\lambda^k & 0 \\ 0 & a\lambda^k \end{pmatrix} - \begin{pmatrix} ib \cos \theta & -ib \sin \theta \\ ib \sin \theta & ib \cos \theta \end{pmatrix} \right) = 0,$$

$$= (\lambda^{k+1} - a\lambda^k - ibe^{i\theta})(\lambda^{k+1} - a\lambda^k - ibe^{-i\theta}) = 0.$$

If we take

$$\Delta_\theta(\lambda) = \lambda^{k+1} - a\lambda^k - ibe^{i\theta} = 0, \tag{6}$$

then we get

$$F(\lambda) \equiv \Delta_\theta(\lambda)\overline{\Delta_\theta(\bar{\lambda})} = 0, \tag{7}$$

where  $\bar{\lambda}$  is the complex conjugate of  $\lambda$ . Note that  $\overline{\Delta_\theta(\bar{\lambda})} = 0$  implies  $\Delta_\theta(\bar{\lambda}) = 0$ . Also, when  $-\pi < \theta < -\frac{\pi}{2}$ , substituting  $\tilde{\theta} = -\theta$  in  $\Delta_\theta(\lambda) = 0$  and  $\Delta_\theta(\bar{\lambda}) = 0$  implies  $\frac{\pi}{2} < \tilde{\theta} < \pi$ . Thus, it is just enough to consider the case  $\Delta_\theta(\lambda) = 0$  under the condition  $\frac{\pi}{2} < \theta < \pi$  to investigate the roots of (7). We know that system (1) is asymptotically stable iff all the roots of (6) with  $\frac{\pi}{2} < \theta < \pi$  are inside the unit disk. Moreover,  $\Delta_\theta(\lambda)$  has no real root when  $b \neq 0$ . For  $a \neq 0$  and  $\frac{\pi}{2} < \theta < \pi$ , (6) can be written equivalently:

$$\mu^{k+1} - \mu^k - iqe^{i\theta} = 0, \tag{8}$$

where  $\mu = \frac{\lambda}{a}$  and  $q = \frac{b}{a^{k+1}}$ . Thus, all the roots of (6) are inside the unit disk iff all the roots of (8) are inside the disk  $|\mu| < \frac{1}{|a|}$ . Now, some auxiliary lemmas can be given.

### 3. Some auxiliary lemmas and main theorems

**LEMMA 1** (Matsunaga, 2004) Let  $a \neq 0$  and  $q = \frac{b}{a^{k+1}}$ . Then, the zero solution of system (1) is asymptotically stable iff all the roots of (8) are inside the disk  $|\mu| < \frac{1}{|a|}$ .

Now our aim is to calculate the locations according to movement of the roots via root analysis of the system (1) as  $q$  varies. Note that, for  $q = 0$ , the roots of (8) are 0 (multiplicity  $k$ ) and 1 (simple). Furthermore, (8) has no real root when  $q \neq 0$ . Firstly, the existence region for arguments of complex roots of (8) will be found.

**LEMMA 2** Suppose that  $q > 0$  and  $\frac{\pi}{2} < \theta < \pi$ . Let  $re^{i\omega}$  with  $r > 0$  be a complex root of (8). Then, the following are provided:

$$\text{for } -\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{\pi}{2};$$

$$G_m^+ : \frac{\theta + \left(2m - \frac{1}{2}\right)\pi}{k} < \omega < \frac{\theta + \left(2m + \frac{1}{2}\right)\pi}{k+1}, \quad m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor,$$

for  $\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{3\pi}{2}$ ;

$$G_0^- = \frac{\theta - \frac{3\pi}{2}}{k+1} < \omega < 0, \quad G_m^- : \frac{\theta - \left(2m + \frac{3}{2}\right)\pi}{k+1} < \omega < \frac{\theta - \left(2m + \frac{1}{2}\right)\pi}{k}, \quad m = 1, 2, \dots, \left\lfloor \frac{k}{2} \right\rfloor,$$

where  $\| \cdot \|$  denotes greatest integer function.

*Proof* Let  $\mu = re^{i\omega}$  with  $r > 0$ . From (6), we have

$$r^{k+1} e^{i\omega(k+1)} - r^k e^{i\omega k} - iq e^{i\theta} = 0. \tag{9}$$

Thus, we get

$$r^{k+1} e^{i(\omega(k+1)-\theta)} - r^k e^{i(\omega k - \theta)} - iq = 0.$$

From the real part and the imaginary part of (9)

$$r^k [\cos(\omega k - \theta) - r \cos(\omega(k+1) - \theta)] = 0, \tag{10}$$

and

$$r^k [r \sin(\omega(k+1) - \theta) - \sin(\omega k - \theta)] = q, \tag{11}$$

are obtained. From (10), the following equality can be derived

$$r = \frac{\cos(\omega k - \theta)}{\cos(\omega(k+1) - \theta)}. \tag{12}$$

From (10) and (12), we can write

$$q = r^k \frac{\sin \omega}{\cos(\omega(k+1) - \theta)} = r^k \frac{\cos\left(\frac{\pi}{2} - \omega\right)}{\cos(\omega(k+1) - \theta)}. \tag{13}$$

Consider  $q > 0$  and  $-\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{\pi}{2}$ . Using (12) and (13), we derive  $\cos(\omega(k+1) - \theta) > 0$  and  $\cos(\omega k - \theta) > 0$ . Thus, we can write

$$\frac{\theta + \left(2m - \frac{1}{2}\right)\pi}{k} < \omega < \frac{\theta + \left(2m + \frac{1}{2}\right)\pi}{k}, \quad m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor,$$

and

$$\frac{\theta + \left(2m - \frac{1}{2}\right)\pi}{k+1} < \omega < \frac{\theta + \left(2m + \frac{1}{2}\right)\pi}{k+1}, \quad m = 0, 1, 2, \dots, \left\lfloor \frac{k}{2} \right\rfloor.$$

From the above inequalities, the following result is obtained:

$$G_m^+ : \frac{\theta + \left(2m - \frac{1}{2}\right)\pi}{k} < \omega < \frac{\theta + \left(2m + \frac{1}{2}\right)\pi}{k+1}, \quad m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor.$$

Hence, the proof is completed. For  $\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{3\pi}{2}$ , the proof is similar. □

**LEMMA 3** Suppose that  $q < 0$  and  $\frac{\pi}{2} < \theta < \pi$ . Let  $re^{i\omega}$  with  $r > 0$  be a complex root of (8). Then, the following holds:

for  $-\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{\pi}{2}$ ;

$$H_0^+ = 0 < \omega < \frac{\theta - \frac{\pi}{2}}{k+1}, \quad H_m^+ : \frac{\theta + \left(2m - \frac{3}{2}\right)\pi}{k} < \omega < \frac{\theta + \left(2m - \frac{1}{2}\right)\pi}{k+1}, \quad m = 1, 2, \dots, \left\lfloor \frac{k}{2} \right\rfloor,$$

for  $\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{3\pi}{2}$ ;

$$H_m^- : \frac{\theta - \left(2m + \frac{5}{2}\right)\pi}{k+1} < \omega < \frac{\theta - \left(2m + \frac{3}{2}\right)\pi}{k}, \quad m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor.$$

*Proof* For  $q < 0$ , the proof is analogous to the above. □

Since Lemmas 2 and 3, the following notation can be used as whole existence region for arguments of complex roots of (8).

$$W \equiv \left( \bigcup_{m=0}^{\lfloor (k-1)/2 \rfloor} G_m^+ \right) \cup \left( \bigcup_{m=0}^{\lfloor k/2 \rfloor} G_m^- \right) \cup \left( \bigcup_{m=0}^{\lfloor k/2 \rfloor} H_m^+ \right) \cup \left( \bigcup_{m=0}^{\lfloor (k-1)/2 \rfloor} H_m^- \right).$$

The following lemma calculates arguments of complex roots of (8) on the unit circle.

**LEMMA 4** Suppose that  $\frac{\pi}{2} < \theta < \pi$ . Then, arguments of complex roots of (8) on the unit circle are as follows

$$\omega = \eta_n \equiv \frac{2\theta + 2n\pi}{2k+1}, \quad n = 0, \pm 1, \pm 2, \dots, \pm k. \tag{14}$$

*Proof* If we get  $r = 1$  in (10), then we obtain

$$\cos(\omega(k+1) - \theta) - \cos(\omega k - \theta) = 2 \sin\left(\frac{\omega}{2}\right) \sin\left(\frac{(2k+1)\omega}{2} - \theta\right) = 0.$$

Since  $\sin\left(\frac{\omega}{2}\right) \neq 0$ , we obtain  $\sin\left(\frac{(2k+1)\omega}{2} - \theta\right) = 0$ ; this implies (14). □

**LEMMA 5** We suppose that  $\frac{\pi}{2} < \theta < \pi$ . For  $n = 0$ ,

$$0 < \eta_0 < \frac{\theta + \left(2m - \frac{3}{2}\right)\pi}{k}, \quad m = 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor,$$

where  $\omega = \eta_0$  is argument of complex roots of (8) on the unit circle.

*Proof* The proof is obvious. □

Now, the movement of the roots of (8) will be investigated as  $q$  varies on the complex plane.

**LEMMA 6** Suppose that  $\frac{\pi}{2} < \theta < \pi$ . When  $q$  increases to 0, simple root  $\mu = 1$  of (8) is inside the unit circle.

*Proof* It is enough to show that  $\left. \frac{dr}{dq} \right|_{r=1, \omega=0} < 0$ .

$$\frac{dr}{dq} = \frac{dr}{d\omega} \frac{d\omega}{dq} = \frac{dr}{d\omega} \left( \frac{dq}{d\omega} \right)^{-1}.$$

Also,  $q = 0$  is equivalent to  $\omega = 0$  and  $r = 1$  as such. Let  $re^{i\omega}$  be a root of (8). For  $\frac{\pi}{2} < \theta < \pi$ , using (12) and (13)

$$\frac{dr}{d\omega} = \frac{k \sin \omega + \sin(\omega(k+1) - \theta) \cos(k\omega - \theta)}{(\cos(\omega(k+1) - \theta))^2},$$

and

$$\frac{dq}{d\omega} = r^k \frac{k^2 \sin^2 \omega + 2k \cos(k\omega - \theta) \sin \omega \sin(\omega(k+1) - \theta) + \cos^2(k\omega - \theta)}{\cos(k\omega - \theta) (\cos(\omega(k+1) - \theta))^2},$$

are obtained. If we take

$$I(\omega) \equiv k \sin \omega + \sin(\omega(k+1) - \theta) \cos(k\omega - \theta), \tag{15}$$

$$J(\omega) \equiv k^2 \sin^2 \omega + 2k \cos(k\omega - \theta) \sin \omega \sin(\omega(k+1) - \theta) + \cos^2(k\omega - \theta), \tag{16}$$

then

$$\frac{dr}{dq} = \frac{\cos(k\omega - \theta) I(\omega)}{J(\omega)}, \tag{17}$$

for  $\omega = 0$  and  $r = 1$ , we get

$$\left. \frac{dr}{dq} \right|_{\omega=0} = -\sin \theta < 0$$

The proof is completed. □

**LEMMA 7** Suppose that  $\frac{\pi}{2} < \theta < \pi$ . Absolute value of the complex roots of (8) increases while  $|q|$  increases except for  $\left( \frac{2\theta - (4m+5)\pi}{2k+2}, \frac{2\theta - (4m+3)\pi}{2k} \right)$  and the root  $\tilde{\mu}$  satisfying interval  $\left( \omega^+, \frac{2\theta + (4m-1)\pi}{2k+2} \right)$ , where  $\omega^+$  is a root of  $I(\omega) = 0$  inside  $(0, \eta_0)$ .

*Proof* Let  $re^{i\omega}$  be a root of (8) with  $\omega \in W$ . Since  $\frac{dI}{d\omega} > 0$ ,  $I(\omega)$  is strictly increasing for  $\omega \in W$  because of (15) and also  $J(\omega) > 0$  from (16). Firstly,  $q > 0$  is considered;

Case (a)  $-\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{\pi}{2}$ . For  $\omega \in G_m^+$ ,  $m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor$  it is enough to show that  $\frac{dr}{dq} > 0$ . Inequality  $I\left(\frac{\theta + (2m - \frac{1}{2})\pi}{k}\right) = k \sin\left(\frac{\theta + (2m - \frac{1}{2})\pi}{k}\right) > 0$  is provided from  $I(\omega) > 0$  for  $\omega \in G_m^+$ ,  $m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor$ . Also  $\cos(k\omega - \theta) > 0$  for  $\omega \in (0, \pi)$ . Thus, from the increasing property of  $I(\omega)$  and (17) for  $\omega \in (0, \pi)$ ,  $\frac{dr}{dq} > 0$ .

Case (b)  $\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{3\pi}{2}$ . It will be similar to case (a), from the increasing property of  $I(\omega)$  and  $I\left(\frac{\theta - (2m + \frac{3}{2})\pi}{k+1}\right) = -k \sin\left(\frac{\theta - (2m + \frac{3}{2})\pi}{k+1}\right) < 0$  is provided from  $I(\omega) < 0$ , for  $\omega \in G_m^-$ ,  $m = 0, 1, 2, \dots, \left\lfloor \frac{k}{2} \right\rfloor$  and  $\cos(k\omega - \theta) < 0$  for  $\omega \in (-\pi, 0)$ ; thus, from (17) for  $\omega \in (-\pi, 0)$ ,  $\frac{dr}{dq} > 0$ . Now consider  $q < 0$ ;

Case (c)  $-\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{\pi}{2}$ .  $I(0) = -\frac{\sin 2\theta}{2} < 0$  for  $\omega \in (0, \pi)$  and  $I(\eta_0) = \left(k + \frac{1}{2}\right) \sin \frac{2\theta}{2k+1} > 0$  are obtained. Also  $\cos(k\omega - \theta) > 0$  and  $I\left(\frac{\theta + (2m - \frac{3}{2})\pi}{k}\right) = k \sin\left(\frac{\theta + (2m - \frac{3}{2})\pi}{k}\right) > 0$  are provided from  $I(\omega) > 0$  for  $\omega \in H_m^+$ ,  $m = 0, 1, 2, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor$ . These equalities and the increasing property of  $I(\omega)$  show that  $I(\omega) > 0$  for  $\omega \in (0, \omega^+)$  and  $I(\omega) < 0$  for  $\omega \in \left(\omega^+, \frac{\theta - \frac{\pi}{2}}{k+1}\right)$ . This way the claim of this lemma is provided with (17).

Case (d)  $\frac{\pi}{2} < \frac{\pi}{2} - \omega < \frac{3\pi}{2}$ .  $I\left(\frac{\theta - (2m + \frac{3}{2})\pi}{k}\right) = -k \sin\left(\frac{\theta - (2m + \frac{3}{2})\pi}{k}\right) < 0$  is provided from  $I(\omega) < 0$  for  $\omega \in H_m^-$ ,  $m = 0, 1, 2, \dots, \lfloor \frac{k-1}{2} \rfloor$ . Also  $I\left(\frac{\theta - (2m + \frac{5}{2})\pi}{k+1}\right) = -k \sin\left(\frac{\theta - (2m + \frac{5}{2})\pi}{k+1}\right)$  is provided from  $I(\omega) < 0$ , for  $\omega \in H_m^-$ ,  $m = 0, 1, 2, \dots, \lfloor \frac{k-1}{2} \rfloor$ . Thus, from (17),  $\cos(k\omega - \theta) < 0$  and the increasing property of  $I(\omega)$  for  $\omega \in (-\pi, 0)$ ,  $\frac{dr}{dq} < 0$ . The proof is completed.  $\square$

Thus, we show that roots of (8) move continuously away from zero, while  $|q|$  increases except for the root  $\tilde{\mu}$  from Lemmas 6 and 7.

**LEMMA 8** Suppose that  $\frac{\pi}{2} < \theta < \pi$ ,  $a = 1$  and  $\omega = \eta_n$  ( $n = 0, \pm 1, \pm 2, \dots, \pm k$ ) are arguments of complex roots of (8) on the unit circle while  $q < 0$ . Then,

$$q = -\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - \theta}{2k + 1}\right)$$

is obtained, where  $k$  is a nonnegative integer.

*Proof* We should note that  $\omega$  takes minimum value at  $n = 0$ . Also squaring both of (10) and (11), adding side by side to them, we get  $|q| = 2 \sin\left(\frac{\omega}{2}\right)$ , and for  $n = 0$ , we have

$$q(\eta_0) = -2 \sin\left(\frac{\omega}{2}\right) = -2 \sin\left(\frac{\pi}{2} - \frac{\left(k + \frac{1}{2}\right)\pi - \theta}{2k + 1}\right) = -\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - \theta}{2k + 1}\right). \quad \square$$

**THEOREM 2** Assume that  $\frac{\pi}{2} < |\theta| < \pi$ ,  $a = 1$  and  $\omega = \eta_n$  ( $n = 0, \pm 1, \pm 2, \dots, \pm k$ ) are arguments of complex roots of (8) on the unit circle while  $b < 0$ . Then system (1) is asymptotically stable iff

$$-\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - |\theta|}{2k + 1}\right) < b < 0,$$

where  $k$  is a nonnegative integer.

*Proof* Assume that system (1) is asymptotically stable with  $\frac{\pi}{2} < |\theta| < \pi$  as  $a = 1$ . It is known that the minimum value of  $q$  is  $-\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - \theta}{2k + 1}\right)$  when  $\lambda = 1$  from Lemma 8. Thus, we use the continuity  $q$  with respect to  $\lambda$ ; it is written

$$-\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - |\theta|}{2k + 1}\right) < b < 0,$$

for every  $|\lambda| < 1$ .

Now, assume that above inequality is provided. It is known that the minimum value of  $q$  is  $-\cos\left(\frac{\left(k + \frac{1}{2}\right)\pi - \theta}{2k + 1}\right)$  when  $\lambda = 1$  from Lemma 8. Also via Lemma 6 and continuity  $q$  with respect to  $\lambda$ , we can write every root of  $\lambda$  is inside unit disk as  $q$  is in the neighborhood of 0. So system (1) is asymptotically stable.  $\square$

If the matrix  $A$  is chosen  $n$ -dimensionally, that is,  $A$  is a  $n \times n$  constant complex matrix, then the following theorem is obtained:

**THEOREM 3** Assume that  $\frac{\pi}{2} < |\theta| < \pi$  and  $a = 1$ . Let  $q_j e^{i\theta_j}$  ( $j = 1, 2, \dots, n$ ) be the eigenvalues of  $A$ . Then, the system (1) is asymptotically stable iff

$$-\cos \left( \frac{\left(k + \frac{1}{2}\right)\pi - |\theta_j|}{2k + 1} \right) < b < 0 \quad j = 1, 2, \dots, n$$

where  $q_j, \theta_j$  are real numbers and  $\frac{\pi}{2} < |\theta_j| < \pi$ .

*Proof* Let  $b_j e^{i\theta_j}$  ( $j = 1, 2, \dots, n$ ) be the eigenvalues of  $A$ ; the characteristic equation of the system (1) is given by

$$f(\lambda) = \prod_{j=1}^n (\lambda^{k+1} - \lambda^k - ib_j e^{i\theta_j}) = 0.$$

Thus, Theorem 3 can be seen as a result of Theorem 2. □

Now the values  $|q|$  will be found for a root of (8) on the circle  $\mu = \frac{1}{|a|}$ . The following lemma provided these values of  $|q|$  in terms of  $|a|$  and the complex root arguments  $\omega$ .

**LEMMA 9** Let  $re^{i\omega}$  be a root of (8) on the circle  $\mu = \frac{1}{|a|}$ . Then,

$$Q_v(a) \equiv |q| = \frac{(a^2 + 1 - 2|a| \cos \theta)^{1/2}}{|a|^{k+1}}, \quad v = 1, 2, 3, 4. \tag{18}$$

*Proof* Using (12), we can write

$$r = \frac{\cos(k\omega - \theta)}{\cos((k+1)\omega - \theta)} = \frac{1}{|a|},$$

then,

$$\cot(k\omega - \theta) = \frac{\sin \omega}{\cos \omega - |a|},$$

is obtained. From (13), we get

$$q = r^k \frac{\sin \omega}{\cos(\omega(k+1) - \theta)} = r^{k+1} \frac{\sin \omega}{\cos(k\omega - \theta)}.$$

Square of both sides yields

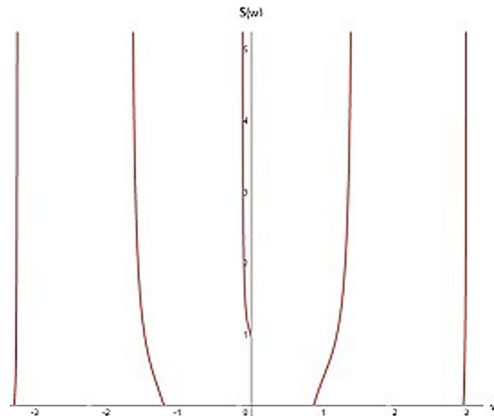
$$\frac{q^2}{r^{2(k+1)}} = \frac{\sin^2 \omega}{\cos^2(k\omega - \theta)} = \sin^2 \omega \left[ \frac{1 + \cot^2(k\omega - \theta)}{\cot^2(k\omega - \theta)} \right].$$

Thus, (18) is obtained. Since  $a^2 - 2|a| \cos \theta + 1 \geq (|a|^2 - 1)^2 \geq 0$ , the square root in Equation (18) is valid. □

Since (18), the value of  $|q|$  increases with respect to  $|\omega|$  and, then the minimum value of  $|q|$  is equivalent to the minimum value of  $|\omega|$  which provides  $r = \frac{1}{|a|}$ . If it is used the notation

$$S(\omega) = \frac{\cos(k\omega - \theta)}{\cos((k+1)\omega - \theta)} = \frac{1}{|a|}, \tag{19}$$

**Figure 3. The graphic of  $S(\omega)$  with  $k = 1$  and  $\theta = \frac{\pi}{3}$  for  $q > 0$ .**



then the minimum value of  $S(\omega)$  becomes equivalent to the minimum value of  $|q|$ . Hence, by means of Lemmas 2, 3 and 7, it is obvious that  $S(\omega)$  is strictly increasing with respect to  $|\omega|$ , and  $S(\omega)$  only has a local minimum at  $\omega^+$ . Now necessary and sufficient conditions for the roots of (8) to be inside the disk  $|\mu| = \frac{1}{|a|}$  are provided. For  $q > 0$ , a graphic of  $S(\omega)$  with  $k = 1$  and  $\theta = \frac{\pi}{3}$  is presented.

**LEMMA 10** Assume that  $a \neq 0$ ,  $\frac{\pi}{2} < \theta < \pi$  and  $q > 0$ . Then, all the roots of (8) are inside the disk  $|\mu| = \frac{1}{|a|}$  iff

$$0 < |a| \leq 1 \text{ and } 0 < q < \frac{Q_1(a)}{|a|^{k+1}},$$

where  $Q_v(a) (v = 1)$  is given in (18).

*Proof*

*Case (i)*  $\frac{1}{|a|} \geq 1$ , i.e.  $0 < |a| \leq 1$ . From Figure 3, there is at least one root  $\omega_1$  of  $S(\omega)$  in  $(\frac{\theta - \frac{3\pi}{2}}{k+1}, 0)$ . Therefore, from Lemmas 6–9, trace of branch  $|\mu| = 1$  shows that this trace is inside the disk  $|\mu| = \frac{1}{|a|}$  while it is increasing from 0 to  $\frac{Q_1(a)}{|a|^{k+1}}$ , it is on the circle  $|\mu| = \frac{1}{|a|}$  when it is at  $\frac{Q_1(a)}{|a|^{k+1}}$  and it is outside the disk  $|\mu| = \frac{1}{|a|}$  while it is increasing from  $\frac{Q_1(a)}{|a|^{k+1}}$ . Hence, our claim is the following: all the roots of (8) are inside the disk  $|\mu| = \frac{1}{|a|}$  iff  $0 < q < \frac{Q_1(a)}{|a|^{k+1}}$ .

*Case (ii)*  $0 < \frac{1}{|a|} < 1$ , i.e.  $|a| > 1$ . From Figure 3,  $S(\omega) \geq \frac{1}{|a|}$  for  $\omega \in G_0^-$ . This shows that there is a root of (8) belonging to the branch  $|\mu| = 1$  as  $|\mu| \geq \frac{1}{|a|}$ .  $\square$

Now for the following lemma for  $q < 0$ , a graphic of  $S(\omega)$  with  $k = 1$  and  $\theta = \frac{\pi}{3}$  is presented.

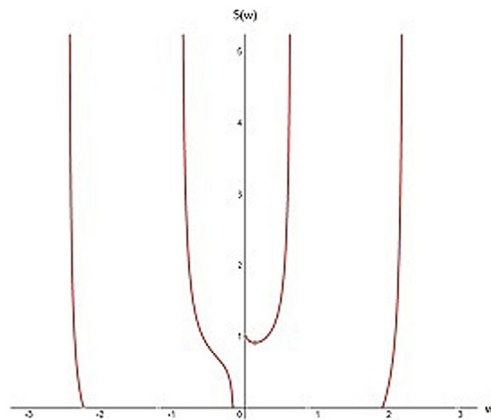
**LEMMA 11** Assume that  $a \neq 0$ ,  $\frac{\pi}{2} < \theta < \pi$  and  $q < 0$ . Then, all the roots of Equation (8) are inside the disk  $|\mu| = \frac{1}{|a|}$  iff

$$0 < |a| < \frac{1}{S(\omega^+)} \text{ and } \frac{-Q_4(a)}{|a|^{k+1}} < q < \frac{-Q_3(a)}{|a|^{k+1}},$$

or

$$0 < |a| < 1 \text{ and } \frac{-Q_2(a)}{|a|^{k+1}} < q < 0,$$

**Figure 4. The graphic of  $S(\omega)$  with  $k = 1$  and  $\theta = \frac{\pi}{3}$  for  $q < 0$ .**



where  $S(\omega)$  is given in (18),  $Q_\nu(a)$  ( $\nu = 2, 3, 4$ ) is given in (18) and  $\omega^+$  is a root of  $I(\omega) = 0$  inside  $(0, \eta_0)$ .

*Proof* Firstly, assume that  $-Q_4(a) < -Q_3(a) < -Q_2(a)$ . It is known that the value of  $\omega$  satisfying  $r = S(\omega) = 1$  inside  $H_0^+$  is  $\eta_0$ . Now, the locations of the roots of (8) are investigated. In case of  $q = 0$ , the roots of (8) are 0 and 1. A branch of  $|\mu| = 1$  is in the region  $H_0^+$  inside while it is increasing to 0 from  $q$ . (at a very small neighborhood of  $q = 0$ ) There are three cases:

Case (i)  $\frac{1}{|a|} > 1$ , i.e.  $0 < |a| < 1$ . From Figure 4, there is at least one root  $\omega_2$  of  $S(\omega)$  in  $(\eta_0, \frac{\theta - \frac{\pi}{2}}{k+1})$ . So, from Lemmas 6–9, trace of branch  $|\mu| = 1$  shows that this trace is inside the disk  $|\mu| = \frac{1}{|a|}$  while  $q$  is decreasing from 0 to  $\frac{-Q_2(a)}{|a|^{k+1}}$ , it is on the circle  $|\mu| = \frac{1}{|a|}$  when it is at  $\frac{-Q_2(a)}{|a|^{k+1}}$  and it is outside the disk  $|\mu| = \frac{1}{|a|}$  while it is increasing from  $\frac{-Q_2(a)}{|a|^{k+1}}$ . Here our claim is:

All the roots of (8) are inside the disk  $|\mu| = \frac{1}{|a|}$  iff  $\frac{-Q_2(a)}{|a|^{k+1}} < q < 0$ .

Case (ii)  $S(\omega^+) < \frac{1}{|a|} < 1$ , i.e.  $1 < |a| < \frac{1}{S(\omega^+)}$ . From Figure 4, there is at least one root  $\omega_3$  and  $\omega_4$  of  $S(\omega)$ , respectively, at intervals of  $(0, \omega^+)$  and  $(\omega^+, \eta_0)$ . Then, from Lemmas 6–9, trace of branch  $|\mu| = 1$  shows that this trace is inside the disk  $|\mu| = \frac{1}{|a|}$  while  $q$  is increasing from  $\frac{-Q_4(a)}{|a|^{k+1}}$  to  $\frac{-Q_3(a)}{|a|^{k+1}}$ , it is on the circle  $|\mu| = \frac{1}{|a|}$  when it is at  $q = \frac{-Q_4(a)}{|a|^{k+1}}$  or  $q = \frac{-Q_3(a)}{|a|^{k+1}}$  and it is outside the disk  $|\mu| = \frac{1}{|a|}$  while it is increasing from  $\frac{-Q_3(a)}{|a|^{k+1}}$ . Here our claim is: all the roots of (8) are inside the disk  $|\mu| = \frac{1}{|a|}$  iff  $\frac{-Q_4(a)}{|a|^{k+1}} < q < \frac{-Q_3(a)}{|a|^{k+1}}$ .

Case (iii)  $0 < \frac{1}{|a|} \leq 1$ , i.e.  $|a| \geq 1$ . From Figure 4,  $S(\omega) \geq \frac{1}{|a|}$  for  $\omega \in H_0^+$ . This shows that there is a root of (8) belonging to the branch  $|\mu| = 1$  as  $|\mu| \geq \frac{1}{|a|}$ . □

Next, necessary and sufficient conditions for (6) are given.

**THEOREM 4** Assume that  $\frac{\pi}{2} < |\theta| < \pi$ ,  $\omega_\nu$  ( $\nu = 1, 2, 3, 4$ ) are roots of  $S(\omega) = \frac{1}{|a|}$  and  $\omega^+$  is first coordinate of local minimum  $S(\omega)$ . Then, system (1) is asymptotically stable provided

$$\omega_1 < 0 < \omega_4 < \omega^+ < \omega_3 < \frac{2\theta}{2k+1} < \omega_2,$$

where  $k$  is a nonnegative integer.

*Proof* Using Lemmas 10 and 11, it can be easily seen. □

**THEOREM 5** Assume that  $a \neq 0$ ,  $\frac{\pi}{2} < |\theta| < \pi$ , and  $k$  is odd. System (1) is asymptotically stable if and only if one of the following situations is provided:

- (i)  $\frac{1}{|a|} > 1$  and  $0 < b < Q_1(a)$ .
- (ii)  $\frac{1}{|a|} \geq 1$  and  $-Q_2(a) < b < 0$ .
- (iii)  $S(\omega^+) < \frac{1}{|a|} < 1$  and  $-Q_4(a) < b < -Q_3(a)$ .

*Proof* Since  $k$  is odd and  $a \neq 0$ , we know that  $a^{k+1} > 0$ . Thus, the sign of  $q$  varies depending on  $b$ . We consider the following cases:

*Case 1:*  $b > 0$ . Then,  $q = \frac{b}{a^{k+1}} > 0$  and hence, using Lemma 10, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} > 1 \text{ and } 0 < b < Q_1(a).$$

*Case 2:*  $b < 0$ . Then,  $q = \frac{b}{a^{k+1}} < 0$  and hence, using Lemma 11, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} \geq 1 \text{ and } -Q_2(a) < b < 0 \text{ or } S(\omega^+) < \frac{1}{|a|} < 1 \text{ and } -Q_4(a) < b < -Q_3(a). \quad \square$$

**THEOREM 6** Assume that  $a \neq 0$ ,  $\frac{\pi}{2} < |\theta| < \pi$ , and  $k$  is even. System (1) is asymptotically stable if and only if one of the following situations is provided:

- (i)  $\frac{b}{a} \geq 0$ ,  $\frac{1}{|a|} > 1$  and  $|b| < Q_1(a)$ .
- (ii)  $\frac{b}{a} < 0$ ,  $\frac{1}{|a|} \geq 1$  and  $|b| < Q_2(a)$ .
- (iii)  $\frac{b}{a} < 0$ ,  $S(\omega^+) < \frac{1}{|a|} < 1$  and  $Q_3(a) < |b| < Q_4(a)$ .

*Proof* Since  $k$  is even and  $a \neq 0$ , thus, the sign of  $q$  varies depending on both of  $a$  and  $b$ ; we consider the following cases.

*Case 1:*  $a > 0$  and  $b > 0$ . Then,  $q = \frac{b}{a^{k+1}} > 0$  and hence, using Lemma 10, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} > 1 \text{ and } 0 < b < Q_1(a).$$

*Case 2:*  $a > 0$  and  $b < 0$ . Then,  $q = \frac{b}{a^{k+1}} < 0$  and hence, using Lemma 11, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} \geq 1 \text{ and } -Q_2(a) < b < 0 \text{ or } S(\omega^+) < \frac{1}{|a|} < 1 \text{ and } -Q_4(a) < b < -Q_3(a).$$

*Case 3:*  $a < 0$  and  $b > 0$ . Then,  $q = \frac{b}{a^{k+1}} < 0$  and hence, using Lemma 11, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} \geq 1 \text{ and } 0 < b < Q_2(a) \text{ or } S(\omega^+) < \frac{1}{|a|} < 1 \text{ and } Q_3(a) < b < Q_4(a).$$

*Case 4:*  $a < 0$  and  $b < 0$ . Then,  $q = \frac{b}{a^{k+1}} > 0$  and hence, using Lemma 10, we can see that the zero solution of system (1) is asymptotically stable iff

$$\frac{1}{|a|} > 1 \text{ and } -Q_1(a) < b < 0. \quad \square$$

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