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Fractional Langevin equations with multi-point and non-local integral boundary conditions

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Abstract: In this paper, we investigate a non-linear Langevin equation with periodic, multi-point and non-local fractional integral boundary conditions. The contraction mapping theorem is employed to determine sufficient conditions for the uniqueness of the solution. Also, different results in the existence of solution are demonstrated by using Krasnoselskii and Leray-Schauder theorems. Finally, some examples are provided as applications of the theorems in order to support the main outcomes of this paper.

Keywords: fractional Langevin equations; existence and uniqueness; multi-point boundary conditions
MSC: 26A33; 34A08; 34A12

1. Introduction

The area of fractional calculus has been expanded by many references supported by wide scientific reviews. Accumulated works on fractional differential equations, titled with different kinds of both initial and boundary value problems, reflect its importance in mathematical studies (see (Ahmad, 2010; Bai & Sun, 2012; Liang & Zhang, 2011; Zhang et al., 2013)). Obviously, there are several areas where fractional calculus can be applied such as aerodynamics, biomathematics, control theory, ecology, electrodynamics, etc. (see (Kilbas et al., 2006; Li et al., 2019; Podlubny, 1999; Tomovski et al., 2010)).

The Langevin equation is a perfect way to describe mathematical physics, which can help physicians effectively to describe processes like anomalous diffusion in a descent manner. In the economy field, processes include price index fluctuations (West & Picozzi, 2002). In the theory of critical dynamics, the general form of the Langevin equation for noise sources with correlations plays an important role (Hohenberg & Halperin, 1977). Diverse general Langevin equations have

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PUBLIC INTEREST STATEMENT

The Langevin equation is an excellent technique to describe some phenomena which can help physicians, engineers, economists, etc., effectively to describe processes. The Langevin equation (drafted for first by Langevin in 1908) is obtained to be an accurate tool to describe the development of physical phenomena. The non-linear fractional Langevin equations have been modernized by Mainardi and Pironi in 1996. In this research article, we discuss the existence of the non-linear fractional Langevin differential equations by Krasnoselskii and Leray-Schauder theorems. Proof of the unique solutions is studied by Banach fixed point theorem. These applications are illustrated by examples for each theorem.

been restricted to the kind of dynamical processes in media, given such Langevin equation in general (Lutz, 2001; Wang & Tokuyama, 1999). It is noticeable that the nonlinear fractional Langevin equations have been modernized by Mainardi and Pironi (Mainardi & Pironi, 1996).

According to the recent published papers, Ahmad et al. (Ahmad et al., 2019) provided sufficient conditions for the existence of solutions for a nonlinear Langevin equation subject to conditions involving a generalized integral operator. Salem et al. (Salem, Alzahrani, Alnegga et al., 2020) have discussed both the existence and uniqueness of the solution to a coupled system involving multi-fractional orders. Not long ago, various contributions related to the existence and uniqueness of the solution to Langevin equations have been published (see (H. Baghani, 2018; O. Baghani, 2017; Baghani & Nieto, 2019; Darzi et al., 2020; Gao et al., 2016; Kiataramkul et al., 2016; Mahmudov, 2020; Vojta et al., 2019) and the references given therein).

The studying of differential equations with integral boundary conditions designates an extremely useful and interesting class of boundary value problems. Several problems in chemical engineering, population dynamics, heat conduction, thermoelasticity, underground water flow, and plasma physics are presented (see (Cetin & Topa, 2013)). Also, there are many published contributions concern with fractional boundary value problems with integral boundary conditions (see (Salem et al., 2019; Zhou & Qiao, 2018) and references given therein). Multi-point boundary value problems for differential equations become apparent naturally in scientific applications. For illustration, given a dynamical system with m degrees of freedom, there may be available exactly m cases spotted at m distinct times. A mathematical depiction of such problems in an m-point boundary value problem. Multi-point problems for differential equations are a special class of interface problems, and hence solvable with variant techniques. The studying of fractional differential equations with multi-point boundary conditions has been drawn the attention of many contributors (see (Derbazi1 et al., 2019; Lv, 2020; Salem & Alghamdi, 2019; Salem, Alzahrani, Alghamdi et al., 2020) and references given therein).

Motivated with the significance of multi-point, integral boundary conditions, and fractional Langevin equations in different branches of science and engineering, we discuss the existence and uniqueness of the solution to the following value problem with the modern techniques of functional analysis used in (Ahmad et al., 2019). Currently, we deal with the nonlinear fractional Langevin equation

$$ {}^{c}D^{\beta}({}^{c}D^{\alpha}+\lambda)x(t)=f(t,x(t)), \qquad t\in[0,1] \tag{1.1} $$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Liouville–Caputo's fractional derivatives of orders $0<\alpha\leq 1$, $1<\beta\leq 2$, $\lambda\in\mathbb{R}$ and the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuously differentiable. This equation is subjected to periodic, multi-point and non-local integral boundary terms.

$$ x(0)=x(1), \quad {}^{c}D^{\alpha}x(0)=0, \quad \sum_{i=1}^{m}a_i x(\zeta_i)=\mu\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}x(s)ds \tag{1.2} $$

where $\gamma>0$, $\mu\in\mathbb{R}\backslash\{0\}$, $0<\eta<\zeta_1<\zeta_2<\cdots<\zeta_m<1, m\in\mathbb{N}$, and $a_i\in\mathbb{R}\backslash\{0\}$ where $i=1,2,\cdots,m$.

We are eager to make sure that both non-local multi-point and Riemann-Liouville fractional integral are taken at a proper interval $(0,\eta)\subset[0,1]$ and is able to be expressed as the case of a linear combination of values of an unknown function $\zeta_i\in(0,1)$ which is proportional to the third boundary condition of an unknown function. We take here the function f in the general form which constitutes from the position $x(t)$ of the particle at time t. This function may contain external force field, position-dependent phenomenological fluid friction coefficient, intensity of the stochastic force, or zero-mean Gaussian white noise term.

Our goal is to determine sufficient conditions for the existence and uniqueness of the solution of Langevin equation in a non-linear case, including non-local integral boundary and non-local multi-point. Here, we use some mathematical analysis technique by applying famous fixed-point theorems: contraction mapping principle, Krasnoselskii, and Leray-Schauder.

Our research paper is organized as follows. Section 2 includes the main lemmas and definitions related to the expected answer of the linearly fractional Langevin equation including two fractional orders. The third section demonstrates the existence and uniqueness of a non-linear Langevin equation which have boundary value conditions mentioned above by applying Krasnoselskii's theorem, contraction mapping principle, and Leray-Schauder theorems.

2. Basic concepts and relevant lemmas

In this part, we start by giving the basic definitions of fractional calculus and the initially needed results. We are grateful to have found important terminology in the references (Kilbas et al., 2006; Podlubny, 1999). Let $C^n[a,b]$ and $AC^n[a,b]$ be the classes of all continuous and absolutely continuous functions up to n and $n-1$ order derivative, respectively.

Definition 2.1. *If $x(t) \in C[a,b]$. Then, the R-L fractional integral with order $p > 0$ exists almost everywhere on $[a,b]$ and can be represented in the form*

$$I^p x(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} x(s) ds.$$

Definition 2.2. *If $x(t) \in AC^n[a,b]$ and $n \in \mathbb{N}$. The Liouville-Caputo fractional derivative of order $n-1 < p \leq n$ exists almost everywhere on $[a,b]$ and can be represented in the form*

$${}^cD^p x(t) = \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} x^{(n)}(s) ds.$$

Lemma 2.1. *Let $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. If $x \in AC^n[a,b]$ and ${}^cD^\alpha x \in C[a,b]$, then*

$$I^{\alpha\, c}D^\alpha x(t) = x(t) + c_0 + c_1(t-a) + \cdots + c_{n-1}(t-a)^{n-1}$$

Lemma 2.2. *Let α be a positive real. Then we have*

$$I^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}, \qquad m > -1, t \in [0,b].$$

Theorem 2.1 (Krasnoselskii's theorem (Krasnoselskii, 1955)). *Let M be a bounded, closed, convex and nonempty subset of a Banach space $\mathcal{C}$. Let A and B be operators such that:*

(i) $Ax + By \in M$ whenever $x, y \in M$.

(ii) A is compact and continuous.

(iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.2 (Leray-Schauder non-linear alternating (Granas & Dugundji, 2003)). *Let $\mathcal{C}$ be a Banach space, $\mathcal{D}$ be a closed, convex, and $\mathcal{U}$ be a nonempty open subset of $\mathcal{D}$ with $0 \in \mathcal{U}$. If the operator $T : \overline{\mathcal{U}} \to \mathcal{D}$ is relatively compact. Then either:*

(i) T has a fixed point $x^* \in \overline{\mathcal{U}}$ or

(ii) there exists $x^* \in \partial\mathcal{U}$, and $\delta \in (0,1)$ such that $\delta T(x) = x$.

Lemma 2.3. *Consider the linear fractional Langevin equation*

$${}^cD^\beta({}^cD^\alpha + \lambda)x(t) = h(t), \qquad t \in [0,1] \tag{2.1}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $\lambda \in \mathbb{R}$, $x \in AC^3[0,1]$ and the function $h \in C[0,1]$. Then the boundary value problem (2.1) with the boundary conditions (1.2) *has a unique solution*

$$
\begin{aligned}
x(t) = {} & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \mu A(t) \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds \\
& - A(t) \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds + \lambda A(t) \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\
& + B(t) \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds - \lambda B(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds.
\end{aligned} \tag{2.2}
$$

where

$$\Delta = \lambda S(\alpha) - \lambda S(\alpha+1) + \Gamma(\alpha+1)S(0) \neq 0,$$

$$\Delta_1 = 1 + \frac{\lambda S(\alpha+1)}{\Delta},$$

$$A(t) = \frac{1}{\Delta}[\lambda t^\alpha (1-t) + \Gamma(\alpha+1)],$$

$$B(t) = \frac{S(\alpha+1)}{\Delta}(\lambda t^\alpha + \Gamma(\alpha+1)) - \lambda \Delta_1 t^{\alpha+1},$$

$$S(\alpha) = \sum_{i=1}^m a_i \zeta_i^\alpha.$$

Proof. By applying I^β on both sides of (2.1) with using Lemma 2.1, we have

$$D^\alpha x(t) = -\lambda x(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(t) + c_2 t + c_1. \tag{2.3}$$

Again, operate I^α on both sides of (2.3) to become

$$x(t) = -\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(t) + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(t) + c_2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + c_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + c_0. \tag{2.4}$$

As a results of the condition $D^\alpha x(0) = 0$, we get $c_1 = \lambda c_0$. By using the last two conditions of (1.2), the values of constants c_0, and c_1 can be evaluated as

$$
\begin{aligned}
\frac{c_1}{\Gamma(\alpha+1)} = \frac{\lambda}{\Delta} \Bigg[& \mu \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds - \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds \\
& + \lambda \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\
& + S(\alpha+1) \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds - \lambda S(\alpha+1) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \Bigg].
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
\frac{c_2}{\Gamma(\alpha+2)} = \lambda \Delta \Bigg[& \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds - \mu \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds - \lambda \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \Bigg] \\
& - \Delta_1 \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds + \lambda \Delta_1 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds.
\end{aligned} \tag{2.5}
$$

By direct substitution (2.5) and (2.6) in (2.4), the expression (2.2) has been satisfied. Conversely, (2.2) is a solution of (2.1) by direct substitution. ∎

Lemma 2.4. *For all* $t \in [0, 1]$, *we have*

$$A = \max_{t\in[0,1]} |A(t)| = \frac{1}{|\Delta|} \left[\frac{|\lambda| \alpha^\alpha}{(1+\alpha)^{1+\alpha}} + \Gamma(\alpha+1) \right],$$

$$B = \max_{t\in[0,1]}|B(t)| = \frac{|\lambda|\alpha^\alpha}{(1+\alpha)^{1+\alpha}}|\Delta_1|^\alpha\left|\frac{S(\alpha+1)}{\Delta}\right|^{1+\alpha} + \frac{|S(\alpha+1)|\Gamma(\alpha+1)}{|\Delta|}.$$

Proof. Put $f(t) = t^\alpha(1-t)$, which has been discussed in (Zhou & Qiao, 2018). Thus, we can deduce the first one. The second is similar. ⊠

3. Main results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ be a Banach space of all absolutely continuous functions with the following norm defined by

$$||x|| = \sup_{t\in[0,1]}|x(t)|.$$

Now, consider the following operator $T:\mathcal{C}\to\mathcal{C}$

$$\begin{aligned}(Tx)(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds - \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds + \mu A(t)\int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}x(s)ds \\ &\quad - A(t)\sum_{i=1}^m a_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds + \lambda A(t)\sum_{i=1}^m a_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds \\ &\quad + B(t)\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds - \lambda B(t)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds. \end{aligned} \tag{3.1}$$

We would like to introduce some mathematical computations in order to have more a convenient view.

$$Q_1 = \frac{A|S(\alpha+\beta)| + B + 1}{\Gamma(\alpha+\beta+1)}, \tag{3.2}$$

$$Q_2 = \frac{|\lambda|(A|S(\alpha)| + B + 1)}{\Gamma(\alpha+1)} + \frac{|\mu|A\eta^\gamma}{\Gamma(\gamma+1)}. \tag{3.3}$$

Here, we divide the operator $(Tx)(t)$ as follows:

$$(Tx)(t) = (T_1x)(t) + (T_2x)(t) \tag{3.4}$$

where

$$\begin{aligned}(T_1x)(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds - A(t)\sum_{i=1}^m a_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds \\ &\quad + B(t)\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(T_2x)(t) &= -\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds + \mu A(t)\int_0^\eta\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}x(s)ds \\ &\quad + \lambda A(t)\sum_{i=1}^m a_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \lambda B(t)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds. \end{aligned} \tag{3.6}$$

The first result of the existence of a solution is obtained in the following theorem via applying Krasnoselskii's theorem.

Theorem 3.1. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Consider the assumption*

(ω_1) $|f(t,x)| \le \rho(t)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$ *with* $\rho\in C([0,1],\mathbb{R}_+)$.

Then, (1.1)–(1.2) *has a solution under the following condition*:

$$Q_2 < 1 \tag{3.7}$$

where Q_2 *is defined in* (3.3).

Proof. Consider $\mathcal{B}_r = \{x \in \mathcal{C} : \| x \| \leq r\}$ with

$$r \geq \| \rho \| Q_1 (1 - Q_2)^{-1}, \quad Q_2 < 1. \tag{3.8}$$

From (3.4), we obtain

$$|(Tx)(t)| \leq |(T_1 x)(t)| + |(T_2 x)(t)|. \tag{3.9}$$

By taking the norm over all $t \in [0, 1]$, we have

$$\| Tx \| \leq \| T_1 x \| + \| T_2 x \| . \tag{3.10}$$

Let $x \in \mathcal{B}_r$. Then

$$|(T_1 x)(t)| = \left| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, x(s)) ds - A(t) \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, x(s)) ds \right.$$

$$\left. + B(t) \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, x(s)) ds \right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s, x(s))| ds + |A(t)| \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s, x(s))| ds$$

$$+ |B(t)| \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s, x(s))| ds$$

$$\leq |\rho(t)| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds + A \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds + B \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds$$

$$= \frac{|\rho(t)| \left[t^{\alpha+\beta} + A|S(\alpha+\beta)| + B \right]}{\Gamma(\alpha+\beta+1)}$$

By taking the norm over all $t \in [0, 1]$, we obtain

$$\| T_1 x \| = \sup_{t \in [0,1]} \left| \frac{|\rho(t)| \left[t^{\alpha+\beta} + A|S(\alpha+\beta)| + B \right]}{\Gamma(\alpha+\beta+1)} \right| = \frac{\| \rho \| \left[1 + B + A|S(\alpha+\beta)| \right]}{\Gamma(\alpha+\beta+1)}. \tag{3.11}$$

Similarly, if $y \in \mathcal{B}_r$. Then

$$|(T_2 y)(t)| \leq |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds + |\mu| |A(t)| \int_0^\eta \frac{(\eta - s)^{\gamma-1}}{\Gamma(\gamma)} |y(s)| ds$$

$$+ |\lambda| |A(t)| \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds + |\lambda| |B(t)| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds$$

$$\leq |\lambda| r \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\mu| A r \int_0^\eta \frac{(\eta - s)^{\gamma-1}}{\Gamma(\gamma)} ds$$

$$+ |\lambda| A r \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\lambda| B r \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$$

$$=\frac{|\mu|Ar\eta^{\gamma}}{\Gamma(\gamma+1)}+\frac{|\lambda|rt^{\alpha}+|\lambda S(\alpha)|Ar+|\lambda|Br}{\Gamma(\alpha+1)}.$$

this implies that

$$\| T_2y \|=\sup_{t\in[0,1]}\left|\frac{|\mu|Ar\eta^{\gamma}}{\Gamma(\gamma+1)}+\frac{|\lambda|rt^{\alpha}+|\lambda S(\alpha)|Ar+|\lambda|Br}{\Gamma(\alpha+1)}\right|=\frac{|\mu|Ar\eta^{\gamma}}{\Gamma(\gamma+1)}+\frac{|\lambda|r+|\lambda S(\alpha)|Ar+|\lambda|Br}{\Gamma(\alpha+1)}. \tag{3.12}$$

Inserting (3.11) and (3.12) into (3.10) gives

$$\| T_1x+T_2y \| \leq \| \rho \| Q_1+rQ_2 \ \leq \ r.$$

Thus, the first item of Theorem 2.1 has been satisfied well. Suppose $x,y\in\mathcal{B}_r$ and $t\in[0,1]$. Then

$$\begin{aligned}|(T_2x)(t)-(T_2y)(t)| \ &\leq \ |\lambda|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds+|\mu|A\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}|x(s)-y(s)|ds\\ &\quad+|\lambda|A\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds+|\lambda|B\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)\\ &\quad-y(s)|ds\\ &=\left[\frac{|\mu|A\eta^{\gamma}}{\Gamma(\gamma+1)}+\frac{|\lambda|+|\lambda S(\alpha)|A+|\lambda|B}{\Gamma(\alpha+1)}\right]|x(t)-y(t)|.\end{aligned} \tag{3.13}$$

By taking the norm over all $t\in[0,1]$, we get

$$\| (T_2x)-(T_2y) \|=\sup_{t\in[0,1]}|(T_2x)(t)-(T_2y)(t)|=Q_2 \| x-y \| . \tag{3.14}$$

Since f is given as a continuous function, this implies that the operator T_1 is continuous on $\mathcal{B}_r$. Moreover, T_1x is uniformly bounded by (3.11). Suppose that $0 \ \leq \ t_1<t_2 \ \leq \ 1$. Then

$$|A(t_2)-A(t_1)| \ \leq \ \frac{|\lambda|}{|\Delta|}\left[|t_2^{\alpha}-t_1^{\alpha}|+|t_2^{\alpha+1}-t_1^{\alpha+1}|\right] \tag{3.15}$$

and

$$|B(t_2)-B(t_1)| \ \leq \ \frac{|\lambda S(\alpha+1)|}{|\Delta|}|t_2^{\alpha}-t_1^{\alpha}|+|\lambda\Delta_1||t_2^{\alpha+1}-t_1^{\alpha+1}| \tag{3.16}$$

which can be used to find that

$$\begin{aligned}|(T_1x)(t_2)-(T_1x)(t_1)| \ &\leq \ \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))ds\right|+|A(t_2)\\ &\quad-A(t_1)|\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))|ds+|B(t_2)-B(t_1)|\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))|ds\\ &\leq \| \rho\\ &\| \left[\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+\int_0^{t_1}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+|A(t_2)\right.\\ &\quad\left.-A(t_1)|\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+|B(t_2)-B(t_1)|\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\right]\\ &\leq \ \frac{\| \rho \|}{\Gamma(\alpha+\beta+1)}\left[|t_2^{\alpha+\beta}-t_1^{\alpha+\beta}|+\frac{|\lambda S(\alpha+1)|}{|\Delta|}|t_2^{\alpha}-t_1^{\alpha}|+|\lambda\Delta_1||t_2^{\alpha+1}-t_1^{\alpha+1}|+\frac{|\lambda|}{|\Delta|}\left[|t_2^{\alpha}-t_1^{\alpha}|+|t_2^{\alpha+1}-t_1^{\alpha+1}|\right]\right].\end{aligned}$$

which does not count on x and goes to zero uniformly when $t_2\to t_1$. This shows that T_1 is relatively compact on $\mathcal{B}_r$. Hence, by Arzeliá-Ascoli theorem, T_1 is completely continuous on $\mathcal{B}_r$. All terms of theorem 2.1 have been satisfied well. Accordingly, theorem 2.1 guarantees that (1.1)–(1.2) has a solution on $[0,1]$.□

Theorem 3.2. *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and the following condition holds*

(ω_2) $|f(t,x)-f(t,y)| \le L|x-y|, \forall (t,x)\in[0,1]\times\mathbb{R}.$

Then, (1.1)–(1.2) *has a unique solution if*

$$J = LQ_1 + Q_2 < 1$$

where Q_1 *and* Q_2 *are given in* (3.2) *and* (3.3) *respectively.*

Proof. Consider $\mathcal{B}_r = \{x\in\mathcal{C} : \| x \| \le r\}$ with

$$r \ge (MQ_1)/(1-\mathbf{J})$$

where

$$M = \sup_{t\in[0,1]} |f(t,0)|.$$

For each $t\in[0,1]$ and $x\in\mathcal{B}_r$

$$|f(t,x(t)| = |f(t,x(t)) - f(t,0) + f(t,0)|$$

$$\le |f(t,x(t)) - f(t,0)| + |f(t,0)| \le L|x(t)| + |f(t,0)|$$

which implies that

$$\sup_{t\in[0,1]} |f(t,x(t)| \le Lr + M.$$

Briefly, Our aim is to show that $\forall x\in\mathcal{B}_r$ implies that $T(\mathcal{B}_r)\subseteq\mathcal{B}_r$. The norm of the operators T_1 and T_2 are given as follows

$$\| T_1x \| = \sup_{t\in[0,1]}\left|\frac{(Lr+M)\left[t^{\alpha+\beta}+A|S(\alpha+\beta)|+B\right]}{\Gamma(\alpha+\beta+1)}\right| = \frac{(Lr+M)[1+B+A|S(\alpha+\beta)|]}{\Gamma(\alpha+\beta+1)}.$$

and

$$\| T_2x \| = \frac{|\mu|Ar\eta^{\gamma}}{\Gamma(\gamma+1)} + \frac{|\lambda|r + |\lambda S(\alpha)|Ar + |\lambda|Br}{\Gamma(\alpha+1)}.$$

By using (3.10), it yields

$$\| Tx \| \le \mathbf{J}r + MQ_1 \le r.$$

It remains to show that T is a contraction mapping, we have already shown that T_2 satisfies the contraction principle as it has been detailed in (3.14). It suffices to demonstrate that T_1 is a contraction mapping. Let $x,y\in\mathcal{B}_r$ and $t\in[0,1]$, we have

$$\begin{aligned}|(T_1x)(t)-(T_1y)(t)| \le\ & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))-f(s,y(s))|ds \\ & + |A(t)|\sum_{i=1}^m |a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))-f(s,y(s))|ds \\ & + |B(t)|\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))-f(s,y(s))|ds \\ \le\ & L\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(s)-y(s)|ds + LA\sum_{i=1}^m|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(s)-y(s)|ds \\ & + LB\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|x(s)-y(s)|ds\end{aligned}$$

$$= \frac{L\left[t^{\alpha+\beta} + B + A|S(\alpha+\beta)|\right]}{\Gamma(\alpha+\beta+1)} |x(s) - y(s)|. \tag{3.17}$$

Taking the norm over all $t \in [0,1]$

$$\| T_1x - T_1y \| = \sup_{t\in[0,1]} \left[\left(\frac{L\left[t^{\alpha+\beta} + B + A|S(\alpha+\beta)|\right]}{\Gamma(\alpha+\beta+1)} \right) |x(s) - y(s)| \right]$$
$$= \left[\frac{L[1 + B + A|S(\alpha+\beta)|]}{\Gamma(\alpha+\beta+1)} \right] \| x - y \| . \tag{3.18}$$

By plugging (3.14) and (3.18) into (3.10) gives

$$\| (Tx) - (Ty) \| \leq \mathbf{J} \| x - y \| .$$

By employing the Banach fixed point theorem, we conclude that T has a unique fixed point in $\mathcal{B}_r$. Thus, the boundary value problem (1.1)-(1.2) possess a unique solution on $[0,1]$. □

Theorem 3.3. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, assume the following conditions hold:*

(ω_3) *There is a function $g \in C([0,1], \mathbb{R}^+)$, and non-decreasing one* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x)| \leq g(t)\varphi(\| x \|), \quad \forall (t,x) \in [0,1] \times \mathbb{R}.$$

(ω_4) *There is a constant $\Theta > 0$ such that*

$$\frac{\Theta}{\| g \| \varphi(\Theta)Q_1 + Q_2\Theta.} > 1$$

where Q_1 and Q_2 are shown in (3.2) *and* (3.3) *respectively.*

Then, there exists at least one solution in $[0,1]$ *for the boundary value problem* (1.1)- (1.2).

Proof. Let $T : \mathcal{C} \to \mathcal{C}$ be an operator defined in (3.1). Indeed,

$$\| T_1x \|$$
$$= \sup_{t\in[0,1]} \left| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))ds - A(t) \sum_{i=1}^m a_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))ds + B(t) \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))ds \right|$$
$$\leq \sup_{t\in[0,1]} \left[\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s))|ds + A \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s))|ds + B \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s))|ds \right]$$
$$\leq \frac{\| g \| \varphi(\| x \|)[1 + B + A|S(\alpha+\beta)|]}{\Gamma(\alpha+\beta+1)} \tag{3.19}$$

Inserting (3.12) and (3.19) into (3.10) to estimate

$$\| Tx \| \leq \frac{|\mu|Ar\eta^\gamma}{\Gamma(\gamma+1)} + \frac{[A|S(\alpha+\beta)| + B + 1] \| g \| \varphi(\| x \|)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|[A|S(\alpha)| + B + 1]r}{\Gamma(\alpha+1)} = \| g \| \varphi(\| x \|)Q_1 + Q_2r. \tag{3.20}$$

In view of Theorem 3.1, the operator T_1 is equicontinuous. To show that T_2 is equicontinuous operator. Let $0 \leq t_1 < t_2 \leq 1$. Then,

$$|(T_2x)(t_2) - (T_2x)(t_1)| \leq |\lambda| \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)|ds + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)|ds \right.$$

$$-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)|ds\Bigg]+|A(t_2)-A(t_1)|$$

$$\left(|\mu|\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}|x(s)|ds+|\lambda|\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)|ds\right)$$

$$+|B(t_2)-B(t_1)|\left(|\lambda|\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)|ds\right)$$

$$\leq\frac{|\lambda|r}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})+\left[\frac{|\mu|\eta^{\gamma}r}{\Gamma(\gamma+1)}+\frac{|\lambda||S(\alpha)|r|}{\Gamma(\alpha+1)}\right]|A(t_2)-A(t_1)|$$

$$+\left(\frac{|\lambda|r}{\Gamma(\alpha+1)}\right)|B(t_2)-B(t_1)|$$

$$\leq\frac{|\lambda|r}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})+\left[\frac{|\mu|\eta^{\gamma}r}{\Gamma(\gamma+1)}+\frac{|\lambda||S(\alpha)|r|}{\Gamma(\alpha+1)}\right]\frac{|\lambda|}{|\Delta|}\left[|t_2^{\alpha}-t_1^{\alpha}|\right.$$

$$\left.+|t_2^{\alpha+1}-t_1^{\alpha+1}|\right]+\frac{|\lambda S(\alpha+1)|}{|\Delta|}|t_2^{\alpha}-t_1^{\alpha}|+|\lambda\Delta_1||t_2^{\alpha+1}-t_1^{\alpha+1}|.$$

The right-hand side approaches zero uniformly once $t_2\to t_1$ which illustrates that the right-hand side does not rely on x at all. By Arzelá-Ascoli theorem, the operator T_2 is completely continuous and so does the operator T.

To complete the proof, we illustrate that the solution of the equation $x=\sigma Tx,\forall\sigma\in[0,1]$ is bounded. Indeed, we see that

$$\| x \|=\| \sigma Tx \| \leq \| g \| \varphi(\| x \|)Q_1+Q_2 \| x \| .$$

Then

$$\frac{\| x \|}{\| g \| \varphi(\| x \|)Q_1+Q_2 \| x \|} \leq 1$$

.
According to (ω_4), there is a positive real number Θ such that

$\| x \| \neq\Theta$. Set

$W=\{x\in C([0,1],\mathbb{R}) :\| x \| <\Theta\}$.

Then, the operator $T:\overline{W}\to C([0,1],\mathbb{R})$ is completely continuous. Since W has been chosen, impossibly, we are able to have at least one element such $x\in\partial W$ solving $x=\sigma Tx$ for some $\sigma\in[0,1]$. As a result, by theorem2.2, we conclude that T has a fixed point $x\in\overline{W}$ which gives a solution to (1.1)–(1.2).□

The fourth outcome of the current paper depends on Leray-Schauder degree theorem.

Theorem 3.4. *Suppose* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *such that*

$$|f(t,x)| \leq a|x|+b \quad \forall t\in[0,1] \quad \forall x\in C[0,1],$$

such that a and b are positive real numbers with

$$0 \leq a < 1 - Q_2 Q_1$$

where Q_1 and Q_2 are defined in (3.2) *and* (3.3) *respectively. Then,* (1.1)–(1.2) *has a solution.*

Proof. Let $T : C[0,1] \to C[0,1]$ be an operator as in (3.1). Define

$$T : \overline{\mathcal{B}_r} \to C[0,1]$$

such that

$$x \neq \delta Tx \quad \forall x \in \partial \mathcal{B}_r, \quad \forall \delta \in [0,1] \tag{3.21}$$

where $\mathcal{B}_r = \{x \in C[0,1] : \| x \| < r\}$.

By Arzelá-Ascoli theorem, the operator $h_\delta(x) = x - \delta Tx$ is a complete continuous. In the case of (3.21) is completely correct, then by Leray-Schauder degrees are well defined. Moreover, by homotopy invariance of Leray-Schauder of topological degree, it leads to get

$$deg(h_\delta, \mathcal{B}_r, 0) = deg(I - \delta T, \mathcal{B}_r, 0) = deg(h_1, \mathcal{B}_r, 0) = deg(h_0, \mathcal{B}_r, 0) = deg(I, \mathcal{B}_r, 0) = 1 \neq 0; \quad 0 \in \mathcal{B}_r.$$

where I is the unit one. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - Tx = 0$ has at least one solution in $\mathcal{B}_r$. It remains to have the proof of (3.21). So, suppose by way of contradiction that $x = \delta Tx$ for some $\delta \in [0,1] \forall t \in [0,1]$, we get

$$\| x \| = \| \delta Tx \| \leq Q_1(a \| x \| + b) + Q_2 \| x \| .$$

which leads to

$$\| x \| \leq \frac{Q_1 b}{1 - (aQ_1 + Q_2)}$$

with $0 < a < (1 - Q_2)/Q_1$. Choose

$$r = \frac{Q_1 b}{1 - (aQ_1 + Q_2)} + 1,$$

which it asserts that (3.21) is held. The proof has been done.

4. Numerical example

Example 4.1. *Let us study the uniqueness of the following boundary value problem*:

$$\begin{cases} {}^cD^{\frac{4}{3}}\left({}^cD^{\frac{1}{2}} + \frac{1}{20}\right)x(t) = \frac{1}{(t+4)^2}\frac{1}{(3+x^2)}, \qquad 0 \leq t \leq 1 \\ x(0) = x(1),\ D^{\frac{1}{2}}x(0) = 0,\ \frac{1}{3}x\left(\frac{1}{2}\right) + \frac{1}{2}x\left(\frac{2}{3}\right) + \frac{2}{3}x\left(\frac{3}{4}\right) = \frac{1}{4}\int_0^{\frac{1}{6}} \frac{\left(\frac{1}{6}-s\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} x(s)ds. \end{cases} \tag{4.1}$$

Obviously, $\beta = \frac{4}{3}$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{20}$, $\eta = \frac{1}{6}$, $\gamma = \frac{3}{2}, \mu = \frac{1}{4}$, $\zeta_1 = \frac{1}{2}$, $\zeta_2 = \frac{2}{3}, \zeta_3 = \frac{3}{4}, a_1 = \frac{1}{3}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$. *By applying Theorem* 3.2, *we find* $Q_1 \approx 1.199413464$, $Q_2 \approx 0.142317173$, $L = \frac{1}{16}$ *and* $J \approx 0.217280514$, *which means* (4.1) *has a uniqueness property for its solution.*

Example 4.2. *In this example, we will apply Theorem* 3.1. *So, let's reconsider the example* (4.1). *By checking its conditions, we obtain* $\rho(t) = \frac{1}{(t+4)^2}$. *Furthermore, it satisfies theorem* 3.1 *since* $Q_2 \approx 0.142317173$ *Thus,* (4.1) *has a solution.*

Example 4.3. *Discuss the existence of the example below*:

$$\begin{cases} {}^{c}D^{\frac{4}{3}}\left({}^{c}D^{\frac{1}{2}}+\frac{1}{5}\right)x(t)=\frac{e^{-t^2}(2+x^2)}{9\sqrt{1+t^4}}, \quad 0\le t\le 1 \\ x(0)=x(1),\ D^{\frac{1}{2}}x(0)=0,\ \frac{1}{3}x(\frac{1}{2})+\frac{1}{2}x(\frac{2}{3})+\frac{2}{3}x(\frac{3}{4})=\frac{1}{2}\int_0^{\frac{1}{6}}\frac{(\frac{1}{6}-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}x(s)ds. \end{cases} \tag{4.2}$$

Clearly, $|f(t,x)|\le\frac{e^{-t^2}(2+x^2)}{9\sqrt{1+t^4}}, \beta=\frac{4}{3}, \alpha=\frac{1}{2}, \lambda=\frac{1}{5}, \eta=\frac{1}{6}, \gamma=\frac{3}{2}, \mu=\frac{1}{2}, \zeta_1=\frac{1}{2}, \zeta_2=\frac{2}{3}, \zeta_3=\frac{3}{4}, a_1=\frac{1}{3}, a_2=\frac{1}{2}, a_3=\frac{2}{3}$.

By using Theorem 3.3, *we have* $g(t)=\frac{e^{-t^2}}{9\sqrt{1+t^4}}$ *with* $\| g \|=\frac{1}{9}$, $\varphi(\| x \|)=2+\| x\|^2$, $Q_1\approx 1.204720615$, *and* $Q_2\approx 0.555750967$. *Using* ω_4 *of Theorem* 3.3 *which indicates there are two positive numbers* $\Theta_1\approx 6.711295635$, *and* $\Theta_2\approx 0.2980050513$. *Theorem* 3.3 *confirms the existence of a solution for* (4.2) *in* $[0,1]$.

Example 4.4. *Here, we apply Theorem* 3.4 *to study the existence of a solution as follows.*

$$\begin{cases} {}^{c}D^{\frac{4}{3}}({}^{c}D^{\frac{1}{2}}+\frac{1}{5})x(t)=\frac{1}{(t+2)^2}|x|+\frac{5}{t+1}, 0\le t\le 1 \\ x(0)=x(1),\ D^{\frac{1}{2}}x(0)=0,\ \frac{1}{3}x(\frac{1}{2})+\frac{1}{2}x(\frac{2}{3})+\frac{2}{3}x(\frac{3}{4})=\frac{1}{2}\int_0^{\frac{1}{6}}\frac{(\frac{1}{6}-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}x(s)ds. \end{cases} \tag{4.3}$$

Clearly, $|f(t,x)|\le\frac{1}{4}|x|+5, \beta=\frac{4}{3}, \alpha=\frac{1}{2}, \lambda=\frac{1}{5}, \eta=\frac{1}{6}, \gamma=\frac{3}{2}, \mu=\frac{1}{2}, \zeta_1=\frac{1}{2}, \zeta_2=\frac{2}{3}, \zeta_3=\frac{3}{4}, a_1=\frac{1}{3}, a_2=\frac{1}{2}, a_3=\frac{2}{3}$.

By using Theorem 3.4, *we see that* $a=0.25, Q_1\approx 1.204720615$ *and* $Q_2\approx 0.555750967$. *It shows that* $a<\frac{1-Q_2}{Q_1}\approx 0.368756894$. *Thus, the Theorem* 3.4 *guarantees that the existence of a solution to example* (4.4) *in* $[0,1]$.

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